$\lambda\phi^4$ theory in the nonrelativistic limit

M. A. B. Bég and R. C. Furlong
The Rockefeller University, New York, New York 10021
(Received 9 October 1984)

We show that the nonrelativistic limit of the $\lambda\phi^4$ theory is trivial in $1+3$ dimensions; the renormalized coupling constant vanishes and the S matrix reduces to the unit matrix. Our result is consistent with, though not sufficient to establish, the triviality of the Lorentz-invariant theory.

I. INTRODUCTION

It is generally believed that the relativistic $\lambda\phi^4$ theory in $1+3$ dimensions is a trivial field theory, characterized by a vanishing renormalized coupling constant and a unit S matrix.[1] The result is presumed to be true for a real field as well as several real or complex fields; in the many-field case, an overall symmetry such as O(N) is assumed to ensure that there is but one quartic coupling. As recently emphasized,[2,3] this triviality problem is of more than academic interest; it modifies the traditional interpretation of the Salam-Weinberg theory; the requirement that the canonical formulation of the electroweak synthesis be consistent leads to upper bounds for fermion and Higgs masses.

In view of the importance of triviality, it is worth reminding oneself that no firm mathematical proof is yet in hand. We are dealing here with what, strictly speaking, must be regarded as an article of faith. However, as noted elsewhere,[3] one may predicate one's faith on certain results that may be deemed to have been established with a measure of rigor; a judicious selection is listed below. It should be borne in mind though that to complete the argument leading to some of the following results, it is necessary to postulate that the solutions of the field equations are in accord with the Osterwalder-Schrader axioms;[4] this is to permit construction of the Minkowski space (real time) theory from the Euclidean formulation, the customary starting point in the functional approach.

(i) Triviality cannot be proven in any finite order of perturbation theory.

This almost obvious result can be put on a firm basis, using the work of Glimm and Jaffe.[5]

(ii) The renormalized coupling constant lies in a bounded interval:[6]

$$0 \le \lambda_{\text{ren}} < \lambda_{\max} \quad \text{for } d \le 4 \; , \tag{1.1}$$

where d is the dimensionality of the space in which the theory is defined.

A necessary condition for triviality is thus satisfied.

(iii) For the theory in the symmetric phase, triviality for $d>4$ has been established by Aizenman[7] and Fröhlich.[8]

(iv) For $d=4$, Fröhlich[8] has noted that triviality can be established if Z_3—the wave-function renormalization constant—vanishes.

(v) The continuum limit of the lattice theory is trivial, or consistent with triviality, in all existing calculations.[9] When other nonperturbative calculational techniques are available, notably the $1/N$ expansion for the O(N)-symmetric theory, triviality again follows in the limit of infinite cutoff.[10]

The preceding paragraph summarizes what appear to be, at this time, the most compelling reasons for believing that the theory is indeed trivial.

Our purpose in this note is to add one more result to the above list:

(vi) The nonrelativistic limit of the theory is trivial in $1+3$ dimensions. The collision matrix vanishes and so does the renormalized coupling; the S matrix thus reduces to the unit matrix.

While our result is fully consistent with the conjecture that the Lorentz-invariant $\lambda\phi^4$ theory is trivial, it does not by itself shed any light on the status of the relativistic theory. The reader who wishes to boost his way to a proof is cautioned that an essential feature of nonrelativistic dynamics is the absence of production processes; in relativistic quantum field theory, however, the Aks theorem[11] tells us that scattering *implies* production; thus, the no-production constraint, unless it can somehow be relaxed, would—without benefit of any input about the nature of the field theory and the couplings therein—ensure a trivial theory in the relativistic domain, in a not very meaningful way.

Some readers may notice a similarity between the arguments leading to our result and the ones that underlay an early attempt,[12] made before the gauge-theoretic gospel found proper formulation and universal acceptance, to tame weak interactions. This similarity, while amusing, should not be taken too seriously; the physics, in the two situations, is totally unrelated.

