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Chiral anomalies and zeta-function regularization

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The zeta-function method for regularizing determinants is used to calculate the chiral anomalies of several field-theory models. In SU(N) gauge theories without γ_5 couplings, the results of perturbation theory are obtained in an unambiguous manner for the full gauge theory as well as for the corresponding external-field problem. If axial-vector couplings are present, different anomalies occur for the two cases. The result for the full gauge theory is again uniquely determined; for its nongauge analog, however, ambiguities can arise. The connection between the basic path integral and the operator used to construct the heat kernel is investigated and the significance of its Hermiticity and gauge covariance are analyzed. The implications of the Wess-Zumino conditions are considered.

I. INTRODUCTION

Since the discovery that in spinor field theories the axial-vector current is not necessarily conserved even if mass terms are absent,[1] various regularization schemes have been devised to calculate such anomalies. Here we only mention the point-separation method[2] and the Pauli-Villars regulator scheme,[3] which were applied not only to the simple *VVA* triangle of Abelian and non-Abelian vector gauge theories, but also, for instance, to an SU(3)$\times$SU(3) theory with external vector, axial-vector, scalar, and pseudoscalar fields. Recently, Fujikawa[4,5] gave a new, very compact and nondiagrammatic derivation for such anomalies in theories which may also contain axial-vector couplings. In the framework of path-integral quantization he showed that the chiral noninvariance of the fermionic integration measure is the origin of the anomalous Ward-Takahashi identities or, equivalently, the anomalous divergence of the (axial-) vector current. However, as was first realized by McKay and Young,[6] his method reproduces the results of perturbation theory only if there are no γ_5 couplings involved. The anomaly for a simple Abelian model, where only the left-handed fermions couple to the gauge field, is in contradiction (by a factor of $\frac{1}{3}$) with the well-known results obtained within the other schemes. The crucial point in Fujikawa's derivation is the interpretation of an ill-defined infinite sum, which has to be regularized by introducing a convergence factor $\exp(-C/M^2)$, where C is an *a priori* unknown cutoff operator. The value of this sum is not independent of the choice of C and so the question arises as to which operator is the correct one. For the SU(N) vector gauge theory (e.g., for QCD), the Adler-Bell-Jackiw anomaly is reproduced by setting $C=\not{D}^2$, with D_μ being the covariant derivative. However, if we were to use $C=\partial^2$, the sum still would be regular, but we would obtain no anomaly at all.[7] The preferred role played by $\not{D}^2$ is that it is gauge covariant, whereas ∂^2 is not. Thus the additional requirement of gauge covariance is sufficient to obtain a unique result from Fujikawa's approach (just as in perturbation theory).

The situation becomes more complicated for theories in which left- and right-handed fermions couple differently to the gauge fields, as, for example, in the Glashow-Weinberg-Salam model of electroweak interaction. It was demonstrated by Einhorn and Jones[8] and by Hu, Young, and McKay[9] that in such theories one may not use the gauge-covariant cutoff operators advocated by Fujikawa if one wants to reproduce the expressions found by point-separation methods or by Pauli-Villars regularization. Because of this contradiction between the path-integral approach in Refs. 4 and 5 and the more conventional methods, it seems desirable to have a nonperturbative method which automatically produces well-defined expressions at every stage of the calculation, i.e., where *ad hoc* modifications of divergent quantities are not necessary. A technique which meets this requirement is the zeta-function regularization which was introduced into field theory by Dowker and Critchley[10] and by Hawking[11] as a method to evaluate determinants of elliptic operators appearing in the evaluation of path integrals. Using a variant of this method, the Adler-Bell-Jackiw anomaly was rederived by Schroer and others,[12] and for a *VA* theory, the result of Bardeen[2] was reproduced by Balachandran *et al.*[13] In these works, a heat-kernel regularization of the above-mentioned sum is employed, but, just as in Refs. 4 and 5, the cutoff operator, i.e., the operator which appears in the heat equation, is not deduced from first principles. As we shall see, it is this way from the fundamental path integral to the heat kernel which is responsible for the different conclusions reached in Ref. 5 and Refs. 8 and 9.

In this paper we will use "zeta-function regularization" according to its original meaning as a way to regularize determinants. Starting from the basic definition[11,14–16]

$$\det A=\exp[-\zeta'(A\mid 0)] \tag{1.1}$$

we construct a generating functional which, upon suitable differentiations, leads to regularized matrix elements of the relevant currents and their divergences. A second approach is to use (1.1) to compute the functional Jacobian for chiral transformations, which in turn implies the desired anomalous divergences or Ward-Takahashi identities.

In Secs. II and III, we explain these methods in some detail for a pure vector gauge theory and show that both of them correctly yield the Adler-Bell-Jackiw anomaly term. Then, in Sec. IV, we repeat our analysis for a simple model containing a γ_5 coupling, and, in Sec. V, for an $SU(N)\times SU(N)$ model with vector, axial-vector, scalar, and pseudoscalar fields. It turns out there that the fully quantized gauge theory (for $N=2$) and the associated nongauge theory must be treated on a different footing. In the first case, one obtains a unique result in accord with the findings in Refs. 4 and 5, but in the second case, ambiguities occur. Finally, in Sec. VI, we formulate the Wess-Zumino consistency conditions for our model and investigate how they restrict possible choices for the cutoff operator.

II. THE GENERATING FUNCTIONAL

As a first example to demonstrate our method, we consider now an SU(N) vector gauge theory. The calculational details for this model are presented at some length because they are also typical of the more complicated models to be discussed later.

Our Lagrangian for N_f flavors of massive Dirac fermions reads

$$\mathscr{L}=\bar{\psi}(i\not{D}-m)\psi-\tfrac{1}{4}F^a_{\mu\nu}F^{a\mu\nu} \tag{2.1}$$

with

$$D_\mu\equiv\partial_\mu+igA_\mu\equiv\partial_\mu+igA^a_\mu T^a \tag{2.2}$$

and

$$F^a_{\mu\nu}=\partial_\mu A^a_\nu-\partial_\nu A^a_\mu-gf^{abc}A^b_\mu A^c_\nu\ . \tag{2.3}$$

Flavor indices are suppressed throughout and the generators T^a of the fundamental representation of SU(N) are normalized according to

$$\mathrm{tr}(T^aT^b)=\tfrac{1}{2}\delta^{ab}\ . \tag{2.4}$$

Next, we perform a Wick rotation $[x^0\to ix^4, A_0\to iA_4, \gamma^0\to -i\gamma^4,\ g=(+---)\to g=(----)]$ to Euclidean space-time and define for real functions Q^μ and K the functional

$$Z[Q^\mu,K]=\int[d\psi\, d\bar{\psi}\, dA]\times\exp\left[\int d^4x[\,\bar{\psi}(i\not{D}-m-i\not{Q}\gamma_5-K\gamma_5)\psi-\tfrac{1}{4}F^a_{\mu\nu}F^{a\mu\nu}]\right] \tag{2.5}$$

with the measure $[dA]$ containing the gauge-fixing term in the Lagrangian and the corresponding Faddeev-Popov determinant. By suitably choosing the boundary conditions, we now can express matrix elements of $j^\mu_5\equiv\bar{\psi}\gamma^\mu\gamma_5\psi$ as a derivative of Z with respect to Q^μ:

$$\partial^\mu_w\langle\bar{\psi}(w)\gamma_\mu\gamma_5\psi(w)\rangle=i\partial^\mu_w\left.\frac{\delta Z[Q,K]}{\delta Q^\mu(w)}\right|_{Q=K=0}\ . \tag{2.6}$$

To proceed, we integrate over the fermion fields in (2.5) with the result

$$Z[Q,K]=\int[dA]\det(i\not{D}-m-i\not{Q}\gamma_5-K\gamma_5)\times\exp\left[-\tfrac{1}{4}\int d^4xF^a_{\mu\nu}F^{a\mu\nu}\right]\ . \tag{2.7}$$

For the ζ-function technique to be applicable, we have to rewrite (2.7) in terms of the determinant of a positive operator. To this end, we use

$$\det(i\not{D}-m-i\not{Q}\gamma_5-K\gamma_5)=\det(-i\not{D}-m+i\not{Q}\gamma_5-K\gamma_5) \tag{2.8}$$

leading to

$$\begin{aligned}Z[Q,K]&=\int[dA][\det(i\not{D}-m-i\not{Q}\gamma_5-K\gamma_5)\det(-i\not{D}-m+i\not{Q}\gamma_5-K\gamma_5)]^{1/2}\exp\left[-\tfrac{1}{4}\int d^4xF^a_{\mu\nu}F^{a\mu\nu}\right]\\&\equiv\int[dA]\exp\left[\tfrac{1}{2}\ln\det\Omega(A,Q,K)-\tfrac{1}{4}\int d^4xF^a_{\mu\nu}F^{a\mu\nu}\right]\ .\end{aligned} \tag{2.9}$$

