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Quark model of light mesons with dynamically broken chiral symmetry

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We study the meson spectrum in a model with a confining Lorentz-vector—and hence chiral-invariant—interaction between massless quark fields. As shown in a previous work, chiral invariance is spontaneously broken. In the case of the harmonic oscillator, as the Fourier transform of the potential is the Laplacian of a δ function, the Bethe-Salpeter (BS) equation—a system of linear integral equations in general—splits into a system of differential equations that we solve in the broken vacuum. Without appealing to any spin-spin interaction, we find, besides the massless pseudoscalar, a vector meson at the right scale and an excited pion and two vectors in the 1–2-GeV region. Moreover, we find a large L-S splitting with the expected ordering for a vector interaction. We study in detail the BS wave function for the pion in motion, necessary to compute axial-vector-current matrix elements, and recover well known relations of current algebra. We compute f_π and find on general grounds that $f_{\pi'}=0$ in the chiral limit, where π' is any radially excited pion. The pion satisfies the expected dispersion law for a Goldstone boson, $\omega(p)\rightarrow cp$ $(p\rightarrow 0)$.

I. INTRODUCTION

Since the retrospectively impressive work of Nambu and Jona-Lasinio,[1] many papers have appeared on dynamical breaking of chiral symmetry, i.e., spontaneous breaking by fermion-pair condensation, without appealing to elementary scalars.

Although these ideas have been popular for many years, the more implicit approach of current algebra[2]—using general symmetry principles or effective Lagrangians realizing these ideas—has been the dominant point of view. The emergence of QCD as the fundamental theory of the strong interactions has put forward again the problem of the precise dynamical mechanism responsible for the pair condensation, either in the continuum or on a lattice.

The studies of chiral-symmetry breaking for QCD on the lattice are quite interesting in the different versions (Susskind fermions,[3] Wilson formulation,[4] staggered fermions,[5] SLAC fermions[6]) but the results reached so far are not conclusive due to the difficulties linked to the doubling of fermions.[7]

In continuum QCD a number of approaches have been applied to the problem of chiral-symmetry breaking. There are asymptotic results[8] that exploit the asymptotic freedom of QCD to study the large-Q^2 behavior of quantities such as the dynamical or the current masses. On the other hand, chiral perturbation theory[9] gets results on deviations from the exact chiral limit where $M_\pi=0$. However, in these works (as well as for the QCD sum rules)[10] dynamical chiral-symmetry breaking is an assumption, a starting point.

Our approach is closer in spirit to the original one of Nambu and Jona-Lasinio: given a chiral-invariant interaction between massless quarks, the problem is to see if there is indeed a lowest-energy solution of the gap equation (or the equivalent Schwinger-Dyson equation for the self-mass) noninvariant under chiral symmetry, and then solve the Bethe-Salpeter (BS) equation to study the light-meson spectrum. In this spirit, there are also a number of partial results for QCD. Instability analyses in the Landau[11] or Coulomb gauges show indeed that the chiral-invariant vacuum is unstable for $\alpha_s>\alpha_s^{\text{crit}}$, with $\alpha_s^{\text{crit}}\sim 1$, its precise value depending on the approximations adopted.[12] A deeper study of the problem—solving the gap equation and studying the pion properties[13]—has been put forward in a series of papers by Finger, Mandula, and co-workers,[14] which have proposed to work in the Coulomb gauge. One important advantage of this gauge is that the renormalization-group corrections can be easily implemented: the Coulomb propagator corrections give the complete QCD β function. The Coulomb gauge allows also the extension of the study to confining potentials that simulate the QCD long-distance effects if the confining potential is assumed to be a piece of the time component of a Lorentz-vector exchange.

In QCD, one needs a renormalized form of the gap equation. Finger, Mandula, and co-workers[14] propose as a renormalization prescription taking the Wick ordering in the chiral-invariant vacuum—thus preserving chiral invariance and avoiding infinities in the fermion self-energy. We have shown[15] that this prescription leads to a paradox for confining potentials: for a positive r^α $(\alpha>0)$ interaction, the chiral-invariant state is stable, a true vacuum. We have shown that this result is an artifact of the normal ordering. If one adopts, on the contrary, the original Hamiltonian without normal ordering, one is led to compute self-energy loops in the variational calculation. The result is that the massless-fermion self-energy is negative and infrared singular, behaving like $-1/k^\alpha$, and overcomes the positive potential energy, destabilizing the vacuum by pair condensation. Moreover, the gap equation has a different form than the one with normal ordering. Adler and Davis[16] have recently proved, starting from the renormalized Dyson equations for the vector and axial-vector vertices, and using the Ward identities, that the prescription of Finger and Mandula is only correct for the case of pure Coulomb exchange. The gap equation of Adler and Davis agrees with ours in the infrared and both are identical for a pure confining interaction, as it is ultraviolet finite.

Moreover, we have solved this gap equation in the particular case of the harmonic oscillator:[15] its Fourier transform being proportional to $\Delta_{\vec{k}}\delta(\vec{k})$, the gap equation (a nonlinear integral equation, in general) reduces to a second-order differential equation of the sine-Gordon type. We have computed the shift in energy density between the invariant and the noninvariant vacua, the mass gap, and $\langle\bar{\psi}\psi\rangle$. In spite of the infrared-singular behavior of the potential that reflects in the behavior of the self-energy, all these quantities turn out to be infrared finite, and we have shown the precise cancellation between the singularities.[15,16]

We want to proceed further with our program and compute the meson spectrum by solving the BS equation. Although this equation has often been used in the study of the meson spectrum,[17] there are no results to our knowledge for the meson masses in the case of dynamical chiral-symmetry breaking (with the main exception of the original paper by Nambu and Jona-Lasinio). We have here a very different starting point (or zeroth-order perturbation) than in the old naive SU(6) limit[18]: $M_\pi = M_\rho$, $M_\delta = M_{A_1} = M_B = M_{A_2}$. The splittings were then attributed to short-distance spin-spin and spin-orbit one-gluon-exchange effects.[19] Here we have, on the contrary, as a starting point, $M_\pi = 0$, $M_\rho \neq 0$, and the $L=1$ mesons split. We will have $M_\pi = 0$, $M_\rho \neq 0$, independently of a spin-spin interaction, as a consequence of dynamical symmetry breaking. Moreover, this model satisfies the pion low-energy theorems, unlike the naive quark model.