II. NOTATION

We shall consider a theory described by the Lagrangian

$$L=\int d^3\mathbf{x}[\partial^\mu\phi^\dagger\partial_\mu\phi-M_0{}^2\phi^\dagger\phi-\lambda_0(\phi^\dagger\phi)^2] \; , \tag{2.1}$$

where $\phi\equiv\phi(\mathbf{x},t)$ is a complex spin-0 field at space-time point $(\mathbf{x},t)$ and M_0 is the mass of its (bare) quanta. If the Hamiltonian corresponding to L is to be bounded from below or, in other words, if the system is to be stable

$$\lambda_0 > 0 \ . \qquad (2.2)$$

Without any significant loss of generality, in the present context, we may work with the theory in the symmetric phase, so that $M_0{}^2 > 0$ and the relevant order parameter $\langle \phi \rangle$ vanishes. This means, in particular, that N, the ϕ number, is a constant of the motion and that the ground or vacuum state corresponds to $N=0$; the physical states are thus characterized by definite values of N.

Conservation of N plays a crucial role in the transition to the nonrelativistic limit; without it one cannot envisage a nonrelativistic many-ϕ system. For any such system would collapse spontaneously into a highly relativistic system with fewer ϕ particles. Note that we distinguish between many-ϕ systems and many-$\phi,\bar{\phi}$ systems, $\bar{\phi}$ being the charge conjugate to ϕ. The latter are intrinsically relativistic and do not lend themselves to a nonrelativistic treatment. In the following, we shall write

$$N = N(\phi) - N(\bar{\phi}) \qquad (2.3)$$

and restrict our discussion to systems with either $N(\bar{\phi})=0$ or $N(\phi)=0$; later on in our paper, this restriction will emerge naturally, in the nonrelativistic limit.

III. THE NONRELATIVISTIC LIMIT

The logically correct way to take the nonrelativistic limit (hereafter called the NR limit) of a theory is to first calculate the S matrix and then go to the large-mass limit for all the particles in the theory. (Coulombic interactions, if any, are to be understood in terms of action at a distance rather than photon exchange; *real* photons do not exist in the NR limit.) This is in general an impossible task, however, since the S matrix does not lend itself to explicit calculation. What is done in practice, in atomic physics and low-energy nuclear physics, for example, is to take the NR limit of the Hamiltonian and then calculate the S matrix. This is a tractable procedure but essential radiative effects, such as coupling constant and mass renormalization, have to be put in by hand. A formal proof of the validity of this procedure may be sought, albeit only in finite orders of perturbation theory, using logic analogous to that in the decoupling arguments of Appelquist and Carazzone.[13] We shall not delve into it here, however; instead, we take the sanction of usage in, say, atomic physics—the NR limit of the electrodynamics of electrons and protons—to be sufficient justification for taking the NR limit in this way.

Our first task, therefore, is to take the NR limit of the field operators and the Lagrangian. To this end, we introduce an auxiliary field $\chi_{(+)}$ via

$$\chi_{(+)} = (2M_0)^{1/2} e^{iM_0 t} \phi(\mathbf{x},t) \qquad (3.1)$$

and evaluate it in the large-M_0 limit. (In actuality, with self-energy effects included, it is the renormalized mass M that is relevant. *Vide infra*.) The suffix on χ is to indicate that only the positive-frequency part of ϕ will survive in this limit; the negative-frequency part will oscillate rapidly and may be set equal to zero. (The argument can be rendered precise by, for example, working with fields averaged over a small time interval and using the Riemann-Lebesgue lemmas.[14]) To project out $\chi_{(-)}$, the negative frequency part of the ϕ in the NR limit, all we need do is change the sign of M_0 in the exponential in Eq. (3.1). The limit yielding $\chi_{(+)}$ takes us to a sector of the Hilbert space in which $N(\bar{\phi})=0$; likewise the $\chi_{(-)}$ projection takes us to the sector $N(\phi)=0$; in the NR limit, there is, of course, no way to communicate between the two sectors.