In order to compute the determinant of Ω, we construct the generalized ζ function[10,11] of the Hermitian operator

$$\begin{aligned}\Omega_x=m^2&+[\not{D}_x-\not{Q}(x)\gamma_5]^2\\&+[m-i\not{D}_x+i\not{Q}(x)\gamma_5]K(x)\gamma_5\\&+K(x)\gamma_5[m+i\not{D}_x-i\not{Q}(x)\gamma_5]+K^2(x)\end{aligned} \tag{2.10}$$

which is defined by

$$\zeta(\Omega\,|\,s)=\sum_n\Lambda_n{}^{-s}\ , \tag{2.11}$$

where the $\{\Lambda_n\}$ are the eigenvalues of Ω and the sum runs over the whole spectrum. (Recall that the Euclidean $\not{D}$ is Hermitian and that $\gamma^\dagger_\mu=-\gamma_\mu$ if we use the conventions of Ref. 17.) We note that $\Lambda_n>0$ for small values of $Q^\mu(x)$ and $K(x)$; this is sufficient for the method to be applicable, because one needs the generating functional only for infinitesimal values of the sources. Because the spectrum is not known explicitly, ζ cannot be calculated from its definition (2.11), so that we have to take another way. We first consider a complete set $\{\phi_n\}$ of orthonormalized eigenfunctions of Ω with eigenvalues $\{\Lambda_n\}$, which is assumed to exist because of the Hermiticity of Ω. Furthermore, for the case of vanishing sources Q and K, i.e., for $\Omega=\not{D}^2+m^2$, we define $\phi_n(Q=K=0)=\varphi_n$ and $\Lambda_n(Q=K=0)=\lambda_n$ as the eigenfunctions and eigenvalues, respectively. Now one constructs the heat kernel

$$G_{\alpha\beta,ij}(x,y;t)=\sum_n e^{-\Lambda_n t}\phi_{n\alpha i}(x)\phi^\dagger_{n\beta j}(y)\ , \tag{2.12}$$

where α,β (i,j) denote spinor (color) indices. In terms of G, the zeta function (2.11) is expressed as the Mellin transform

$$\zeta(\Omega\,|\,s)=\frac{1}{\Gamma(s)}\int_0^\infty dt\,t^{s-1}\mathrm{tr}_{\gamma c}\int d^4x\,G(x,x;t)\ , \tag{2.13}$$

whereby $\mathrm{tr}_{\gamma c}$ means the trace in both spinor and color space. It is easy to see that due to the definition of the $\{\phi_n\}$, the function G obeys the heat equation

$$\left[\Omega_x+\frac{\partial}{\partial t}\right]G(x,y;t)=0 \tag{2.14}$$

with the initial condition $G(x,y;0)=\delta(x-y)$. The function defined by (2.11) or (2.13) for $\mathrm{Re}(s)$ sufficiently large can be analytically continued to a meromorphic function on the whole complex s plane; in particular, it is possible to define the regularized determinant of Ω by[11,14]

$$\det\Omega=\exp[-\zeta'(\Omega\,|\,0)]\ . \tag{2.15}$$

Hence, Eq. (2.9) can be written as

$$Z[Q,K]=\int[dA]\exp[-\tfrac{1}{2}\zeta'(\Omega\,|\,0)-\tfrac{1}{4}F^a_{\mu\nu}F^{a\mu\nu}]\ . \tag{2.16}$$

Thus it follows from (2.6) that

$$\begin{aligned}\partial^\mu_w\langle\bar\psi(w)\gamma_\mu\gamma_5\psi(w)\rangle&=\int[d\psi\,d\bar\psi\,dA]\left[-\frac{i}{2}\partial^\mu_w\frac{\delta\zeta'(\Omega\,|\,0)}{\delta Q^\mu(w)}\bigg|_{Q=K=0}\right]\exp\left[\int d^4x[\bar\psi(i\not{D}-m)\psi-\tfrac{1}{4}F^a_{\mu\nu}F^{a\mu\nu}]\right]\\&\equiv\left\langle-\frac{i}{2}\partial^\mu_w\frac{d}{ds}\bigg|_0\frac{\delta\zeta(\Omega\,|\,s)}{\delta Q^\mu(w)}\bigg|_{Q=K=0}\right\rangle\ .\end{aligned} \tag{2.17}$$

As always, analytical continuation with respect to s is understood. Equation (2.13) shows that one has to calculate the derivative of G with respect to the sources:

$$\frac{\delta\zeta(\Omega\,|\,s)}{\delta Q^\mu(w)}=\frac{1}{\Gamma(s)}\int_0^\infty dt\,t^{s-1}\mathrm{tr}_{\gamma c}\int d^4x\frac{\delta G(x,x;t)}{\delta Q^\mu(w)}\ . \tag{2.18}$$

For this purpose, we use the variation of the heat equation

$$\left[\Omega_x+\frac{\partial}{\partial t}\right]\frac{\delta G(x,y;t)}{\delta Q^\mu(w)}=-\frac{\delta\Omega_x}{\delta Q^\mu(w)}G(x,y;t) \tag{2.19}$$

together with the initial condition $\delta G(x,y;0)/\delta Q^\mu(w)=0$. The solution of this problem was already given by Hawking[11] in a similar context. Inserting it in (2.18) yields, after some simple manipulations,

$$\frac{\delta\zeta(\Omega\,|\,s)}{\delta Q^\mu(w)}=-s\sum_n\Lambda_n{}^{-(1+s)}\int d^4z\,\phi_n^\dagger(z)\frac{\delta\Omega_z}{\delta Q^\mu(w)}\phi_n(z)\ . \tag{2.20}$$

For the operator Ω of (2.10), this becomes

$$\frac{\delta\zeta(\Omega\,|\,s)}{\delta Q^\mu(w)}\bigg|_{K=Q=0}=s\sum_n\lambda_n{}^{-(1+s)}[\varphi_n^\dagger(w)\gamma_\mu\gamma_5\not{D}\varphi_n(w)-\varphi_n^\dagger(w)\overleftarrow{\not{D}}\gamma_\mu\gamma_5\varphi_n(w)] \tag{2.21}$$

with $\overleftarrow{\not{D}}=\overleftarrow{\not\partial}-ig\not{A}$. Using the eigenvalue equation of the $\{\varphi_n\}$, we get for the divergence of this expression

$$\partial^\mu_w\frac{\delta\zeta(\Omega\,|\,s)}{\delta Q^\mu(w)}\bigg|_{K=Q=0}=4s\sum_n\frac{m^2-\lambda_n}{\lambda_n{}^{1+s}}\varphi_n^\dagger(w)\gamma_5\varphi_n(w)\ . \tag{2.22}$$

Repeating the above derivation for variations of Z with respect to the second source function K shows that the first term in (2.22), which contains m^2, is essentially the pseudoscalar density $\bar\psi\gamma_5\psi$. Hence we get from (2.17)

$$\partial^\mu_w\langle\bar\psi(w)\gamma_\mu\gamma_5\psi(w)\rangle=2im\langle\bar\psi(w)\gamma_5\psi(w)\rangle+\left\langle 2i\frac{d}{ds}\bigg|_0 s\sum_n\lambda_n{}^{-s}\varphi_n^\dagger(w)\gamma_5\varphi_n(w)\right\rangle\ . \tag{2.23}$$

The last task is the evaluation of the second term in (2.23). In the spirit of ζ-function regularization, we do the sum over n for sufficiently large values of $\mathrm{Re}(s)$ and then analytically continue to $s=0$. We start by writing

$$\begin{aligned}I(x)&\equiv 2i\frac{d}{ds}\bigg|_0 s\sum_n\lambda_n{}^{-s}\varphi_n^\dagger(x)\gamma_5\varphi_n(x)\\&=2i\frac{d}{ds}\bigg|_0 s\sum_n\varphi_n^\dagger(x)\gamma_5(\not{D}^2+m^2)^{-s}\varphi_n(x)\ .\end{aligned} \tag{2.24}$$

The completeness of the $\{\varphi_n\}$ now implies

$$\begin{aligned}I(x)&=2i\frac{d}{ds}\bigg|_0 s\lim_{x'\to x}\mathrm{tr}_{\gamma c}\gamma_5(\not{D}_x{}^2+m^2)^{-s}\delta(x-x')\\&\equiv 2i\frac{d}{ds}\bigg|_0 s\,\mathrm{tr}_{\gamma c}[\gamma_5R(s)]\ .\end{aligned} \tag{2.25}$$

To evaluate $R(s)$, we return to the integral representation of the operator power [cf. Eq. (2.18)] and insert the Fourier representation of the δ function:

$$\begin{aligned}R(s)&=\lim_{x'\to x}\frac{1}{\Gamma(s)}\int_0^\infty dt\,t^{s-1}e^{-(\not{D}_x{}^2+m^2)t}\int\frac{d^4k}{(2\pi)^4}e^{ik\cdot(x-x')}\\&=\frac{1}{\Gamma(s)}\int_0^\infty dt\,t^{s-1}e^{-m^2t}\int\frac{d^4k}{(2\pi)^4}\exp\left[-\left[-k^2+2ik\cdot D+D^2-\frac{i}{2}g\gamma_\mu\gamma_\nu F^{\mu\nu}\right]t\right]\ .\end{aligned} \tag{2.26}$$

Now one transforms both the t and the k variable; first we define $t \equiv \tau s^2$ and then rescale k according to $k' \equiv ks\sqrt{\tau}$:

$$R(s)=\frac{1}{\Gamma(s)}(s^2)^s\frac{1}{s^4}\int_0^\infty d\tau\,\tau^{s-3}e^{-m^2 s^2\tau}\int\frac{d^4k'}{(2\pi)^4}e^{k'^2}\exp\left[-2ik'\cdot Ds\sqrt{\tau}-\left[D^2-\frac{i}{2}g\gamma_\mu\gamma_\nu F^{\mu\nu}\right]s^2\tau\right]. \quad (2.27)$$

The following steps are very similar to the standard procedure,[4,5] i.e., one expands the last exponential in (2.27) for small values of s to order s^4, does the k' integration, and evaluates the Dirac trace. It is then seen that the continuation to the point $s=0$ is indeed possible; returning to Minkowski space-time, the result reads

$$I(x)=\frac{g^2}{8\pi^2}\mathrm{tr}_c(F_{\mu\nu}{}^*F^{\mu\nu}) \quad (2.28)$$

with ${}^*F^{\mu\nu}=\frac{1}{2}\epsilon^{\mu\nu\rho\sigma}F_{\rho\sigma}$ the dual-field-strength tensor. Hence, with (2.23), we recover the famous equation (in operator language)

$$\partial^\mu(\bar\psi\gamma_\mu\gamma_5\psi)=2im(\bar\psi\gamma_5\psi)+\frac{g^2N_f}{8\pi^2}\mathrm{tr}_c(F_{\mu\nu}{}^*F^{\mu\nu}). \quad (2.29)$$

The factor N_f is due to the trace in flavor space, which was not written out above.