The interaction that we assume is instantaneous and we are thus in the simpler case of the Salpeter equation, our starting point in Sec. III, after a brief review of our previous results on the gap equation in Sec. II. We will show that the existence of a chiral-noninvariant lowest-energy solution of the gap equation implies the existence of a Goldstone boson. In the Appendices we will detail the formalism of the BS equation and the deduction of the Salpeter equation by integration over the relative energies.[20] In Sec. IV we will make the Dirac analysis of the Salpeter equation, and particularize to the case of the harmonic oscillator: the equation reduces then to a system of second-order linear differential equations. In Sec. V we make the expansion in partial waves and we identify the different quantum numbers. In Sec. VI we outline the method of integration and give the numerical results for the spectrum. In Sec. VII we come back to the chiral-invariant vacuum and we see how instability manifests itself by the existence of tachyons. Moreover, around this "vacuum," the states correspond to chiral doublets; as we show, this is also true for the asymptotic high-energy spectrum in the broken vacuum. In Sec. VIII we study in detail the moving-pion wave function to compute axial-vector-current matrix elements, we recover well known current-algebra relations, and we compute the dispersion law for the Goldstone boson $\omega(p) \to cp$ $(p \to 0)$, where c is the pion velocity, $c \neq 1$ in our noncovariant model. Some results of this section have been obtained independently by Govaerts, Mandula, and Weyers[21] and by Adler and Davis.[16] We finally conclude in Sec. IX.

II. GAP EQUATION

For the sake of completeness and to make clear the object of this paper, let us give a brief summary of our framework and the formulation of the gap equation through the Bogoliubov-Valatin (BV) variational method. All details can be found in Ref. 15.

Let us start from the chiral-invariant Hamiltonian for massless quark fields interacting through an instantaneous fourth-component Lorentz-vector color-confining potential:

$$\mathscr{H} = \sum_{\vec{x}} \psi^\dagger(\vec{x})(-i\vec{\alpha}\cdot\vec{\nabla})\psi(\vec{x}) + \frac{1}{2}\sum_{\vec{x},\vec{y},a} V(\vec{x}-\vec{y}) \left[\psi^\dagger(\vec{x})\frac{\lambda^a}{2}\psi(\vec{x})\right] \times \left[\psi^\dagger(\vec{y})\frac{\lambda^a}{2}\psi(\vec{y})\right], \tag{2.1}$$

where $V(\vec{x}) = -V_0^{1+\alpha}|\vec{x}|^\alpha$. The lattice formalism does not play an essential role here and at the end we will take the continuum infinite-volume limit:

$$a^3\sum_{\vec{x}} \to \int d\vec{x}, \quad \frac{1}{(an)^3}\sum_{\vec{k}} \to \int \frac{d\vec{k}}{(2\pi)^3}.$$

For the sake of simplifying the notations, we consider only one fermion flavor. The generalization to the realistic case of two massless flavors is straightforward. We leave aside any discussion of the U(1) problem, which is beyond our approach.

Let us first perform a BV transformation: it consists in writing the quark fields no longer in terms of a massless-spinor base, but in terms of arbitrary spinors u,v,

$$\psi(\vec{x}) = \frac{1}{n^{3/2}}\sum_{\vec{k},s} [u_s(\vec{k})b_s(\vec{k}) + v_s(\vec{k})d_s^\dagger(-\vec{k})]e^{i\vec{k}\cdot\vec{x}}. \tag{2.2}$$

These spinors are not necessarily solutions of the Dirac equation, but obey the usual normalization conditions, preserving in this way the canonical anticommutation relations. The BV methods consist in writing the Hamiltonian in terms of the new creation and annihilation operators of the trial base. What characterizes the BV approximation is the linear character of the relation between the old (corresponding to massless fermions) and the new creation and annihilation operators. The method amounts to use as a trial state a coherent superposition of pairs of massless fermions. In the formulation that we adopt here, the new spinors u,v are trial spinors to be varied to look for the stationary states of the theory. To obtain a useful expression, we need to rewrite the Hamiltonian in terms of normal-ordered operators relatively to the new base. One obtains applying Wick's theorem

$$\mathscr{H} = \mathscr{E} + :H_2: + :H_4:, \tag{2.3}$$

where

$$\mathscr{E}=3\sum_{\vec{k}}\mathrm{Tr}[(\vec{\alpha}\cdot\vec{k})\Lambda_-(\vec{k})]+4\frac{1}{(an)^3}\frac{1}{2}\sum_{\vec{k},\vec{k}'}\tilde{V}(\vec{k}-\vec{k}')\mathrm{Tr}[\Lambda_+(\vec{k})\Lambda_-(\vec{k}')]\ , \tag{2.4}$$

$$H_2=\frac{4}{3}\frac{1}{(an)^3}\frac{1}{2}\sum_{\vec{k},\vec{x},\vec{y}}V(\vec{x}-\vec{y})e^{i\vec{k}\cdot(\vec{x}-\vec{y})}\times\{\psi^\dagger(\vec{x})[\Lambda_+(\vec{k})-\Lambda_-(\vec{k})]\psi(\vec{y})\}+\sum_{\vec{x}}\psi^\dagger(\vec{x})(-i\vec{\alpha}\cdot\vec{\nabla})\psi(\vec{x})\ , \tag{2.5}$$

$$H_4=\frac{1}{2}\sum_{\vec{x},\vec{y},a}V(\vec{x}-\vec{y})\left[\psi^\dagger(\vec{x})\frac{\lambda^a}{2}\psi(\vec{x})\right]\left[\psi^\dagger(\vec{y})\frac{\lambda^a}{2}\psi(\vec{y})\right]. \tag{2.6}$$

$\mathscr{E}$ is the energy of the trial BV states (we will call it the "vacuum" energy) and H_2 and H_4 are the bilinear and quadrilinear terms in the quark fields. The algebraic factors come from color, and $\Lambda_\pm(\vec{k})$ are the projectors

$$\Lambda_+(\vec{k})=\sum_s u_s(\vec{k})u_s^\dagger(\vec{k})\ , \tag{2.7}$$

$$\Lambda_-(\vec{k})=\sum_s v_s(\vec{k})v_s^\dagger(\vec{k})\ . \tag{2.8}$$

If all the invariances of the original interaction but chiral invariance are not spontaneously broken, one may see that $\Lambda_\pm$ must be of the form

$$\Lambda_\pm(\vec{k})=\tfrac{1}{2}[1\pm\sin\varphi(k)\beta\pm\cos\varphi(k)\vec{\alpha}\cdot\hat{k}]\ , \tag{2.9}$$

where $\varphi(k)$ is a function of $k=|\vec{k}|$ since we assume rotational invariance to be preserved. Notice that although $\Lambda_\pm$ has a mass term, the Hamiltonian in terms of $\Lambda_\pm$ remains, of course, chiral invariant: the Hamiltonian is invariant but we allow for possible noninvariant vacua.