The Lagrangian corresponding to the positive-frequency projection may be written as

$$L_{(+)} = \tfrac{1}{2} \int d^3\mathbf{x} \left[i(\chi^\dagger_{(+)}\dot{\chi}_{(+)} - \dot{\chi}^\dagger_{(+)}\chi_{(+)}) - \frac{1}{M_0}\nabla\chi^\dagger_{(+)}\cdot\nabla\chi_{(+)} - \frac{\lambda_0}{2M_0{}^2}(\chi^\dagger_{(+)}\chi_{(+)})^2 \right] + \frac{1}{2M_0}\int d^3\mathbf{x}(\dot{\chi}^\dagger_{(+)}\dot{\chi}_{(+)}) \ . \qquad (3.2)$$

Here we have simply used Eq. (3.1) to eliminate ϕ from Eq. (2.1). The term with two time derivatives, in Eq. (3.2), is of an order traditionally ignored in NR mechanics (i.e., v^2/c^2) and will henceforth be omitted. We implement a Legendre transformation, on the rest of the Lagrangian, to obtain the Hamiltonian:

$$H_{(+)} = -L_{(+)} + \int d^3\mathbf{x} \left[\frac{\delta L_{(+)}}{\delta\dot{\chi}_{(+)}}\dot{\chi}_{(+)} + \frac{\delta L_{(+)}}{\delta\dot{\chi}^\dagger_{(+)}}\dot{\chi}^\dagger_{(+)} \right] \qquad (3.3)$$

or equivalently,

$$H_{(+)} = -\frac{1}{2M_0}\int d^3\mathbf{x}\,\chi^\dagger_{(+)}(\mathbf{x},t)\nabla^2\chi_{(+)}(\mathbf{x},t) + \frac{\lambda_0}{4M_0{}^2}\int d^3\mathbf{x}\,d^3\mathbf{x}'\chi^\dagger_{(+)}(\mathbf{x},t)\chi^\dagger_{(+)}(\mathbf{x}',t)\delta^3(\mathbf{x}-\mathbf{x}')\chi_{(+)}(\mathbf{x}',t)\chi_{(+)}(\mathbf{x},t) \ . \qquad (3.4)$$

Here we explicitly indicate the space-time dependence of the field operators. If we choose to work in, say, the Schrödinger picture, we may simply set $t=0$ in Eq. (3.4) and work with time-independent operators.

The presence of radiative effects, that do not vanish in the NR limit, obliges us to rescale the field operators

$$\chi_{(+)} = \psi\sqrt{Z} \ , \qquad (3.5)$$

where ψ is normalized such that

$$\langle 0 | \psi(\mathbf{x},0) | \mathbf{k} \rangle = \frac{1}{(2\pi)^{3/2}} e^{i\mathbf{k}\cdot\mathbf{x}} , \tag{3.6}$$

$|\mathbf{k}\rangle$ being a one-particle state with momentum $\mathbf{k}$. The field renormalization constant Z, as well as all other renormalization constants of the relativistic theory, can be absorbed in the definition of M and λ, the renormalized mass and coupling constant, respectively. We, therefore, rewrite Eq. (3.4) as

$$H_{(+)} = -\frac{1}{2M} \int d^3\mathbf{x}\, \psi^\dagger(\mathbf{x},t)\nabla^2\psi(\mathbf{x},t) + \frac{\lambda}{4M^2} \int d^3\mathbf{x}\, d^3\mathbf{x}'\, \psi^\dagger(\mathbf{x},t)\psi^\dagger(\mathbf{x}',t)\delta^3(\mathbf{x}-\mathbf{x}')\psi(\mathbf{x}',t)\psi(\mathbf{x},t) . \tag{3.7}$$

The Hamiltonian (3.7) is identical to that of a second-quantized Schrödinger field;[15] the forces between the particles are two-body forces generated by the potential

$$V(\mathbf{x},\mathbf{x}') = g\delta^3(\mathbf{x}-\mathbf{x}') , \tag{3.8}$$

where

$$g = \lambda/2M^2 . \tag{3.9}$$

With the derivation of Eqs. (3.7)—(3.9), we have completed the reduction of the Lorentz-invariant $\lambda\phi^4$ theory to its NR limit; the mass M may now be regarded as a fixed parameter—rather than one tending to infinity—that admits of an operational definition within the framework of NR mechanics. To have a coupling constant which lends itself to a similar definition, we shall renormalize it once again in the next section.