Our derivation of the anomaly can also be used to give a simple proof of the Atiyah-Singer index theorem. (For a review, see Ref. 12.) To this end, one separates those n (if any) from the sum in (2.22) with $\lambda_n=m^2$, because they correspond to the zero modes of the Euclidean Dirac operator $\not{D}$. [Recall that the $\{\varphi_n\}$ are eigenfunctions of $\Omega_x(Q=K=0)=m^2+\not{D}^2$.] The analytical continuation $s\to 0$ gives a mass-independent coefficient for this part of the sum and, after inserting (2.22) in (2.17), one has

$$\partial^\mu(\bar\psi\gamma_\mu\gamma_5\psi)=-2im^2\frac{d}{ds}\bigg|_0 s\sum_n{}''\lambda_n{}^{-(s+1)}\varphi_n^\dagger\gamma_5\varphi_n +i\frac{g^2}{8\pi^2}\mathrm{tr}(F_{\mu\nu}{}^*F^{\mu\nu})-2i\sum_n{}'\varphi_n^\dagger\gamma_5\varphi_n . \quad (2.30)$$

The primed sum is extended only over the n with $\lambda_n=m^2$, the double-primed one only over those with $\lambda_n>m^2$. If one now sets $m=0$, the first class of eigenfunctions becomes the zero modes of $\not{D}$, and because of $\{\gamma_5,\not{D}\}=0$ they can be chosen to be also chirality eigenstates. Then integrating (2.30) over all space-time and dropping surface terms shows that the Pontryagin index of the considered gauge field is equal to the difference between the number of positive- and negative-chirality zero modes of $\not{D}$.[12]

Just as in Fujikawa's approach, (2.29) was derived without any reference to perturbation theory. Fujikawa, however, obtains the anomaly term as an ill-defined infinite sum. Without regularization, his result reads (for $m=0$)

$$\partial^\mu(\bar\psi\gamma_\mu\gamma_5\psi)=2i\sum_n\tilde\varphi_n^\dagger\gamma_5\tilde\varphi_n , \quad (2.31)$$

where the $\{\tilde\varphi_n\}$ are a complete set of eigenfunctions of $\not{D}$. Clearly, the right-hand side of (2.31) is an indeterminate expression. This difficulty is overcome by the artificial introduction of a cutoff factor for the large eigenvalues according to

$$\sum_n\tilde\varphi_n^\dagger\gamma_5\tilde\varphi_n\to\lim_{M\to\infty}\sum_n\tilde\varphi_n^\dagger\gamma_5 e^{-\not{D}^2/M^2}\tilde\varphi_n . \quad (2.32)$$

This regularization procedure leads to the correct result (2.29). If $\sum_n\varphi_n^\dagger\gamma_5\varphi_n$ were a well-defined quantity, we could perform the s differentiation of (2.23) to get (for massless fermions)

$$\partial^\mu(\bar\psi\gamma_\mu\gamma_5\psi)=2i\sum_n\varphi_n^\dagger\gamma_5\varphi_n . \quad (2.33)$$

Apart from the fact that the $\{\varphi_n\}$ are eigenfunctions of $\not{D}^2$, this is exactly (2.31); however, using the zeta-function method, one automatically gets the regularized expression (2.23). Moreover, this technique provides us with a uniquely determined cutoff operator, namely, $\not{D}^2$, which has not been introduced in an *ad hoc* manner. (Of course, as stressed in Refs. 4 and 5, the Atiyah-Singer index theorem may be used as a guiding principle.)

The zeta-function method can also be used to calculate the divergence of the vector current. In this case, the variation of ζ is given by an expression similar to (2.21), but with the γ_5 matrices on the right-hand side absent; its evaluation gives a vanishing divergence for $\bar\psi\gamma^\mu\psi$.

III. THE JACOBIAN OF CHIRAL TRANSFORMATIONS

Next we briefly reconsider the previous proof of the axial anomaly from the viewpoint of anomalous Jacobians, i.e., we show how the nontrivial transformation properties of the Grassmann integration measure $[d\psi d\bar\psi]$ under the chiral transformations

$$\psi(x)\to\tilde\psi(x)=e^{i\alpha(x)\gamma_5}\psi(x) ,$$
$$\bar\psi(x)\to\tilde{\bar\psi}(x)=\bar\psi(x)e^{i\alpha(x)\gamma_5} \quad (3.1)$$

arise in our approach, which uses (1.1) as the basic definition. To calculate the Jacobian factor $J[\alpha]$ defined by

$$[d\psi d\bar\psi]\to[d\tilde\psi d\tilde{\bar\psi}]\equiv J[\alpha][d\psi d\bar\psi] , \quad (3.2)$$

we first note the change of the spinor part of the Lagrangian (2.1) under (3.1):

$$\tilde{\bar\psi}(i\not{D}-m)\tilde\psi=\bar\psi(i\not{D}-m)\psi-(\partial^\mu\alpha)\bar\psi\gamma_\mu\gamma_5\psi -2im\alpha\bar\psi\gamma_5\psi+O(\alpha^2) . \quad (3.3)$$

For our present purpose, it suffices to consider only the fermionic part of the integral (2.5) for vanishing sources. Because its value is invariant under transformations of the variables of integration, (3.2) and (3.3) imply

$$\int [d\tilde{\psi}\, d\tilde{\bar{\psi}}]\exp\left[\int d^4x\, \tilde{\bar{\psi}}(i\not{D}-m)\tilde{\psi}\right]=J[\alpha]\int [d\psi\, d\bar{\psi}]\exp\left[\int d^4x[\bar{\psi}(i\not{D}-m)\psi-(\partial^\mu\alpha)\bar{\psi}\gamma_\mu\gamma_5\psi-2im\alpha\bar{\psi}\gamma_5\psi]\right] . \quad (3.4)$$

On both sides of this equation we have the standard Gauss-type Grassmann integral leading to[14]

$$\det(i\not{D}-m)=J[\alpha]\det(i\not{D}-m-\partial^\mu\alpha\gamma_\mu\gamma_5-2im\alpha\gamma_5) . \quad (3.5)$$

Recalling that

$$\det(i\not{D}-m)=\det(-i\not{D}-m)$$

and

$$\det[i\not{D}-m-(\partial^\mu\alpha)\gamma_\mu\gamma_5-2im\alpha\gamma_5]=\det[-i\not{D}-m+(\partial^\mu\alpha)\gamma_\mu\gamma_5-2im\alpha\gamma_5]$$

the quantity $J[\alpha]$ can be expressed in a form which is suitable for a ζ-function evaluation:

$$J[\alpha]=\left\{\frac{\det(i\not{D}-m)\det(-i\not{D}-m)}{\det[i\not{D}-m-(\partial^\mu\alpha)\gamma_\mu\gamma_5-2im\alpha\gamma_5]\det[-i\not{D}-m+(\partial^\mu\alpha)\gamma_\mu\gamma_5-2im\alpha\gamma_5]}\right\}^{1/2}$$

$$\equiv\left\{\frac{\det(\not{D}^2+m^2)}{\det\Omega'}\right\}^{1/2} \quad (3.6)$$

with the operator

$$\Omega'_z=\not{D}_z{}^2+m^2+i\not{D}_z\{[\partial_z^\mu\alpha(z)]\gamma_\mu\gamma_5-2im\alpha(z)\gamma_5\}$$
$$+i\{[\partial_z^\mu\alpha(z)]\gamma_\mu\gamma_5+2im\alpha(z)\gamma_5\}\not{D}_z$$
$$+4im^2\alpha(z)\gamma_5+O(\alpha^2) \quad (3.7)$$

which is Hermitian and positive for vanishing α. From (3.6) and (3.7) we infer that $\ln J[\alpha]$ (to first order in α) is just the linear term in the Volterra series of $\ln\det\Omega'[\alpha]$:

$$\ln J[\alpha]=-\tfrac{1}{2}[\ln\det\Omega'-\ln\det(\not{D}^2+m^2)]+O(\alpha^2)$$
$$=-\tfrac{1}{2}\int d^4w\frac{\delta}{\delta\alpha(w)}\ln\det\Omega'\Big|_{\alpha=0}$$
$$\times\alpha(w)+O(\alpha^2) . \quad (3.8)$$

Taking (1.1) as the regularized determinant, we get

$$\ln J[\alpha]=\int d^4w\tfrac{1}{2}\frac{\delta\zeta'(\Omega'\,|\,0)}{\delta\alpha(w)}\Big|_{\alpha=0}\alpha(w)+O(\alpha^2) . \quad (3.9)$$

Because α is treated as an infinitesimal quantity, we may assume that the existence of a complete set of orthonormalized eigenfunctions of Ω' is not spoiled by the terms containing α and we can repeat the steps leading to (2.30) for this new operator. It is then easy to see that one obtains for $\ln J[\alpha]$ just the anomaly term on the right-hand side of Eq. (2.23), which was already evaluated in the last section. Thus we get the final result (for N_f flavors)

$$J[\alpha]=\exp\left[(-i)\frac{g^2N_f}{8\pi^2}\int d^4x\,\alpha(x)\mathrm{tr}_c[F_{\mu\nu}(x)^*F^{\mu\nu}(x)]\right] , \quad (3.10)$$

which is also correct in Minkowski space-time. Now inserting (3.10) into (3.4) and expanding to first order in α, we again obtain (2.29). Furthermore, one can add external sources for the spinor field to (3.4), which, upon functional differentiation, leads to the well-known anomalous Ward-Takahashi identities.[1,4]

Obviously, our first derivation of (2.29) via the generating functional (2.5) is completely equivalent to the second one. Nevertheless, it is worthwhile to present both of them here because it turns out that in the theories to be considered in the next sections they are not necessarily equivalent.