The gap equation is just the condition of stationary $\mathscr{E}$, the vacuum energy. $\mathscr{E}$ is a functional of the projector $\Lambda_-(\vec{k})$. Differentiating relatively to it, we obtain

$$\delta\mathscr{E}=3\sum_{\vec{k}}\mathrm{Tr}\left[\delta\Lambda_-(\vec{k})\left[\vec{\alpha}\cdot\vec{k}+\frac{4}{3}\frac{1}{(an)^3}\frac{1}{2}\sum_{\vec{k}'}\tilde{V}(\vec{k}-\vec{k}')[1-2\Lambda_-(\vec{k}')]\right]\right] \tag{2.10}$$

with $\delta\Lambda_-$ satisfying the projector constraint

$$\Lambda_-\delta\Lambda_-+\delta\Lambda_-\Lambda_-=\delta\Lambda_-\ . \tag{2.11}$$

The condition of extremum is then that the operator in brackets in (2.10) must be diagonal by blocks. Then, the gap equation can be written as the two coupled equations

$$H(\vec{k})=\vec{\alpha}\cdot\vec{k}+\frac{4}{3}\frac{1}{(an)^3}\frac{1}{2}\sum_{\vec{k}'}\tilde{V}(\vec{k}-\vec{k}')[1-2\Lambda_-(\vec{k}')]\ , \tag{2.12}$$

$$[\Lambda_-(\vec{k}),H(\vec{k})]=0\ . \tag{2.13}$$

$H(\vec{k})$ corresponds simply to the Hamiltonian of a free Dirac particle, the bilinear part of (2.3). The second condition means that Λ_- and H can be diagonalizable simultaneously; both operators fix the fermion propagator in the broken theory, as we explain in Appendix A. Using the explicit form (2.9) and (2.12), $H(\vec{k})$ can be written in the form

$$H(\vec{k})=A(k)\beta+B(k)\vec{\alpha}\cdot\hat{k} \tag{2.14}$$

with

$$A(k)=\tfrac{1}{2}\times\tfrac{4}{3}\times\frac{1}{(an)^3}\sum_{\vec{k}'}\tilde{V}(\vec{k}-\vec{k}')\sin\varphi(k')\ ,$$
$$B(k)=k+\tfrac{1}{2}\times\tfrac{4}{3}\times\frac{1}{(an)^3}\sum_{\vec{k}'}\tilde{V}(\vec{k}-\vec{k}')\cos\varphi(k')(\hat{k}\cdot\hat{k}')\ . \tag{2.15}$$

The second condition implies then

$$A(k)=E(k)\sin\varphi(k),\quad B(k)=E(k)\cos\varphi(k)\ . \tag{2.16}$$

This system (equivalent to the Schwinger-Dyson equation for the self-mass in the ladder approximation[15]) can be reduced to a single nonlinear integral equation for $\varphi(k)$ since $A/B=\tan\varphi$.

$$\tfrac{1}{2}\times\tfrac{4}{3}\times\frac{1}{(an)^3}\times\sum_{\vec{k}'}\tilde{V}(\vec{k}-\vec{k}')[\sin\varphi(k')\cos\varphi(k)-\cos\varphi(k')\sin\varphi(k)(\hat{k}\cdot\hat{k}')]=k\sin\varphi(k)\ . \tag{2.17}$$

The gap equation can be written as a single equation for $\varphi(k)$ since the vacuum energy depends only on φ through $\Lambda_\pm$. Once we know a solution of (2.17), $A(k)$ and $B(k)$ are given by (2.16), and the energy of a fermion is then given by

$$E(k)=\frac{4}{3}\frac{1}{(an)^3}\frac{1}{2}\sum_{\vec{k}'}\tilde{V}(\vec{k}-\vec{k}')\frac{\sin\varphi(k')}{\sin\varphi(k)}\ . \tag{2.18}$$

As we have seen,[15] in the case of a harmonic oscillator, $\alpha=2$, the gap equation reduces to a nonlinear differential equation, since the Fourier transform of the potential is just the Laplacian of a delta function. We have established in this case the existence of chiral-nonvariant solutions.

Indeed, the Fourier transform of $V(r)=-V_0{}^3r^2$ is

$$\tilde{V}(\vec{k})=V_0{}^3(2\pi)^3\Delta_{\vec{k}}\delta(\vec{k}) \tag{2.19}$$

and the gap equation (2.17) becomes a nonlinear differential equation

$$\tfrac{4}{3}V_0^3(k^2\varphi')'=2k^3\sin\varphi-\tfrac{4}{3}V_0^3\sin 2\varphi\ . \qquad (2.20)$$

For completeness, let us give the fermion energy $E(k)$, the energy density shift between the chiral-invariant and the new broken vacuum $\Delta\epsilon$, and $\langle\bar\psi\psi\rangle$:

$$E(k)=k\cos\varphi-\tfrac{4}{3}V_0^3\frac{1}{k^2}\cos^2\varphi-\tfrac{1}{2}\times\tfrac{4}{3}V_0^3(\varphi')^2\ , \qquad (2.21)$$

$$\Delta\epsilon=\frac{3}{2\pi^2}\int_0^\infty dk\left[2k^3(1-\cos\varphi)-\tfrac{4}{3}V_0^3\left[\sin^2\varphi-\frac{k^2}{2}(\varphi')^2\right]\right]\ , \qquad (2.22)$$

$$\langle\bar\psi\psi\rangle=-\frac{3}{\pi^2}\int_0^\infty k^2dk\sin\varphi\ . \qquad (2.23)$$

The lowest-energy solution of the gap equation behaves like $\varphi(k)\to(\pi/2)+Ck$ as $k\to 0$, and is plotted in Ref. 15. We found, then, for this solution

$$\Delta\epsilon=-\frac{3}{2\pi^2}(\tfrac{4}{3}V_0^3)^{4/3}\times 0.208\ , \qquad (2.24)$$

$$\langle\bar\psi\psi\rangle=-\frac{3}{\pi^2}(\tfrac{4}{3}V_0^3)\times 0.372\ , \qquad (2.25)$$

in terms of the energy scale $(\tfrac{4}{3}V_0^3)^{1/3}$. Using the value fitted by Feynman, Kislinger, and Ravndal,[20]

$$\tfrac{4}{3}V_0^3\cong(368\text{ MeV})^3$$

we obtain

$$\Delta\epsilon=-(155\text{ MeV})^4=-73\text{ MeV/fm}^3\ ,$$
$$\langle\bar\psi\psi\rangle=-(178\text{ MeV})^3\ . \qquad (2.26)$$

$\langle\bar\psi\psi\rangle$ is to be compared with the phenomenological value[22] $-(250\text{ MeV})^3$ (one flavor).