Note that the second term in Eq. (3.7) contributes to the four-point function but not to the two-point function, as it would if ψ had both positive and negative frequency parts. This, of course, is why self-energy effects had to be handled separately.

IV. THE COLLISION MATRIX

Separating out the center of mass motion, the Hamiltonian for the $\phi-\phi$ system may be written as

$$H = H_0 + V , \tag{4.1}$$

where

$$H_0 = -(1/M)\nabla_r{}^2 \tag{4.2}$$

and

$$V = g\delta^3(\mathbf{r}) , \tag{4.3}$$

$\mathbf{r}$ being the separation between the two particles. The suffix r on the Laplacian indicates that it operates in $\mathbf{r}$ space.

Equations (4.1)—(4.3) follow from Eq. (3.7) specialized to the $N=2$ sector. In Eq. (4.2), we have used the fact that the reduced mass is $M/2$.

We shall formally *define*[16] the collision matrix T, at energy E, through the Lippmann-Schwinger equation[15]

$$T = V + VG_0T , \tag{4.4}$$

where

$$G_0 = (E + i\epsilon - H_0)^{-1} . \tag{4.5}$$

In coordinate space,[17] one has the explicit representation for G_0

$$\langle \mathbf{r}' | G_0 | \mathbf{r} \rangle = M \int \frac{d^3\mathbf{q}}{(2\pi)^3} \frac{e^{i\mathbf{q}\cdot(\mathbf{r}-\mathbf{r}')}}{k^2+i\epsilon-q^2} \tag{4.6}$$

$$= -\frac{M}{4\pi} \frac{e^{ik|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|} . \tag{4.7}$$

Here, and hereafter, it is understood that $k \equiv |\mathbf{k}|$, $q \equiv |\mathbf{q}|$, etc.

When V is a singular short-range potential, it is necessary to introduce some regularization procedure to give meaning to Eq. (4.4). We define a regulated Green's function[18] via

$$\langle \mathbf{r}' | G_0^{\text{reg}} | \mathbf{r} \rangle = M \int \frac{d^3\mathbf{q}}{(2\pi)^3} e^{i\mathbf{q}\cdot(\mathbf{r}-\mathbf{r}')} \left[\frac{1}{k^2+i\epsilon-q^2} + \frac{1}{\Lambda^2+q^2} \right] \tag{4.8}$$

$$= -\frac{M}{4\pi} \frac{1}{|\mathbf{r}-\mathbf{r}'|} (e^{ik|\mathbf{r}-\mathbf{r}'|} - e^{-\Lambda|\mathbf{r}-\mathbf{r}'|}) . \tag{4.9}$$

This Green's function is regular at $\mathbf{r}=\mathbf{r}'$ for all finite Λ. Also, its imaginary part is independent of Λ; this ensures that only the *real* part of the inverse collision matrix is affected by regularization. We emphasize that the introduction of Λ is a purely mathematical artifice; no physical significance should be attached to it.

To calculate the collision matrix, we use the modified Lippmann-Schwinger equation

$$T = V + VG_0^{\text{reg}}T \tag{4.10}$$

with the understanding that Λ is to be taken to infinity at the end of the calculation. Taking the matrix elements of both sides of Eq. (4.10) in momentum space, we obtain, after a little manipulation,

$$\langle \mathbf{k}' | T | \mathbf{k} \rangle = \frac{g}{(2\pi)^3} + Mg \int \frac{d^3\mathbf{p}}{(2\pi)^3} \left[\frac{1}{k^2+i\epsilon-p^2} + \frac{1}{\Lambda^2+p^2} \right] \times \langle \mathbf{p} | T | \mathbf{k} \rangle . \tag{4.11}$$