IV. THE U(1) AXIAL-VECTOR THEORY

To illustrate the problems caused by γ_5 couplings, we first consider a simple model similar to that in Refs. 5 and 8, viz., the U(1) gauge theory described by the Lagrangian

$$\mathscr{L}=\bar{\psi}i\not{D}\psi+G\bar{\psi}(\phi_1+i\gamma_5\phi_2)\psi-\tfrac{1}{4}F_{\mu\nu}F^{\mu\nu}+\mathscr{L}_{\text{Higgs}} , \quad (4.1a)$$

$$F_{\mu\nu}=\partial_\mu A_\nu-\partial_\nu A_\mu , \quad (4.1b)$$

$$\not{D}\equiv\gamma^\mu D_\mu\equiv\gamma^\mu(\partial_\mu+igA_\mu\gamma_5) , \quad (4.1c)$$

where the ϕ's are real Higgs fields interacting with a single Dirac field ψ. For our purposes, it is not necessary to specify the form of the Higgs Lagrangian $\mathscr{L}_{\text{Higgs}}$. $\mathscr{L}$ is invariant under the following local gauge transformation:

$$\psi(x)\to\tilde{\psi}(x)=e^{i\alpha(x)\gamma_5}\psi(x) , \quad (4.2a)$$

$$\bar{\psi}(x)\to\tilde{\bar{\psi}}(x)=\bar{\psi}(x)e^{i\alpha(x)\gamma_5} , \quad (4.2b)$$

$$(\phi_1+i\phi_2\gamma_5)\to(\tilde{\phi}_1+i\tilde{\phi}_2\gamma_5)$$
$$=e^{-2i\alpha\gamma_5}(\phi_1+i\phi_2\gamma_5) , \quad (4.3)$$

$$A_\mu\to\tilde{A}_\mu=A_\mu-\frac{1}{g}\partial_\mu\alpha . \quad (4.4)$$

If only (4.2) is applied to (4.1), $\mathscr{L}$ changes according to

$$\tilde{\mathscr{L}}=\mathscr{L}-\partial_\mu\alpha\bar{\psi}\gamma^\mu\gamma_5\psi+2i\alpha G\bar{\psi}\gamma_5(\phi_1+i\gamma_5\phi_2)\psi . \quad (4.5)$$

To investigate the anomalies of this model, we use the method described in Sec. III to represent the Jacobian of

(4.2) as the functional derivative of a regularized determinant. By applying the transformations (4.2) to the Euclidean path integral one immediately has for the Jacobian J_A of the axial-vector gauge transformations

$$J_A[\alpha]=\frac{\det[i\mathscr{D}]}{\det[i\mathscr{D}-\partial_\mu\alpha\gamma^\mu\gamma_5+2i\alpha G(\phi_1+i\phi_2\gamma_5)]} . \tag{4.6}$$

Here we have introduced the non-Hermitian operator

$$\mathscr{D}\equiv \not{D}-iG(\phi_1+i\phi_2\gamma_5) . \tag{4.7}$$

Its adjoint with respect to the scalar product

$$(\psi_1,\psi_2)=\int d^4x\, \psi_{1\alpha}^*(x)\psi_{2\alpha}(x) \tag{4.8}$$

(α is a spinor index) reads

$$\mathscr{D}^\dagger=\not{\partial}-ig\not{A}\gamma_5+iG(\phi_1-i\phi_2\gamma_5) . \tag{4.9}$$

Because we would like to use (1.1) for the evaluation of (4.6), we are now confronted with the problem of casting this equation into a form in which there are only (at least for $\alpha=0$) Hermitian operators with a positive spectrum involved. These conditions are essential for the existence of the heat kernels of these operators. It is this crucial step from (4.6) to the ζ function which is responsible for the different anomaly terms obtained in the literature.[6,8,5] A first possibility of proceeding is to multiply through the ratio (4.6) with $\det(-i\mathscr{D}^\dagger)$, which yields

$$J_A[\alpha]=\frac{\det[\mathscr{D}^\dagger\mathscr{D}]}{\det[\mathscr{D}^\dagger\mathscr{D}+\Delta]} , \tag{4.10}$$

where

$$\Delta\equiv i\mathscr{D}^\dagger\partial_\mu\alpha\gamma^\mu\gamma_5+2G\mathscr{D}^\dagger\alpha\gamma_5(\phi_1+i\phi_2\gamma_5) . \tag{4.11}$$

The fundamental operator $\mathscr{D}^\dagger\mathscr{D}$ thus obtained is Hermitian, positive (at least for $\phi_1,\phi_2\neq0$) and, under (4.2)—(4.4), it covariantly transforms as

$$(\mathscr{D}^\dagger\mathscr{D})^\sim=e^{+i\alpha\gamma_5}\mathscr{D}^\dagger\mathscr{D}e^{-i\alpha\gamma_5} . \tag{4.12}$$

Moreover, it is the only operator with these properties we could get from (4.6). Now (1.1) is applicable and we may write

$$\begin{aligned}\ln J_A[\alpha]&=-\{\ln\det[\mathscr{D}^\dagger\mathscr{D}+\Delta]-\ln\det[\mathscr{D}^\dagger\mathscr{D}]\}\\&=\int d^4w\,\alpha(w)\frac{\delta\zeta'(\mathscr{D}^\dagger\mathscr{D}+\Delta\,|\,0)}{\delta\alpha(w)}\bigg|_{\alpha=0}+O(\alpha^2) .\end{aligned} \tag{4.13}$$

The following steps are analogous to Sec. II and lead to

$$\ln J_A[\alpha]=\int d^4w\,\alpha(w)\frac{d}{ds}\bigg|_0(-s)\times\sum_n\lambda_n{}^{-(s+1)}\int d^4z\,\varphi_n^\dagger(z)\times\frac{\delta(\mathscr{D}^\dagger\mathscr{D}+\Delta)_z}{\delta\alpha(w)}\varphi_n(z) , \tag{4.14}$$

where the $\{\varphi_n\}$ now denote a complete set of orthonormalized functions satisfying

$$\mathscr{D}^\dagger\mathscr{D}\varphi_n=\lambda_n\varphi_n . \tag{4.15}$$

For the evaluation of (4.13) one also needs a complete set of eigenfunctions of $\mathscr{D}\mathscr{D}^\dagger$:

$$\mathscr{D}\mathscr{D}^\dagger\phi_n=\lambda_n\phi_n . \tag{4.16}$$

The phases of the ϕ_n have to be chosen such that

$$\sqrt{\lambda_n}\phi_n=\mathscr{D}\varphi_n . \tag{4.17}$$

That this is indeed always possible can be seen by recalling that any finite-dimensional matrix M can be diagonalized by two unitary matrices U and V as $M=V\Lambda U^\dagger$ with Λ a diagonal matrix.[18] The column vectors of U, denoted by φ_n, are just the eigenvectors of the Hermitian matrix $M^\dagger M$ with eigenvalues $\lambda_n>0$; correspondingly, the columns of V, denoted by ϕ_n, are the eigenvectors of $MM^\dagger$ with the same eigenvalues. Hence Λ has diagonal elements Λ_{nn} with $|\Lambda_{nn}|^2=\lambda_n$. Inserting the explicit form of U and V in $\Lambda=V^\dagger MU$, we get $\Lambda_{nn}=(\phi_n,M\varphi_n)$ for all n. From this we infer that it is possible to redefine the phases of ϕ_n or φ_n so as to obtain $\Lambda_{nn}=+\sqrt{\lambda_n}>0$. Thus we may assume $\sqrt{\lambda_n}=(\phi_n,M\varphi_n)$ for all n. Next observe that ϕ_n and $M\varphi_n$ obey relations of the form $\phi_n=\alpha_nM\varphi_n$ for some α_n because the equality sign is valid in the Cauchy-Schwarz inequality

$$\begin{aligned}\lambda_n&=|(\phi_n,M\varphi_n)|^2\leq(\phi_n,\phi_n)(M\varphi_n,M\varphi_n)\\&=(\varphi_n,M^\dagger M\varphi_n)=\lambda_n .\end{aligned}$$

From $\sqrt{\lambda_n}=(\phi_n,M\varphi_n)$ it is clear that $\alpha_n=1/\sqrt{\lambda_n}$ and therefore $\sqrt{\lambda_n}\phi_n=M\varphi_n$ for all n. An analogous consideration for $\mathscr{D}$ establishes (4.17).

Now it is a matter of simple algebra to get from (4.14)

$$\begin{aligned}\ln J_A[\alpha]&=(-i)\int d^4x\,\alpha(x)\frac{d}{ds}\bigg|_0 s\sum_n\lambda_n{}^{-s}(\varphi_n^\dagger\gamma_5\varphi_n+\phi_n^\dagger\gamma_5\phi_n)\\&=(-i)\int d^4x\,\alpha(x)\frac{d}{ds}\bigg|_0 s\lim_{x'\to x}\mathrm{tr}_\gamma\gamma_5[(\mathscr{D}_x^\dagger\mathscr{D}_x)^{-s}+(\mathscr{D}_x\mathscr{D}_x^\dagger)^{-s}]\delta(x-x') .\end{aligned} \tag{4.18}$$

In the last line, the eigenvalue equations and the completeness of the $\{\varphi_n\}$ and the $\{\phi_n\}$ were used. The following steps are analogous to Sec. II: after introducing a parameter integral for the operator powers and inserting the Fourier representation of the δ function, one rescales the integration variables and expands the resulting exponential. Then one obtains for the Jacobian of the chiral transformations (4.2)

$$J_A[\alpha]=\exp\left[(-i)\int d^4x\frac{g^2}{8\pi^2}F_{\mu\nu}(x)^*F^{\mu\nu}(x)\right] \tag{4.19}$$

giving rise to the following anomalous divergence of the axial-vector current:

$$\partial^\mu(\bar\psi\gamma_\mu\gamma_5\psi)=[i]\frac{g^2}{8\pi^2}F_{\mu\nu}{}^*F^{\mu\nu}-2iG\bar\psi\gamma_5(\phi_1+i\phi_2\gamma_5)\psi\ . \tag{4.20}$$

(The factor $[i]$ is absent in Minkowski space-time.) Now we could repeat this calculation for vector transformations $\psi\to\exp[i\alpha(x)]\psi$. The result would be a nonanomalous Jacobian $J_V[\alpha]\equiv 1$ and hence a conserved vector current.