In order to define an effective dynamical quark mass m_q let us expand the projector $\Lambda_+(\vec{k})$ (2.9) for small k and compare it to the free quark propagator[16]

$$\Lambda_+(\vec{k})\cong\tfrac{1}{2}\left[1+\beta-\left[\frac{d\varphi}{dk}\right]_{k=0}\vec{\alpha}\cdot\vec{k}\right]=\tfrac{1}{2}\left[1+\beta+\frac{\vec{\alpha}\cdot\vec{k}}{m_q}\right]$$

since $\sin\varphi(k)\to 1$ as $k\to 0$. From

$$\left[\frac{d\varphi}{dk}\right]_{k=0}=C=-2.037\times\left[\tfrac{4}{3}V_0^3\right]^{-1/3}=-\frac{1}{m_q}$$

we obtain

$$m_q=(\tfrac{4}{3}V_0^3)^{1/3}\frac{1}{2.037}\ .$$

We obtain $m_q=180$ MeV for the above-mentioned scale. Notice that m_q is somewhat small (we would like it to be about 300 MeV), consistently with the value found for $\langle\bar\psi\psi\rangle$. We get, however, a nice result for the ratio

$$\frac{(-\langle\bar\psi\psi\rangle)^{1/3}}{m_q}=0.985\ ;$$

that means a reasonable dynamical quark mass of 250 MeV if we adopt for $\langle\bar\psi\psi\rangle$ the empirical value $-(250\text{ MeV})^3$.

III. SALPETER EQUATION AND GOLDSTONE BOSON

In the case of an instantaneous interaction like ours, the integration over the relative energies can be done, and the BS equation reduces to the simpler Salpeter equation. To make the reading of this part easier, we will begin setting this equation for the bound-state wave function. We detail in Appendix B its deduction from the general inhomogeneous BS equation.

The Salpeter equation for the $q\bar{q}$ bound-state wave function with center-of-mass momentum $\vec{p}$, $\chi_{\vec{p}}(\vec{k})$, where $\vec{k}$ is the internal momentum, reads

$$H\left[\vec{k}+\frac{\vec{p}}{2}\right]\chi_{\vec{p}}(\vec{k})-\chi_{\vec{p}}(\vec{k})H\left[\vec{k}-\frac{\vec{p}}{2}\right]-\frac{4}{3}\frac{1}{(an)^3}\sum_{\vec{k}'}\tilde{V}(\vec{k}-\vec{k}')\left[\chi_{\vec{p}}(\vec{k}')\Lambda_-\left[\vec{k}-\frac{\vec{p}}{2}\right]-\Lambda_-\left[\vec{k}+\frac{\vec{p}}{2}\right]\chi_{\vec{p}}(\vec{k}')\right]=\omega(p)\chi_{\vec{p}}(\vec{k})\ . \qquad (3.1)$$

$\omega(p)$ is the bound-state energy, and H,Λ_- have been defined in Sec. II. $\chi_{\vec{p}}(\vec{k})$ obeys, moreover, the constraints

$$\Lambda_+\left[\vec{k}+\frac{\vec{p}}{2}\right]\chi_{\vec{p}}(\vec{k})\Lambda_+\left[\vec{k}-\frac{\vec{p}}{2}\right]=\Lambda_-\left[\vec{k}+\frac{\vec{p}}{2}\right]\chi_{\vec{p}}(\vec{k})\Lambda_-\left[\vec{k}-\frac{\vec{p}}{2}\right]=0\ , \qquad (3.2)$$

where the normalized wave functions should verify

$$\frac{1}{(an)^3}\sum_{\vec{k}}\mathrm{Tr}\left[\chi_{\vec{p}}(\vec{k})\Lambda_-\left[\vec{k}-\frac{\vec{p}}{2}\right]\chi^\dagger_{\vec{p}}(\vec{k})-\Lambda_-\left[\vec{k}+\frac{\vec{p}}{2}\right]\chi_{\vec{p}}(\vec{k})\chi^\dagger_{\vec{p}}(\vec{k})\right]=1 \qquad (3.3)$$

for states such that

$$\langle\vec{p}\,|\,\vec{p}\,'\rangle=(2\pi)^3\delta(\vec{p}-\vec{p}\,')$$

(Appendix C).

Let us now see that the gap equation possesses a chiral invariance that implies the existence, in the broken theory, of a massless pseudoscalar solution of the Salpeter equation. Under a chiral transformation

$$\psi(\vec{x})\to e^{i\theta\gamma_5}\psi(\vec{x})\ ,$$

the projectors transform

$$\Lambda_\pm(\vec{k})\to e^{i\theta\gamma_5}\Lambda_\pm(\vec{k})e^{-i\theta\gamma_5}\ .$$

From (2.4) we see that the vacuum energy $\mathscr{E}(\Lambda_\pm(\vec{k}))$ is invariant under this transformation:

$$\mathscr{E}(\Lambda_\pm(\vec{k}))=\mathscr{E}(e^{i\theta\gamma_5}\Lambda_\pm(\vec{k})e^{-i\theta\gamma_5}) . \quad (3.4)$$

This equation expresses the condition of degeneracy of the chiral-noninvariant BV states: the energy is the same for all chiral-transformed states, and we will obtain from (3.4) a condition of chiral invariance of the gap equation. We will see now that this invariance implies a massless solution for the equation (3.1).

Let us perform an infinitesimal chiral transformation on the quark fields. The operators $\Lambda_\pm(\vec{k})$ and $H(\vec{k})$ transform then in the form

$$\Lambda_\pm(\vec{k})\to\Lambda_\pm(\vec{k})+i\delta\theta[\gamma_5,\Lambda_\pm(\vec{k})] ,$$

$$H(\vec{k})\to H(\vec{k})+i\delta\theta[\gamma_5,H(\vec{k})] .$$

The invariance (3.4) reflects on the gap equation under the form

$$[\gamma_5,H(\vec{k})]-\frac{4}{3}\frac{1}{(an)^3}\sum_{\vec{k}'}\tilde{V}(\vec{k}-\vec{k}')[\gamma_5,\Lambda_-(\vec{k}')]=0 \quad (3.5)$$

and the condition (2.13) gives, to first order in $\delta\theta$,

$$[[\gamma_5,H(\vec{k})],\Lambda_\pm(\vec{k})]=-[H(\vec{k}),[\gamma_5,\Lambda_\pm(\vec{k})]] . \quad (3.6)$$

Taking now the commutator of (3.5) with $\Lambda_-(\vec{k})$ and using (3.6) we obtain

$$[H(\vec{k}),[\gamma_5,\Lambda_-(\vec{k})]]-\frac{4}{3}\frac{1}{(an)^3}\sum_{\vec{k}'}\tilde{V}(\vec{k}-\vec{k}')\times[[\gamma_5,\Lambda_-(\vec{k}')],\Lambda_-(\vec{k})]=0 . \quad (3.7)$$