The right-hand side of this equation has the remarkable feature that it does not depend on $\mathbf{k}'$. Hence,

$$\langle \mathbf{k}' | T | \mathbf{k} \rangle = \langle \mathbf{p} | T | \mathbf{k} \rangle = f(k) \frac{1}{(2\pi)^3} \tag{4.12}$$

and we obtain the solution

$$f(k)=\frac{g}{1+(M/4\pi)g(\Lambda+ik)}\ . \tag{4.13}$$

Now, in a theory with purely quadrilinear interactions, a natural way to define the renormalized coupling constant is to equate it to the value of the reduced collision amplitude—the amplitude up to inessential factors of 2π—at some judiciously chosen momentum, say $k=i\kappa$. (The momentum must be such that the amplitude is real.) We therefore write for $g_{\rm ren(NR)}$, the renormalized four-field coupling in the NR limit,

$$\begin{aligned} g_{\rm ren(NR)}&=f(i\kappa)\\ &=g\Big/\left[1+\frac{M}{4\pi}g\Lambda\right] \quad \text{for } \kappa\ll\Lambda\ . \end{aligned} \tag{4.14}$$

In terms of dimensionless quantities

$$\lambda_{\rm ren(NR)}=\lambda\Big/\left[1+\frac{\lambda}{8\pi}\frac{\Lambda}{M}\right]\ . \tag{4.15}$$

The requirement that the theory be stable implies that $g>0$. This, in turn, implies that $g_{\rm ren(NR)}\geq 0$, that it goes smoothly to zero as $\Lambda\to\infty$ and that the S matrix is unity. The result stated in Sec. I is thus established.

V. CONCLUDING REMARKS

We have shown that if ϕ be a complex spin-0 field, the $\lambda\phi^4$ theory reduces in the nonrelativistic limit to a second-quantized Schrödinger theory with repulsive two-body forces, the potentials being three-dimensional δ functions of the interparticle separation. We have also shown that such potentials lead to a zero collision matrix, and thereby established that the theory reduces to a trivial theory with zero renormalized coupling and unit S matrix. The physical reason for this result is not difficult to see; two point particles interacting via a repulsive δ-function potential will be unable to perceive each other.

Our considerations have their origin in the recent surge of interest in the almost universally accepted conjecture that the relativistic $\lambda\phi^4$ theory is a trivial theory. By proving the triviality of the NR limit of the theory, we have established a consequence of this conjecture though not the conjecture itself. As we have taken pains to emphasize, a Lorentz-invariant field theory cannot be easily reconstructed from its Galilean limit.

ACKNOWLEDGMENTS

It is a pleasure to thank Professor Lowell Brown for an enjoyable discussion. One of us (M.A.B.B.) was a guest of the Aspen Center for Physics while this work was done; this hospitality of the Center is gratefully acknowledged. The work of M. A. B. Bég was supported in part by the Department of Energy under Contract Grant Number DE-AC02-81ER40033B.000. R. C. Furlong is a Seitz Fellow.

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[17]Our normalization convention for one-particle states, in configuration and momentum spaces, respectively, is as follows: $\langle \mathbf{r}\,|\,\mathbf{r}'\rangle=\delta^3(\mathbf{r}-\mathbf{r}')$ and $\langle \mathbf{p}\,|\,\mathbf{p}'\rangle=\delta^3(\mathbf{p}-\mathbf{p}')$.

[18]An alternate procedure would be to regulate the potential [see N. Khuri and A. Pais, Rev. Mod. Phys. **36**, 590 (1964)]. One of us (R.C.F.) has verified the results of our paper using the regulated δ-function potential:

$$V(\mathbf{r})=g\delta(r-a)\delta(\theta)\delta(\phi)/r^2\sin\theta$$

where r, θ, and ϕ are the usual polar coordinates that specify the vector $\mathbf{r}$. A unit S matrix results in the limit $a\to 0$.