The anomaly term in (4.20) is the same as that obtained by the path-integral method in Ref. 5; however, it disagrees by a factor of $\frac{1}{3}$ with the result of Pauli-Villars[3] or point-separation methods.[3] The coincidence of our result with that of Ref. 5 is due to the fact that in both schemes $\mathscr{D}^\dagger\mathscr{D}$ and $\mathscr{D}\mathscr{D}^\dagger$ are used as the basic regularizing operators, which in turn is a consequence of having multiplied Eq. (4.6) with $\det(-i\mathscr{D}^\dagger)$ in order to get positive, Hermitian operators. This method of dealing with (4.6) obviously is the only one producing gauge-covariant cutoff operators, i.e., the only way to preserve gauge covariance at every stage of the calculation. Nevertheless, if one does not insist on a covariant regularization scheme, there is another possibility of proceeding from (4.9). First, one multiplies through with $\det(-i\mathscr{D})$:

$$J_A[\alpha]=\frac{\det[\mathscr{D}^2]}{\det[\mathscr{D}^2+\bar\Delta]} \tag{4.21}$$

with $\bar\Delta$ similar to (4.11). However, $\mathscr{D}^2$ is not positive and Hermitian because $\mathscr{D}$ is not Hermitian. This problem can be solved by an analytic continuation of the gauge field and the scalar Higgs field to purely imaginary values. Then our method again becomes applicable and leads to the Jacobian (for vanishing Higgs fields)

$$J[\alpha]=\exp\left[(-i)\int d^4x\,\alpha(x)\frac{g^2}{8\pi^2}\frac{1}{3}F_{\mu\nu}{}^*F^{\mu\nu}\right], \tag{4.22}$$

which differs from (4.19) by the above-mentioned factor of $\frac{1}{3}$. Before we further comment on this fact, we briefly mention that we could also have started from a generating functional of the form (with the gauge-fixing terms and the Faddeev-Popov determinant again included in $[dA]$)

$$Z[a_\mu,b_\mu]=\int[d\psi\,d\bar\psi\,dA\,d\phi_i]\times\exp\int d^4x[\bar\psi(i\mathscr{D}-\not a-\not b\gamma_5)\psi-\tfrac{1}{4}F_{\mu\nu}{}^*F^{\mu\nu}+\mathscr{L}_{\rm Higgs}] \tag{4.23}$$

to generate matrix elements of the currents and their divergences by differentiating with respect to a_μ and b_μ. The covariant procedure then corresponds to writing

$$\begin{aligned}Z[a_\mu,b_\mu]&=\int[dA\,d\phi_i]\det(i\mathscr{D}-\not a-\not b\gamma_5)\exp\left[\int d^4x(-\tfrac{1}{4}F_{\mu\nu}F^{\mu\nu}+\mathscr{L}_{\rm Higgs})\right]\\&=\int[dA\,d\phi_i]\det(-i\mathscr{D}^\dagger)^{-1}\exp\left[-\zeta'(\mathscr{D}^\dagger\mathscr{D}+i\mathscr{D}^\dagger\not a+i\mathscr{D}^\dagger\not b\gamma_5\,|\,0)+\int d^4x(-\tfrac{1}{4}F_{\mu\nu}F^{\mu\nu}+\mathscr{L}_{\rm Higgs})\right].\end{aligned} \tag{4.24}$$

Proceeding as in Sec. III, we again obtain (4.19) together with $J_V[\alpha]=1$, which is the uniquely determined anomaly for a fully quantized gauge theory, because the use of the cutoff operators $\mathscr{D}^\dagger\mathscr{D}$ and $\mathscr{D}\mathscr{D}^\dagger$ is the only way possible for a gauge-covariant evaluation (at every intermediate step) of (4.6). Incidentally, the model (4.1) is not consistent as a gauge theory, because, due to the anomaly, the current, to which A_μ couples, is not conserved; but in order to decide whether there is an anomaly in the gauge coupling or not, one first has to calculate the possible anomalies in a covariant manner. Therefore, for this model, Eq. (4.23) should be used only for external A_μ and ϕ fields, i.e., the $[dA\,d\phi_i]$ integration should be omitted.

To multiply (4.6) with $\det(-i\mathscr{D})$ and to analytically continue is a method which is allowed only if from the very beginning (4.1) is considered as a theory of quantized fermions interacting with *external* fields. The reason for this is that (i) the operator $\mathscr{D}^2$ is not covariant, which is indispensible for a gauge theory, and (ii) for purely imaginary values of A_μ the $[dA]$ integration in (4.24) would become divergent due to the sign change in the $F_{\mu\nu}{}^2$ term. (We ignore for the moment that this special model is not consistent as a gauge theory.)

In conclusion, we can say that for this Abelian model, if considered as a (would-be) gauge theory, only (4.19) is correct, whereas the associated nongauge theory can be regularized in both ways described. As we shall see in the next section, this ambiguity for the nongauge theory is not present for non-Abelian gauge groups.

V. THE SU(N)$\times$SU(N)— VASP THEORY

Now we are going to apply our ζ-function method based upon (1.1) to a model describing fermions in some representation of SU(N) in interaction with vector (V), axial-vector (A), scalar (S), and pseudoscalar (P) fields. The anomalies of this model as a nongauge theory were treated by Bardeen[2] using point-separation methods, by Brown *et al.*[3] within the Pauli-Villars scheme, and by Hu *et al.*[9] with a method similar to that in Ref. 5, but without using gauge-covariant cutoff operators. A heat-kernel regularized version of this calculation was given by Balachandran *et al.*[13] Employing these techniques, one obtains a "minimal set"[9] of anomalies satisfying the Wess-Zumino[19] consistency conditions, together with "normal parity terms," which one usually removes by suitable counterterms. All these results are well known for the case that we treat V_μ, A_μ, S, and P as external, i.e., unquantized fields, but because the mentioned schemes do not produce gauge-covariant results, it is not immediately clear how the corresponding gauge theory has to be regularized. As will be shown, here again the ζ-function method gives a unique, physically acceptable answer.

Due to the analysis within perturbation theory and in accord with the findings of this section, the anomaly cancellation[20] necessary for renormalizability takes place only if the gauge-group generators T^a obey the trace condition

$$\mathrm{tr}T^a\{T^b,T^c\}=0\ , \tag{5.1}$$

hence, for the fermions transforming according to the fundamental representation our model is renormalizable as a gauge theory only for $N=2$.[20] In the following we therefore consider both the SU(2)×SU(2) gauge theory and the external-field problem for arbitrary N.

The fermionic Lagrangian of our model is given by

$$\mathscr{L}_F=\bar\psi(i\mathscr{D}-m)\psi\ , \tag{5.2}$$

$$\mathscr{D}\equiv\partial\!\!\!/+i\not V+i\not A\gamma_5+iS-\gamma_5P\ , \tag{5.3}$$

where $V_\mu\equiv V_\mu{}^aT^a$, etc. The following calculations are greatly simplified by introducing new fields:

$$B_1^\mu\equiv V^\mu-A^\mu\ ,\qquad B_2^\mu\equiv V^\mu+A^\mu \tag{5.4}$$

and

$$H\equiv-(S+iP)\ ,\qquad H^\dagger=-(S-iP)\ , \tag{5.5}$$

which only couple to the left- (right-) handed fermions defined as

$$\psi_L\equiv\tfrac{1}{2}(1-\gamma_5)\psi\equiv P_L\psi\ ,\qquad \psi_R\equiv\tfrac{1}{2}(1+\gamma_5)\psi\equiv P_R\psi\ . \tag{5.6}$$

In terms of these, the complete Lagrangian reads

$$\mathscr{L}=\bar\psi(i\mathscr{D}-m)\psi-\frac{1}{4g^2}F^a_{\mu\nu}(B_1)F^{a\mu\nu}(B_1)-\frac{1}{4g'^2}F^a_{\mu\nu}(B_2)F^{a\mu\nu}(B_2)+\mathscr{L}_{\mathrm{Higgs}}\ , \tag{5.7a}$$

$$\mathscr{D}=\not D(B_1)P_L+\not D(B_2)P_R-iHP_R-iH^\dagger P_L\ , \tag{5.7b}$$

$$D(B_j)\equiv\partial\!\!\!/+i\not B_j,\ \ j=1,2\ , \tag{5.7c}$$

where we have included gauge-boson terms [with $F^a_{\mu\nu}$ given by (2.3)] and a Higgs Lagrangian. The adjoint of the operator $\mathscr{D}$ with respect to the scalar product

$$(\psi_1,\psi_2)=\int d^4x\ \psi^*_{1i\alpha}(x)\psi_{2i\alpha}(x) \tag{5.8}$$

[i (α) is a color (spinor) index] is given by

$$\mathscr{D}^\dagger=\not D(B_1)P_R+\not D(B_2)P_L+iH^\dagger P_R+iHP_L\ . \tag{5.9}$$

First, we want to derive the Jacobians of the $\mathrm{U}(1)_V$ transformations

$$\psi(x)\to\tilde\psi(x)=e^{i\alpha(x)}\psi(x)\ ,\qquad \bar\psi(x)\to\tilde{\bar\psi}(x)=\bar\psi(x)e^{-i\alpha(x)} \tag{5.10}$$

and the $\mathrm{U}(1)_A$ transformations

$$\psi(x)\to\tilde\psi(x)=e^{i\alpha(x)\gamma_5}\psi(x)\ ,\qquad \bar\psi(x)\to\tilde{\bar\psi}(x)=\bar\psi(x)e^{i\alpha(x)\gamma_5}\ . \tag{5.11}$$

The Lagrangian changes as

$$\mathscr{L}\to\tilde{\mathscr{L}}_V=\mathscr{L}+\bar\psi(-\partial_\mu\alpha\gamma^\mu)\psi \tag{5.12}$$

and

$$\begin{aligned}\mathscr{L}\to\tilde{\mathscr{L}}_A&=\mathscr{L}+\bar\psi[-\partial^\mu\alpha\gamma_\mu\gamma_5+2i\alpha\gamma_5(HP_R+H^\dagger P_L)]\psi+O(\alpha^2)\\&\equiv\mathscr{L}+\bar\psi\omega\psi+O(\alpha^2)\ ,\end{aligned} \tag{5.13}$$

respectively. Writing down the fermionic part of the path integral, performing the above transformations for infinitesimal α, and doing the Gaussian integrations, one gets for the Jacobian of, say, Eq. (5.11)

$$J_A[\alpha]=\frac{\det[i\mathscr{D}]}{\det[i\mathscr{D}+\omega]}\ . \tag{5.14}$$