This equation is nothing else but the Salpeter equation for a massless bound state at $\vec{p}=0$ (3.1):

$$[H(\vec{k}),\chi_0(\vec{k})]-\frac{4}{3}\frac{1}{(an)^3}\sum_{\vec{k}'}\tilde{V}(\vec{k}-\vec{k}')\times[\chi_0(\vec{k}'),\Lambda_-(\vec{k})]=0 \quad (3.8)$$

with the Hermitian wave function

$$\chi_0(\vec{k})=i[\Lambda_-,\gamma_5]=\sin\varphi(k)i\gamma_5\gamma_0 \quad (3.9)$$

corresponding to a pseudoscalar state, the Goldstone pion. The normalization of (3.9) is impossible because the norm, from (3.3), vanishes:

$$\sum_{\vec{k}}\mathrm{Tr}\{[\chi_0(\vec{k}),\Lambda_-(\vec{k})]\chi_0^\dagger(\vec{k})\}=0 . \quad (3.10)$$

This is consistent with the usual covariant normalization of states,

$$\langle\vec{p}\,|\,\vec{p}\,'\rangle=(2\pi)^3\delta(\vec{p}-\vec{p}\,')2\omega(p) , \quad (3.11)$$

and we now have $\omega(p)=0$. To normalize the BS pion wave function we will need to compute it beyond the $\vec{p}=0$ limit. This will be done in Sec. VIII.

IV. DIRAC STRUCTURE OF THE SALPETER EQUATION

To study the meson spectrum for the different quantum numbers, let us write the BS wave function at $\vec{p}=0$ in a base of Dirac matrices

$$\chi(\vec{k})=L_0(\vec{k})+\sum_{i=1}^{3}L_i(\vec{k})\rho_i+\vec{M}(\vec{k})\cdot\vec{\sigma}+\sum_{i=1}^{3}\vec{N}_i(\vec{k})\rho_i\cdot\vec{\sigma} , \quad (4.1)$$

where $\rho_1=\gamma_5$, $\rho_2=i\gamma_5\gamma_0$, $\rho_3=\gamma_0$, and $\vec{\sigma}$ are 4×4 spin matrices.

We have seen that the projections (3.2) vanish. Using the relation between $H(\vec{k})$ and $\Lambda_-(\vec{k})$ via $A(k),B(k)$ (Sec. II),

$$H(\vec{k})=E(k)[1-2\Lambda_-(\vec{k})]=-E(k)[1-2\Lambda_+(\vec{k})] , \quad (4.2)$$

we can easily see that the conditions (3.2), when $\vec{p}=0$, are equivalent to the vanishing of the anticommutator

$$\{\chi(\vec{k}),H(\vec{k})\}=0 . \quad (4.3)$$

From the Dirac decomposition of $H(\vec{k})$, which reads in the new notation

$$H(\vec{k})=A(k)\rho_3+B(k)\rho_1\vec{\sigma}\cdot\hat{k} ,$$

the condition (4.3) leads to a number of constraints

$$L_0=0,\quad L_3=-\frac{B}{A}\vec{N}_1\cdot\hat{k} ,$$

$$\vec{N}_3=-\frac{B}{A}L_1\hat{k},\quad \vec{M}=-\frac{B}{A}(\vec{N}_2\times\hat{k}) ,$$

and we get the general form of $\chi(\vec{k})$ satisfying (3.2)

$$\chi(\vec{k})=L_1\left[\rho_1-\frac{B}{A}\rho_3\vec{\sigma}\cdot\hat{k}\right]+L_2\rho_2+\vec{N}_1\cdot\left[\rho_1\vec{\sigma}-\frac{B}{A}\rho_3\hat{k}\right]+\vec{N}_2\cdot\left[\rho_2\vec{\sigma}+\frac{B}{A}\vec{\sigma}\times\hat{k}\right] . \quad (4.4)$$

At $\vec{p}=0$, enough to study the meson spectrum, the Salpeter equation takes the simple form

$$[H(\vec{k}),\chi(\vec{k})]-\frac{4}{3}\frac{1}{(an)^3}\sum_{\vec{k}'}\tilde{V}(\vec{k}-\vec{k}')[\chi(\vec{k}'),\Lambda_-(\vec{k})]=M\chi(\vec{k}) , \quad (4.5)$$

where M is the meson mass. The commutators in (4.5) can be reduced to commutators of a general operator with $H(\vec{k})$ since we have

$$[\chi(\vec{k}'),\Lambda_-(\vec{k})]=\frac{1}{2E(k)}[H(\vec{k}),\chi(\vec{k}')] . \quad (4.6)$$

The commutator of a generic Dirac matrix with $H(\vec{k})$ writes

$$\frac{1}{2i}\left[H(\vec{k}),\sum_{i=1}^{3}X_i\rho_i+\vec{\sigma}\cdot\vec{Y}+\sum_{i=1}^{3}\vec{Z}_i\cdot\rho_i\vec{\sigma}\right]=-X_2(A\rho_1-B\rho_3\vec{\sigma}\cdot\hat{k})+(AX_1-B\vec{Z}_3\cdot\hat{k})\rho_2$$
$$+(-A\vec{Z}_2-B\vec{Y}\times\hat{k})\cdot\left[\rho_1\vec{\sigma}-\frac{B}{A}\rho_3\hat{k}\right]$$
$$+(A\vec{Z}_1-BX_3\hat{k})\cdot\left[\rho_2\vec{\sigma}+\frac{B}{A}\vec{\sigma}\times\hat{k}\right]. \tag{4.7}$$

From this expression and the expansion (4.4) for the wave function we see that the Salpeter equation for the mesons at rest decouples in a set of two by two coupled linear integral equations:

$$-A(k)L_2(\vec{k})+\frac{A(k)}{2E(k)}\frac{4}{3}\frac{1}{(an)^3}\sum_{\vec{k}'}\tilde{V}(\vec{k}-\vec{k}')L_2(\vec{k}')=\frac{M}{2i}L_1(\vec{k}), \tag{4.8a}$$

$$\frac{E^2(k)}{A(k)}L_1(\vec{k})-\frac{A(k)}{2E(k)}\frac{4}{3}\frac{1}{(an)^3}\sum_{\vec{k}'}\tilde{V}(\vec{k}-\vec{k}')\left[1-\frac{B(k)B(k')}{A(k)A(k')}(\hat{k}\cdot\hat{k}')\right]L_1(\vec{k}')=\frac{M}{2i}L_2(\vec{k}), \tag{4.8b}$$

$$-\frac{E^2(k)}{A(k)}\vec{N}_2(\vec{k})+\frac{B^2(k)}{A(k)}[\vec{N}_2(\vec{k})\cdot\hat{k}]\hat{k}-\frac{A(k)}{2E(k)}\frac{4}{3}\frac{1}{(an)^3}\sum_{\vec{k}'}\tilde{V}(\vec{k}-\vec{k}')\left[-\vec{N}_2(\vec{k}')+\frac{B(k)B(k')}{A(k)A(k')}[\vec{N}_2(\vec{k}')\times\hat{k}']\times\hat{k}\right]$$
$$=\frac{M}{2i}\vec{N}_1(\vec{k}), \tag{4.9a}$$

$$A(k)\vec{N}_1(\vec{k})+\frac{B^2(k)}{A(k)}[\vec{N}_1(\vec{k})\cdot\hat{k}]\hat{k}-\frac{A(k)}{2E(k)}\frac{4}{3}\frac{1}{(an)^3}\sum_{\vec{k}'}\tilde{V}(\vec{k}-\vec{k}')\left[\vec{N}_1(\vec{k}')+\frac{B(k)B(k')}{A(k)A(k')}[\vec{N}_1(\vec{k}')\cdot\hat{k}']\hat{k}\right]$$
$$=\frac{M}{2i}\vec{N}_2(\vec{k}). \tag{4.9b}$$