As for the U(1) model of the last section, we first present an evaluation of (5.14) which is suitable for the full gauge theory. We recall that $\mathscr{L}$ of Eq. (5.2) is invariant under the $\mathrm{SU}(N)_V$ vector gauge transformations

$$\begin{aligned}&\psi(x)\to\tilde\psi(x)=e^{i\lambda(x)}\psi(x),\ \ \lambda(x)\equiv\lambda^a(x)T^a\ ,\\&V_\mu(x)\to\tilde V_\mu(x)=V_\mu-\partial_\mu\lambda+i[\lambda,V_\mu]+O(\lambda^2)\ ,\\&A_\mu(x)\to\tilde A_\mu(x)=A_\mu+i[\lambda,A_\mu]+O(\lambda^2)\ .\end{aligned} \tag{5.15}$$

Both $\mathscr{D}$ and $\mathscr{D}^\dagger$ (and, of course, their products) transform covariantly under (5.15). Furthermore, $\mathscr{L}$ is invariant under the $\mathrm{SU}(N)_A$ axial-vector gauge transformations

$$\begin{aligned}&\psi(x)\to\tilde\psi(x)=e^{i\lambda(x)\gamma_5}\psi(x),\ \ \lambda(x)\equiv\lambda^a(x)T^a\ ,\\&V_\mu(x)\to\tilde V_\mu(x)=V_\mu+i[\lambda,A_\mu]+O(\lambda^2)\ ,\\&A_\mu(x)\to\tilde A_\mu(x)=A_\mu-\partial_\mu\lambda+i[\lambda,V_\mu]+O(\lambda^2)\ ,\end{aligned} \tag{5.16}$$

under which $\mathscr{D}$ and its adjoint transform as

$$\tilde{\mathscr{D}}=e^{-i\lambda\gamma_5}\mathscr{D}e^{-i\lambda\gamma_5}\ ,\qquad \tilde{\mathscr{D}}^\dagger=e^{+i\lambda\gamma_5}\mathscr{D}^\dagger e^{+i\lambda\gamma_5}\ , \tag{5.17}$$

so that $\mathscr{D}^\dagger\mathscr{D}$ and $\mathscr{D}\mathscr{D}^\dagger$ are again covariant (or contravariant, which is equivalent in what follows). Thus we can rewrite (5.14) in terms of positive, Hermitian and covariant operators by multiplying through with $\det(-i\mathscr{D}^\dagger)$ and then exploiting (1.1):

$$\ln J_A[\alpha]=\ln\left[\frac{\det[\mathscr{D}\mathscr{D}^\dagger]}{\det[\mathscr{D}\mathscr{D}^\dagger-i\omega\mathscr{D}^\dagger]}\right]=\int d^4x\,\alpha(x)\frac{\delta\zeta'(\mathscr{D}\mathscr{D}^\dagger-i\omega\mathscr{D}^\dagger\,|\,0)}{\delta\alpha(x)}$$

$$=(-i)\int d^4x\,\alpha(x)\frac{d}{ds}\Big|_0 s\sum_n[\phi_n^\dagger\gamma_5(\mathscr{D}^\dagger\mathscr{D})^{-s}\phi_n+\varphi_n^\dagger\gamma_5(\mathscr{D}\mathscr{D}^\dagger)^{-s}\varphi_n]$$

$$\equiv(-i)\int d^4x\,\alpha(x)A_A(x)\,. \tag{5.18}$$

The last lines are obtained by following the analogous steps in the preceding sections and by defining two new complete sets of eigenfunctions:

$$\mathscr{D}\mathscr{D}^\dagger\varphi_n=\lambda_n\varphi_n\,,$$
$$\mathscr{D}^\dagger\mathscr{D}\phi_n=\lambda_n\phi_n\,. \tag{5.19}$$

The phases have to be chosen such that $\mathscr{D}^\dagger\varphi_n=\sqrt{\lambda_n}\phi_n$ is satisfied for all n. The corresponding expression for the Jacobian of (5.10) reads

$$\ln J_V[\alpha]\equiv(-i)\int d^4x\,\alpha(x)A_V(x)$$

$$=(-i)\int d^4x\,\alpha(x)\frac{d}{ds}\Big|_0 s\sum_n[\phi_n^\dagger(\mathscr{D}^\dagger\mathscr{D})^{-s}\phi_n-\varphi_n^\dagger(\mathscr{D}\mathscr{D}^\dagger)^{-s}\varphi_n]\,. \tag{5.20}$$

Owing to the complicated structure of the cutoff operators the evaluation of (5.18) and (5.20) appears quite tedious, but by suitably inserting chirality projectors, everything can be reduced to known operator traces. Thus, the usual completeness argument leads to

$$A_A(x)=\frac{d}{ds}\Big|_0 s\lim_{x'\to x}\mathrm{tr}_{\gamma c}\gamma_5[(P_L\mathscr{D}^\dagger\mathscr{D})^{-s}+(P_R\mathscr{D}^\dagger\mathscr{D})^{-s}+(P_L\mathscr{D}\mathscr{D}^\dagger)^{-s}+(P_R\mathscr{D}\mathscr{D}^\dagger)^{-s}]_x\delta(x-x') \tag{5.21}$$

and a similar expression for $A_V(x)$. As can be seen by explicitly writing down $\mathscr{D}^\dagger\mathscr{D}$ and $\mathscr{D}\mathscr{D}^\dagger$ the operators $P_{L,R}\mathscr{D}^\dagger\mathscr{D}$ and $P_{L,R}\mathscr{D}\mathscr{D}^\dagger$ are all of the form $\hat{a}P_L+\hat{b}P_R$ or $\hat{a}P_R+\hat{b}P_L$, where $\hat{a}$ and $\hat{b}$ are operators containing only terms with an even and odd number of γ matrices, respectively. Moreover, $\hat{a}$ is always positive. Using the integral representation of the operator power, it is therefore easy to show that (apart from a canceling constant)

$$\mathrm{tr}_\gamma\gamma_5(\hat{a}P_{L,R}+\hat{b}P_{R,L})^{-s}=\mathrm{tr}_\gamma(\gamma_5\hat{a}^{-s}P_{L,R})\,. \tag{5.22}$$

Applying this formula to (5.21) one finds

$$A_A(x)=\frac{d}{ds}\Big|_0 s\lim_{x'\to x}\mathrm{tr}_{\gamma c}\gamma_5\{[\not{D}(B_1)+HH^\dagger]^{-s}+[\not{D}(B_2)+H^\dagger H]^{-s}\}_x\delta(x-x')\,. \tag{5.23}$$

The remaining calculations are the same as in Sec. II, and thus we end up with

$$A_A(x)=\frac{1}{16\pi^2}\mathrm{tr}_c[F_{\mu\nu}(B_1)^*F^{\mu\nu}(B_1)+F_{\mu\nu}(B_2)^*F^{\mu\nu}(B_2)]$$
$$=\frac{1}{16\pi^2}\epsilon_{\mu\nu\alpha\beta}\mathrm{tr}_c(G_V^{\mu\nu}G_V^{\alpha\beta}+G_A^{\mu\nu}G_A^{\alpha\beta})\,. \tag{5.24}$$

Here we have introduced

$$G_V^{\mu\nu}=\partial^\mu V^\nu-\partial^\nu V^\mu+i[V^\mu,V^\nu]+i[A^\mu,A^\nu]\,, \tag{5.25}$$

$$G_A^{\mu\nu}=\partial^\mu A^\nu-\partial^\nu A^\mu+i[V^\mu,A^\nu]-i[V^\nu,A^\mu]\,. \tag{5.26}$$

Correspondingly, one has for the anomaly factor of the vector transformations (5.10)

$$A_V(x)=\frac{1}{16\pi^2}\mathrm{tr}_c[F_{\mu\nu}(B_2)^*F^{\mu\nu}(B_2)-F_{\mu\nu}(B_1)^*F^{\mu\nu}(B_1)]$$
$$=\frac{1}{16\pi^2}\epsilon_{\mu\nu\alpha\beta}\mathrm{tr}_c(G_V^{\mu\nu}G_A^{\alpha\beta}+G_A^{\mu\nu}G_V^{\alpha\beta})\,. \tag{5.27}$$

We see that not only the chiral transformations (5.11) have a nonvanishing anomaly factor, but also the vector transformations (5.10). This gives rise to the following nonconservation laws:

$$\partial^\mu(\bar\psi\gamma_\mu\psi)=[i]\frac{1}{16\pi^2}\mathrm{tr}_c[F_{\mu\nu}(B_2)^*F^{\mu\nu}(B_2)-F_{\mu\nu}(B_1)^*F^{\mu\nu}(B_1)]\,, \tag{5.28a}$$

$$\partial^\mu(\bar\psi\gamma_\mu\gamma_5\psi)=[i]\frac{1}{16\pi^2}\mathrm{tr}_c[F_{\mu\nu}(B_2)^*F^{\mu\nu}(B_2)+F_{\mu\nu}(B_1)^*F^{\mu\nu}(B_1)]-2i\bar\psi\gamma_5(HP_R+H^\dagger P_L)\psi\,. \tag{5.28b}$$

Up to now, we only considered the "neutral" currents $\bar\psi\gamma^\mu\psi$ and $\bar\psi\gamma^\mu\gamma_5\psi$; it is easy, however, to generalize the above derivation for $\bar\psi\gamma^\mu T^a\psi$ and $\bar\psi\gamma^\mu\gamma_5 T^a\psi$. One simply replaces (5.10) by the transformations

$$\psi(x)\to\tilde\psi(x)=e^{i\alpha(x)T^a}\psi(x)\ ,\qquad \bar\psi(x)\to\tilde{\bar\psi}(x)=\bar\psi(x)e^{-i\alpha(x)T^a}\ , \tag{5.29}$$

and (5.11) by

$$\psi(x)\to\tilde\psi(x)=e^{i\alpha(x)\gamma_5T^a}\psi(x)\ ,\qquad \bar\psi(x)\to\tilde{\bar\psi}(x)=\bar\psi(x)e^{i\alpha(x)\gamma_5T^a}\ , \tag{5.30}$$

and then immediately obtains

$$A_V^a(x)=\frac{1}{16\pi^2}\epsilon_{\mu\nu\alpha\beta}\mathrm{tr}T^a(G_V^{\mu\nu}G_A^{\alpha\beta}+G_A^{\mu\nu}G_V^{\alpha\beta}) \tag{5.31a}$$

and

$$A_A^a(x)=\frac{1}{16\pi^2}\epsilon_{\mu\nu\alpha\beta}\mathrm{tr}T^a(G_V^{\mu\nu}G_V^{\alpha\beta}+G_A^{\mu\nu}G_A^{\alpha\beta}) \tag{5.31b}$$

for the vector and the axial-vector transformations, respectively. Just as in the Abelian case, the anomaly terms obtained in this covariant way are completely independent of the Higgs fields. For $N=2$, the expressions (5.28) and (5.31) coincide with those obtained by Fujikawa[5] for the $SU(2)\times SU(2)$ gauge theory. In particular, A_V^a and A_A^a vanish for $N=2$, as required by renormalizability. More generally, they vanish whenever the representation of the fermion fields is such that (5.1) is satisfied. Our findings, however, disagree with the results given in Refs. 2 and 3, for instance. This is not only due to the different regularization schemes used there (point separation, Pauli-Villars) but also stems from the fact that in these works, counterterms are used to remove the anomaly from the vector current, which was not done in the above. [The minimal set could be obtained from (5.31b) by using appropriate counterterms.] Because we multiplied through Eq. (5.14) with $\det(-i\mathscr{D}^\dagger)$, we were led to covariant cutoff operators $\mathscr{D}^\dagger\mathscr{D}$ and $\mathscr{D}\mathscr{D}^\dagger$ and thus covariant results. In this sense, (5.28) is uniquely obtained within the ζ-function scheme: no factor other than $\det(-i\mathscr{D}^\dagger)$ for this multiplication leads to a positive, Hermitian, and covariant cutoff operator.

If we are less restrictive and do not impose the requirement of gauge covariance, we also could analytically continue A_μ^a and S^a to purely imaginary values; this makes $\mathscr{D}$ Hermitian, but covariance is spoiled now. Then the only possible factor to multiply through (5.14) is $\det(-i\mathscr{D}')$ and hence (5.18) is replaced by

$$\begin{aligned}\ln J_A[\alpha]&=\ln\left[\frac{\det[\mathscr{D}'^2]}{\det(\mathscr{D}'^2-i\omega'\mathscr{D}')}\right]\\&=(-i)\int d^4x\,\alpha(x)\frac{d}{ds}\bigg|_0 2s\sum_n\varphi_n^\dagger\gamma_5(\mathscr{D}'^2)^{-s}\varphi_n\\&=(-i)\int d^4x\,\alpha(x)\frac{d}{ds}\bigg|_0 2s\lim_{x'\to x}\\&\quad\times\mathrm{tr}_{\gamma c}\gamma_5(\mathscr{D}_x'^2)^{-s}\delta(x-x')\ ,\end{aligned} \tag{5.32}$$

where the primes indicate the analytical continuation of the fields and the $\{\varphi_n\}$ now are a complete set of eigenfunctions of $\mathscr{D}'^2$. The evaluation of (5.32) yields an axial anomaly consisting of three parts:[9,13]

$$A_A^a(x)=A_{\min}^a(x)+A_{\mathrm{np}}^a(x)+A_H^a(x)\ . \tag{5.33}$$

The first contribution, $A_{\min}^a(x)$, is Bardeen's minimal anomaly given by

$$\begin{aligned}A_{\min}^a(x)=\frac{1}{16\pi^2}\epsilon_{\mu\nu\alpha\beta}\mathrm{tr}_cT^a[&G_V^{\mu\nu}G_V^{\alpha\beta}+\tfrac{1}{3}G_A^{\mu\nu}G_A^{\alpha\beta}\\&+\tfrac{8}{3}i(A^\mu A^\nu G_V^{\alpha\beta}+G_V^{\mu\nu}A^\alpha A^\beta\\&+A^\mu G_V^{\nu\alpha}A^\beta)\\&-\tfrac{32}{3}A^\mu A^\nu A^\alpha A^\beta]\ .\end{aligned} \tag{5.34}$$

The second one, $A_{\mathrm{np}}^a(x)$, is responsible for anomalies occurring in the normal-parity n-point functions[3] and is usually removed by appropriate counterterms. The same applies to the third part, $A_H^a(x)$, which is a consequence of the Higgs fields and vanishes for $H=0$. [Recall that the covariant results above were independent of $H(x)$.] The explicit form of $A_{\mathrm{np}}^a(x)$ and $A_H^a(x)$ are given in Ref. 9. As is easily seen, for this kind of regularization, with $\mathscr{D}'^2$ as the cutoff operator, the Jacobian of the $SU(N)_V$ transformations (5.29) is not anomalous, i.e., it is

$$A_V^a(x)=0\ , \tag{5.35}$$

and hence the vector current is conserved.

As already explained, the analytical continuation to make $\mathscr{D}$ Hermitian is unacceptable for a gauge theory, because ordinary gauge covariance is lost [or stated differently,[13] the gauge group is continued to $GL(N,C)$] and, because the "rotation" of two of the integration variables of the gauge-theory path integral, A_μ and S, is forbidden owing to the sign change in the kinetic term of A_μ, which renders the path integral divergent. [Needless to say, we also could have derived (5.31), but not (5.33), using a generating functional similar to (4.23), which now, for $SU(2)\times SU(2)$, belongs to a renormalizable theory and is meaningful as it stands.]

At first glance, the non-Abelian $SU(N)\times SU(N)$ theory shows the same feature as the U(1) model of Sec. IV: a uniquely determined anomaly for the gauge theory $(N=2)$ and two possibilities to regularize the anomalies of the nongauge theory. However, the Wess-Zumino consistency conditions to be discussed in the next section indicate that it is impossible to use the gauge-covariant scheme for the nongauge theory; it turns out that one may not use covariant cutoff operators for the external-field model.

VI. THE WESS-ZUMINO CONDITIONS

Now consider a system of quantized fermions interacting with external vector and axial-vector fields. (For the sake of simplicity, we omit S^a and P^a now; they would not change our conclusion in any way.) Then the effective action functional W is given by

$$e^{iW[V_\mu,A_\mu]}=\int [d\psi d\bar{\psi}]e^{i\int d^4x\,\mathcal{L}(\psi,\bar{\psi};V_\mu,A_\mu)}\ . \tag{6.1}$$

The integral (6.1) is also equal to the vacuum persistence amplitude $\langle 0_+|0_-\rangle^{V,A}$. In the space of functionals $F=F[V_\mu,A_\mu]$ of the fields V_μ and A_μ, the gauge transforms (5.15) and (5.16) are generated by operators $X^a(x)$ and $Y^a(x)$,[19] i.e., it is

$$\delta_V F=\int d^4x\,\lambda^a(x)X^a(x)F\ ,$$
$$\delta_A F=\int d^4x\,\lambda^a(x)Y^a(x)F \tag{6.2}$$

if we define

$$X^a(x)=\partial_\mu^x\frac{\delta}{\delta V_\mu^a(x)}-f^{abc}V_\mu^b(x)\frac{\delta}{\delta V_\mu^c(x)}-f^{abc}A_\mu^b(x)\frac{\delta}{\delta A_\mu^c(x)}\ ,$$
$$Y^a(x)=\partial_\mu^x\frac{\delta}{\delta A_\mu^a(x)}-f^{abc}A_\mu^b(x)\frac{\delta}{\delta V_\mu^c(x)}-f^{abc}V_\mu^b(x)\frac{\delta}{\delta A_\mu^c(x)}\ . \tag{6.3}$$

As it must be, these operators satisfy the $SU(N)\times SU(N)$ commutator relations

$$[X^a(x),X^b(y)]=f^{abc}\delta(x-y)X^c(x)\ , \tag{6.4a}$$
$$[X^a(x),Y^b(y)]=f^{abc}\delta(x-y)Y^c(x)\ , \tag{6.4b}$$
$$[Y^a(x),Y^b(y)]=f^{abc}\delta(x-y)X^c(x)\ . \tag{6.4c}$$

On the other hand, because $\mathcal{L}$ is invariant under (5.15) and (5.16) and the Jacobians of $[d\psi d\bar{\psi}]$ are assumed to be of the form $\exp[-i\int d^4x\,\lambda^a(x)A^a(x)]$, it is easy to infer the change of W under a transformation of its arguments. With the definitions in (5.18) and (5.20) one finds

$$\delta_V W=-\int d^4x\,\lambda^a(x)A_V^a(x)\ ,$$
$$\delta_A W=-\int d^4x\,\lambda^a(x)A_A^a(x)\ . \tag{6.5}$$

Comparison with (6.2) yields

$$X^a(x)W=-A_V^a(x)\ , \tag{6.6a}$$
$$Y^a(x)W=-A_A^a(x)\ . \tag{6.6b}$$

Now one applies (6.4) to W and uses (6.6) to get

$$X^a(x)A_V^b(y)-X^b(y)A_V^a(x)=f^{abc}\delta(x-y)A_V^c(x)\ , \tag{6.7a}$$
$$X^a(x)A_A^b(y)-Y^b(y)A_V^a(x)=f^{abc}\delta(x-y)A_A^c(x)\ , \tag{6.7b}$$
$$Y^a(x)A_A^b(y)-Y^b(y)A_A^a(x)=f^{abc}\delta(x-y)A_V^c(x)\ . \tag{6.7c}$$

These are the Wess-Zumino consistency conditions which strongly restrict the possible forms of the functions A_V^a and A_A^a. In contrast to their original derivation,[19] in (6.7) we allowed for a nonvanishing vector anomaly A_V^a. Obviously the derivation of (6.7) is nonperturbative and does not depend on a special regularization scheme. Thus we can use these constraints as a check for our different calculations. It has to be stressed, however, that only the anomalies of the nongauge theory have to satisfy (6.7). Because of the presence of gauge-fixing δ functions and Faddeev-Popov determinants in the path integral of the fully quantized theory, (6.7) is of no significance there.