Let us now identify the quantum numbers that correspond to these functions $L_1,L_2,\vec{N}_1,\vec{N}_2$. To this aim, let us deduce, from (4.4), the transformation laws of these functions under P, C, and T. We obtain the following:

	P	C	T
$L_1(\vec{k})$	$-L_1(-\vec{k})$	$L_1(-\vec{k})$	$L_1^*(-\vec{k})$
$L_2(\vec{k})$	$-L_2(-\vec{k})$	$L_2(-\vec{k})$	$-L_2^*(-\vec{k})$
$\vec{N}_1(\vec{k})$	$-\vec{N}_1(-\vec{k})$	$-\vec{N}_1(-\vec{k})$	$-\vec{N}_1^*(-\vec{k})$
$\vec{N}_2(\vec{k})$	$-\vec{N}_2(-\vec{k})$	$-\vec{N}_2(-\vec{k})$	$\vec{N}_2^*(-\vec{k})$

We find therefore the usual formulas: $P=(-1)^{L+1}$, that takes into account the opposite parity of particle and antiparticle; $C=(-1)^{L+S}$. L_1 and L_2 correspond to $S=0$ and $\vec{N}_1,\vec{N}_2$ correspond to $S=1$. The time-reversal transformation laws mean that we can take $L_1,iL_2,\vec{N}_1$, $i\vec{N}_2$ as real, as we will do.

The first equations (4.8) coupling L_1 and L_2 correspond thus to mesons with total quark spin $S=0$ $(\pi,B,\pi',\ldots)$ and the second set (4.9) coupling $\vec{N}_1$ and $\vec{N}_2$ to mesons with $S=1$ $(\rho,\delta,A_1,A_2,\rho',\ldots)$. From the first two equations we rediscover the Goldstone-boson solution discussed in Sec. III. Indeed, we see that

$$L_1(\vec{k})=0,\quad L_2(\vec{k})=\frac{A}{E}=\sin\varphi$$

is a solution for $M=0$, that gives, from (4.4) the wave function

$$\chi(\vec{k})=\sin\varphi\rho_2$$

already discovered in Sec. III.

Let us now particularize the integral equations for the case of the harmonic oscillator, whose Fourier transform is given by (2.19). Note that $\varphi(k)$ is already determined by the gap equation, the nonlinear differential equation (2.20), as we have discussed in Ref. 15.

We will need expressions of the form

$$\nabla_{\vec{k}'}f(k')\,|_{\vec{k}'=\vec{k}}=\nabla_{\vec{k}'}f(k')(\hat{k}\cdot\hat{k}')\,|_{\vec{k}'=\vec{k}}=f'(k)\hat{k},$$
$$\hat{k}\cdot\vec{\nabla}_{\vec{k}}f(k)=f'(k),$$
$$\Delta_{\vec{k}'}f(k')\,|_{\vec{k}'=\vec{k}}=\frac{2}{k}f'(k)+f''(k), \tag{4.10}$$
$$\Delta_{\vec{k}'}f(k')(\hat{k}\cdot\hat{k}')\,|_{\vec{k}'=\vec{k}}=-\frac{2}{k^2}f(k)+\frac{2}{k}f'(k)+f''(k).$$

Let us rename the variables

$$\frac{k}{(\frac{4}{3}V_0{}^3)^{1/3}}\to k,\quad \frac{M}{(\frac{4}{3}V_0{}^3)^{1/3}}\to M.$$

Applying (4.10) to our case, the combination $2k\varphi'+k^2\varphi''$ appears often, and can be simplified using the gap equation (2.20):

$$2k\varphi'+k^2\varphi''=2k^3\sin\varphi-\sin2\varphi.$$

We also need the expression

$$\Delta\{\cot\varphi(k')[\vec{N}(\vec{k}\,')\times\hat{k}\,']\}\,|_{\vec{k}\,'=k}$$
$$=\frac{2}{\sin\varphi}\left[\frac{\cos\varphi}{\sin^2\varphi}(\varphi')^2+\frac{\cos\varphi}{k^2}-k\right](\vec{N}\times\hat{k})$$
$$+\cot\varphi\Delta(\vec{N}\times\hat{k})-\frac{2\varphi'}{\sin^2\varphi}\frac{d}{dk}(\vec{N}\times\hat{k}) \qquad (4.11)$$

and for $\Delta\{\cot\varphi(k')[\vec{N}(\vec{k}\,')\cdot\hat{k}\,']\}$ we get the same expression, replacing $\vec{N}(\vec{k})\times\hat{k}$ by $\vec{N}(\vec{k})\cdot\hat{k}$.

After some calculation we get, in the case of the harmonic oscillator,

$$\sin\varphi(k)\Delta L_2(\vec{k})-2E(k)\sin\varphi(k)L_2(\vec{k})=-iML_1(\vec{k}) , \qquad (4.12a)$$

$$-2\left[\frac{E(k)+k\cos\varphi(k)}{\sin\varphi(k)}-\frac{\cos^2\varphi(k)}{\sin^3\varphi(k)}[\varphi'(k)]^2\right]L_1(\vec{k})+\frac{1}{\sin\varphi(k)}\Delta L_1(\vec{k})-\frac{2\varphi'(k)\cos\varphi(k)}{\sin^2\varphi(k)}\frac{d}{dk}L_1(\vec{k})=iML_2(\vec{k}) , \qquad (4.12b)$$

$$\sin\varphi(k)\Delta\vec{N}_1(\vec{k})+\frac{\cos^2\varphi(k)}{\sin\varphi(k)}\{\Delta[\vec{N}_1(\vec{k})\cdot\hat{k}]\}\hat{k}-\frac{2\cos\varphi(k)\varphi'(k)}{\sin^2\varphi(k)}\left[\frac{d}{dk}[\vec{N}_1(\vec{k})\cdot\hat{k}]\right]\hat{k}-2E(k)\sin\varphi(k)\vec{N}_1(\vec{k})$$
$$+2\cot\varphi(k)\left[\frac{\cos\varphi(k)}{\sin^2\varphi(k)}[\varphi'(k)]^2+\frac{\cos\varphi(k)}{k^2}-k-E\cos\varphi(k)\right][\vec{N}_1(\vec{k})\cdot\hat{k}]\hat{k}=iM\vec{N}_2(\vec{k}) , \qquad (4.13a)$$

$$\sin\varphi(k)\Delta\vec{N}_2(\vec{k})-\frac{\cos^2\varphi(k)}{\sin\varphi(k)}\{\Delta[\vec{N}_2(\vec{k})\times\hat{k}]\}\times\hat{k}+\frac{2\cos\varphi(k)\varphi'(k)}{\sin^2\varphi(k)}\left[\frac{d}{dk}[\vec{N}_2(\vec{k})\times\hat{k}]\right]\times\hat{k}-\frac{2E(k)}{\sin\varphi(k)}\vec{N}_2(\vec{k})$$
$$+\frac{2E(k)\cos^2\varphi(k)}{\sin\varphi(k)}[\vec{N}_2(\vec{k})\cdot\hat{k}]\hat{k}-2\cot\varphi(k)\left[\frac{\cos\varphi(k)}{\sin^2\varphi(k)}(\varphi'(k))^2+\frac{\cos\varphi(k)}{k^2}-k\right][\vec{N}_2(\vec{k})\times\hat{k}]\times\hat{k}=-iM\vec{N}_1(\vec{k}) . \qquad (4.13b)$$

If we call

$$\vec{N}_T=\vec{N}\times\hat{k} ,\quad N_L=\vec{N}\cdot\hat{k} , \qquad (4.14)$$

it is easy to see that in the normalization condition (3.3), for $\vec{p}=0$,

$$\frac{1}{(an)^3}\sum_{\vec{k}}(||\chi_{+-}||^2-||\chi_{-+}||^2)=1$$

$$(\chi_{+-}=\Lambda_+\chi\Lambda_-,\ \chi_{-+}=\Lambda_-\chi\Lambda_+),\ ||\chi_{+-}||^2 , \qquad (4.15)$$

$||\chi_{-+}||^2$ are given by

$$\tfrac{1}{2}||\chi_{+-}||^2=\left|\frac{E}{A}L_1-iL_2\right|^2+\left|\vec{N}_{1T}-i\frac{E}{A}\vec{N}_{2T}\right|^2+\left|\frac{E}{A}N_{1L}-iN_{2L}\right|^2 , \qquad (4.16)$$

$$\tfrac{1}{2}||\chi_{-+}||^2=\left|\frac{E}{A}L_1+iL_2\right|^2+\left|\vec{N}_{1T}+i\frac{E}{A}\vec{N}_{2T}\right|^2+\left|\frac{E}{A}N_{1L}+iN_{2L}\right|^2 . \qquad (4.17)$$

The natural variables will therefore be

$$\frac{E}{A}L_1,\ iL_2,\ \frac{E}{A}N_{1L},\ \vec{N}_{1T},\ iN_{2L},\ i\frac{E}{A}\vec{N}_{2T} . \qquad (4.18)$$

We will expand these functions in partial waves.

V. PARTIAL-WAVE EXPANSION

The expansion into states of definite angular momentum makes no problem for the two equations for $L_1(\vec{k}),L_2(\vec{k})$ (4.12). Defining radial functions

$$-\frac{1}{\sin\varphi(k)}L_1(\vec{k})=\sum_{LM}\frac{v_1(k)}{k}Y_L^M(\hat{k}) ,\quad iL_2(\vec{k})=\sum_{LM}\frac{v_2(k)}{k}Y_L^M(\hat{k}) \qquad (5.1)$$

we obtain

$$\left[-\frac{d^2}{dk^2}+2E+(\varphi')^2+\frac{L(L+1)+2\cos^2\varphi}{k^2}\right]v_{1L}=Mv_{2L} , \qquad (5.2a)$$

$$\left[-\frac{d^2}{dk^2}+2E+\frac{L(L+1)}{k^2}\right]v_{2L}=Mv_{1L} . \qquad (5.2b)$$

These equations still simplify if we take into account the expression for $E(k)$ (2.21):

$$\left[-\frac{d^2}{dk^2}+2k\cos\varphi+\frac{L(L+1)}{k^2}\right]v_{1L}=Mv_{2L} , \qquad (5.3a)$$

$$\left[-\frac{d^2}{dk^2}+2E+\frac{L(L+1)}{k^2}\right]v_{2L}=Mv_{1L} . \qquad (5.3b)$$

These equations correspond to mesons with total quark spin $S=0$, i.e., the Goldstone pion and its radial excitations, the B meson ($J=L=1$), etc.

Let us now expand $\vec{N}_1,\vec{N}_2$ in vector spherical harmonics. We define the radial functions, dependent on J and L only:

$$\vec{N}(\vec{k})=\sum_{JLM}\frac{n_{JLM}(k)}{k}\vec{Y}_{JLM}(\hat{k}) . \quad (5.4)$$

We will need a number of formulas involving the vector spherical harmonics $\vec{Y}_{JLM}(\hat{k})$ that we give in Appendix D. From the linear independence of the $\vec{Y}_{JJM}(\hat{k})$ we obtain two coupled equations for the radial functions n_{iJJ} $(i=1,2)$. These equations greatly simplify if we adopt the new variables, analogous to (5.1) [as we can infer from (4.18)]:

$$m_{1JJ}(k)=n_{1JJ}(k) ,$$
$$m_{2JJ}(k)=-i\frac{n_{2JJ}(k)}{\sin\varphi(k)} \quad (5.5)$$

and we get

$$\left[-\frac{d^2}{dk^2}+2E+\frac{J(J+1)}{k^2}\right]m_{1JJ}=Mm_{2JJ} , \quad (5.6a)$$

$$\left[-\frac{d^2}{dk^2}+2E+(\varphi')^2+\frac{J(J+1)}{k^2}\right]m_{2JJ}=Mm_{1JJ} . \quad (5.6b)$$

These equations correspond to mesons with total quark spin $S=1$ and $L=J$, i.e., for example, the A_1 meson and its radial excitations.