As has long been known, Bardeen's minimal set of anomalies (5.34) satisfies (6.7) together with $A_V^a\equiv 0$. Thus the second way of treating (5.14), i.e., multiplying with $\det(-i\mathcal{D})$ and continuing A_μ^a and S^a, is a sensible regularization procedure for the anomalies of the $SU(N)\times SU(N)$ external field model. The situation is different for the expressions (5.31); one obtains for their change under infinitesimal transformations

$$X^a(x)A_V^b(y)=f^{abc}A_V^c(x)\delta(x-y)\ , \tag{6.8a}$$
$$X^a(x)A_A^b(y)=f^{abc}A_A^c(x)\delta(x-y)\ , \tag{6.8b}$$
$$Y^a(x)A_V^b(y)=f^{abc}A_A^c(x)\delta(x-y)\ , \tag{6.8c}$$
$$Y^a(x)A_A^b(y)=f^{abc}A_V^c(x)\delta(x-y)\ , \tag{6.8d}$$

as was expected from the use of covariant cutoff operators. Obviously, Eq. (6.8a), for example, is in contradiction to (6.7a). More generally, one can easily convince oneself that in any model *every* gauge-invariant cutoff operator would lead to (6.8a) and hence the anomaly factors of the gauge theory must be different from those occurring in the external-field model.

Thus we must conclude that for the nongauge theory (5.32) is the correct transcription of (5.14) in terms of operators for which (1.1) is applicable, whereas the gauge theory requires (5.18) as the starting point of the ζ-function regularization. Now, because the non-Abelian structure of the gauge group implies the constraints (6.7), any ambiguity is removed from our regularization scheme. However, there remains one unpleasant feature. By using $\mathcal{D}'^2$, one not only obtains the welcome terms of $A_{\rm min}^a$, but also normal-parity terms, which one would like to "regularize away" without explicitly using counterterms. The unphysical nature of $A_{\rm np}^a$ can be seen, for instance, as follows. In the presence of strong external fields, fermion pair production can occur, as is indicated by a nonvanishing imaginary part of W. Because ${\rm Im}\,W$ is related to the measurable (in principle, at least) probability to create a pair, it must be a gauge-invariant quantity, i.e., $X^a{\rm Im}\,W=Y^a{\rm Im}\,W=0$. Combining this with (6.6b), one observes that A_A^a must be a real function of the fields. This corresponds to the Jacobian $J_A[\alpha]$ being a pure phase factor, as is naively expected for a unitary transformation. Indeed, $A_{\rm min}^a$ is real for real fields, but not $A_{\rm np}^a$. This was to be expected from the very beginning, because the anomaly (for the nongauge theory) is of the general form

$$\lim_{x'\to x}{\rm tr}_{\gamma c}\gamma_5 T^a C_x{}^{-s}\delta(x-x') \tag{6.9a}$$

or, equivalently,

$${\rm tr}_{\gamma c x}(\gamma_5 T^a C^{-s}) \tag{6.9b}$$

with C the cutoff operator. For this expression to be real, C has to be Hermitian [apart from possible terms vanishing upon ${\rm tr}(\gamma_5 T^a\cdots)$], which is not the case for $C=\mathcal{D}^2$. [We consider (6.9) before the continuation to imaginary fields or after having continued back to the final result to real fields.] On the other hand, the use of $\mathscr{D}^2$ is dictated by the Wess-Zumino conditions. This suggests that if one wants to obtain the minimal anomaly without the normal-parity terms, it is insufficient to use only one cutoff operator. As in (5.18), one could think of a generalization of (6.9) by using more than one operator:

$$\mathrm{tr}_{\gamma c x}[\gamma_5 T^a(C_1{}^{-s}+C_2{}^{-s}+C_3{}^{-s}+\cdots)] \ . \tag{6.10}$$

The simplest choice for the C's producing $A^a_{\min}$ alone is

$$\mathrm{tr}_{\gamma c x}\{\gamma_5 T^a[(\mathscr{D}^2)^{-s}+(\mathscr{D}^{\dagger 2})^{-s}]\} = \mathrm{tr}_{\gamma c x}[\gamma_5 T^a(\mathscr{D}^2)^{-s}] + \{A_\mu \to -A_\mu, S \to -S\} \ . \tag{6.11}$$

The curly brackets denote the preceding expression with the sign of the axial-vector and (if present) scalar field changed. It was shown by Hu, Young, and McKay[9] that all the terms of A^a_{np} and A^a_H are odd under this change, so that (6.11) yields, as desired, the minimal anomaly alone. The vector anomaly regularized with $(\mathscr{D}^2)^{-s}+(\mathscr{D}^{\dagger 2})^{-s}$ still gives a vanishing result. The definition of the axial anomaly by (6.11) is equivalent to the regularization prescription of Hu *et al.*:[9] one expands ψ and $\bar{\psi}$ in terms of eigenfunctions of $\mathscr{D}^2$ and $\mathscr{D}^{\dagger 2}$, respectively, which also leads to the cancellation of A^a_{np} and A^a_H in the Jacobian of $[d\psi d\bar{\psi}]$. Nevertheless, such different treatment of ψ and $\bar{\psi}$ is by no means a necessary consequence of the path-integral formalism. The same applies to the artificial introduction of a second operator in (6.11); if one defines Jacobians as ratios like (4.6) or (5.14), no justification is found within the ζ-function formalism.

VII. CONCLUSION

We have shown that ζ-function regularization based upon the definition (1.1) is an appropriate method for performing the regularization of anomalies in different physical situations from a unified point of view. It produces a unique answer for theories without γ_5 couplings and in fully quantized gauge theories even if axial-vector couplings are present. In nongauge theories with γ_5 couplings ambiguities can occur, which, however, are completely removed for a non-Abelian gauge group by imposing the Wess-Zumino conditions. The important point is the conversion of ratios like (5.14) into expressions containing only positive, Hermitian operators, whose determinants can be calculated by (1.1). This is achieved by multiplying through with $\det(-i\mathscr{D}^\dagger)$ in the gauge theories and with $\det(-i\mathscr{D})$ in the nongauge theories. In the last case, an analytical continuation of the scalar and the axial-vector field is necessary to preserve Hermiticity. This breaks gauge invariance and the result can satisfy the Wess-Zumino conditions. (This fact is also exploited in quite different approaches; see, for example, Ref. 21.)

The different treatment of gauge and nongauge theories was already discussed in Refs. 5 and 6 from another point of view. For a gauge theory to be consistent, both vector and axial-vector (gauge) vertices must be free of anomalies. This fact has to be respected by any sensible regularization scheme. The anomalies derived within the first scheme of Sec. V indeed lead to the well-known anomaly-free criterion of perturbation theory: the factors (5.31) vanish whenever the fermion multiplets are chosen in accordance with (5.1). As a consequence, however, the (external) vector current (5.28a) unavoidably acquires an anomalous divergence. On the other hand, when dealing with external-field problems, one traditionally preserves the conservation property of this current by introducing counterterms. The anomalies so obtained are those which we get from the second scheme of Sec. V, where we used $\mathscr{D}'^2$ as the cutoff operator.

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[1] S. Adler, Phys. Rev. **177**, 2426 (1969); J. Bell and R. Jackiw, Nuovo Cimento **60A**, 47 (1969); for a review see R. Jackiw, in *Lectures on Current Algebra and Its Applications,* edited by D. Gross, R. Jackiw, and S. Treiman (Princeton University Press, Princeton, N.J., 1972); S. Adler, in *Lectures on Elementary Particles and Quantum Field Theory,* edited by Stanley Deser *et al.* (MIT, Cambridge, Mass., 1971), Vol. 1. For an early discussion of the γ_5 problem in the context of the proper-time method, see J. Schwinger, Phys. Rev. **82**, 664 (1951).

[2] W. Bardeen, Phys. Rev. **184**, 1848 (1969).

[3] R. W. Brown, C.-C. Shih, and B.-L. Young, Phys. Rev. **186**, 1491 (1969).

[4] K. Fujikawa, Phys. Rev. Lett. **42**, 1195 (1979); Phys. Rev. D **21**, 2848 (1980).

[5] K. Fujikawa, Phys. Rev. D **29**, 285 (1984).

[6] D. W. McKay and B.-L. Young, Phys. Rev. D **28**, 1039 (1983).

[7] G. A. Christos, Z. Phys. C **18**, 155 (1983); **20**, 186 (1983).

[8] M. B. Einhorn and D. R. T. Jones, Phys. Rev. D **29**, 331 (1984).

[9] S.-K. Hu, B.-L. Young, and D. W. McKay, Phys. Rev. D **30**, 836 (1984).

[10] J. S. Dowker and R. Critchley, Phys. Rev. D **13**, 3224 (1976).

[11] S. W. Hawking, Commun. Math. Phys. **55**, 133 (1977).

[12] B. Schroer, Acta Phys. Austriaca, Suppl. XIX, **155**, (1978); E. Corrigan, P. Goddard, H. Osborn, and S. Templeton, Nucl. Phys. **B159**, 469 (1979).

[13] A. P. Balachandran, G. Marmo, V. P. Nair, and C. G. Trahern, Phys. Rev. D **25**, 2713 (1982).

[14] P. Ramond, *Field Theory* (Benjamin Cummings, Reading, Mass., 1981).

[15] W. Dittrich and M. Reuter, Phys. Lett. **128B**, 321 (1983).

[16] M. Reuter and W. Dittrich, Phys. Lett. **144B**, 99 (1984).

[17] J. D. Bjorken and S. Drell, *Relativistic Quantum Mechanics* (McGraw-Hill, New York, 1964).

[18] K. Fujikawa, Phys. Rev. D **25**, 2584 (1982).

[19] J. Wess and B. Zumino, Phys. Lett. **37B**, 95 (1971).

[20] H. Georgi and S. L. Glashow, Phys. Rev. D **6**, 429 (1972).

[21] A. Andrianov, L. Bonora, and R. Gamboa-Saravi, Phys. Rev. D **26**, 2821 (1982); A. Andrianov and L. Bonora, CERN Report No. Th 3539 (unpublished); Padova IFPD Report No. 16/83, 1983 (unpublished).