From the linear independence of $\vec{Y}_{JJ\pm 1M}(\hat{k})$ we obtain four coupled differential equations involving $n_{iJJ\pm 1}$ $(i=1,2)$. These equations are rather complicated but they simplify dramatically if one adopts the variables [cf. (4.18)]:

$$n_{1+}=-\left[\left[\frac{J}{2J+1}\right]^{1/2}n_{1JJ+1}+\left[\frac{J+1}{2J+1}\right]^{1/2}n_{1JJ-1}\right] ,$$

$$n_{1-}=-\frac{1}{\sin\varphi}\left[\left[\frac{J+1}{2J+1}\right]^{1/2}n_{1JJ+1}-\left[\frac{J}{2J+1}\right]^{1/2}n_{1JJ-1}\right] , \quad (5.7)$$

$$n_{2+}=\frac{i}{\sin\varphi}\left[\left[\frac{J}{2J+1}\right]^{1/2}n_{2JJ+1}+\left[\frac{J+1}{2J+1}\right]^{1/2}n_{2JJ-1}\right], \quad (5.8)$$

$$n_{2-}=i\left[\left[\frac{J+1}{2J+1}\right]^{1/2}n_{2JJ+1}-\left[\frac{J}{2J+1}\right]^{1/2}n_{2JJ-1}\right] .$$

We obtain, in terms of these new variables,

$$\left[-\frac{d^2}{dk^2}+2E+\frac{J(J+1)}{k^2}\right]n_{1+}+\frac{2[J(J+1)]^{1/2}}{k^2}\sin\varphi n_{1-}=Mn_{2+} , \quad (5.9a)$$

$$\left[-\frac{d^2}{dk^2}+2E+(\varphi')^2+\frac{J(J+1)+2\sin^2\varphi}{k^2}\right]n_{1-}+\sin\varphi\frac{2[J(J+1)]^{1/2}}{k^2}n_{1+}=Mn_{2-} , \quad (5.9b)$$

$$\left[-\frac{d^2}{dk^2}+2E+\frac{J(J+1)+2}{k^2}\right]n_{2-}+\frac{2[J(J+1)]^{1/2}}{k^2}\sin\varphi n_{2+}=Mn_{1-} \quad (5.9c)$$

$$\left[-\frac{d^2}{dk^2}+2E+(\varphi')^2+\frac{J(J+1)}{k^2}\right]n_{2+}+\frac{2[J(J+1)]^{1/2}}{k^2}\sin\varphi n_{2-}=Mn_{1+} . \quad (5.9d)$$

These four equations correspond to mesons having total quark spin $S=1$ and orbital angular momentum $L=J\pm 1$. For instance, the ρ ($S=1$, $L=0$, $J=1$) and its radial excitations will correspond to $n_{i10}(k)$, which is coupled to $n_{i12}(k)$, i.e., the orbitally excited $L=2,1^{--}$ state. The A_2 will correspond to $n_{i21}(k)$, coupled to $n_{i23}(k)$. The σ or δ will correspond to n_{i01}: in this case we have only two equations, like for $S=0$.

Looking back at the Dirac decomposition (4.4), we can see that the linear combinations

$$v_l=v_{1L}+v_{2L},\quad m_l=m_{1JJ}+m_{2JJ} ,$$
$$n_{l+}=n_{1+}+n_{2+},\quad n_{l-}=n_{1-}+n_{2-} \quad (5.10)$$

correspond to "large" components, having a nonrelativistic limit. The small components are

$$v_s=v_{1L}-v_{2L},\quad m_s=m_{1JJ}-m_{2JJ} ,$$
$$n_{s+}=n_{1+}-n_{2+},\quad n_{s-}=n_{1-}-n_{2-} . \quad (5.11)$$

With these new variables the equations take a rather symmetric form:

(1) $S=0$:

$$\left[-\frac{d^2}{dk^2}+2E+\tfrac{1}{2}(\varphi')^2+\frac{L(L+1)+\cos^2\varphi}{k^2}\right]v_l+\left[\tfrac{1}{2}(\varphi')^2+\frac{\cos^2\varphi}{k^2}\right]v_s=Mv_l , \quad (5.12a)$$

$$\left[-\frac{d^2}{dk^2}+2E+\tfrac{1}{2}(\varphi')^2+\frac{L(L+1)+\cos^2\varphi}{k^2}\right]v_s+\left[\tfrac{1}{2}(\varphi')^2+\frac{\cos^2\varphi}{k^2}\right]v_l=-Mv_s . \quad (5.12b)$$

(2) $S=1$, $J=L$:

$$\left[-\frac{d^2}{dk^2}+2E+\tfrac{1}{2}(\varphi')^2+\frac{J(J+1)}{k^2}\right]m_l-\tfrac{1}{2}(\varphi')^2m_s=Mm_l , \quad (5.13a)$$

$$\left[-\frac{d^2}{dk^2}+2E+\tfrac{1}{2}(\varphi')^2+\frac{J(J+1)}{k^2}\right]m_s-\tfrac{1}{2}(\varphi')^2 m_l = -Mm_s \ . \quad (5.13b)$$

(3) $S=1$, $J=L\pm 1$:

$$\left[-\frac{d^2}{dk^2}+2E+\tfrac{1}{2}(\varphi')^2+\frac{J(J+1)}{k^2}\right]n_{l+}-\tfrac{1}{2}(\varphi')^2 n_{s+} + \sin\varphi\frac{2[J(J+1)]^{1/2}}{k^2}n_{l-}=Mn_{l+} \ , \quad (5.14a)$$

$$\left[-\frac{d^2}{dk^2}+2E+\tfrac{1}{2}(\varphi')^2+\frac{J(J+1)}{k^2}\right]n_{s+}-\tfrac{1}{2}(\varphi')^2 n_{l+} + \sin\varphi\frac{2[J(J+1)]^{1/2}}{k^2}n_{s-}=-Mn_{s+} \ , \quad (5.14b)$$

$$\left[-\frac{d^2}{dk^2}+2E+\tfrac{1}{2}(\varphi')^2+\frac{J(J+1)+1+\sin^2\varphi}{k^2}\right]n_{l-} + \left[\tfrac{1}{2}(\varphi')^2-\frac{\cos^2\varphi}{k^2}\right]n_{s-}+\sin\varphi\frac{2[J(J+1)]^{1/2}}{k^2}n_{l+} = Mn_{l-} \ , \quad (5.14c)$$

$$\left[-\frac{d^2}{dk^2}+2E+\tfrac{1}{2}(\varphi')^2+\frac{J(J+1)+1+\sin^2\varphi}{k^2}\right]n_{s-} + \left[\tfrac{1}{2}(\varphi')^2-\frac{\cos^2\varphi}{k^2}\right]n_{l-}+\sin\varphi\frac{2[J(J+1)]^{1/2}}{k^2}n_{s+} = -Mn_{s-} \ . \quad (5.14d)$$

VI. MESON SPECTRUM

We need to solve the three systems of linear second-order differential equations [(5.3), (5.6), and (5.9)]. The first two systems, which correspond to mesons $S=0$, $J=L$ and $S=1$, $J=L$, are of the form

$$\left[-\frac{d^2}{dk^2}+\frac{L(L+1)}{k^2}+R(k)\right]\begin{bmatrix} n_{1L} \\ n_{2L}\end{bmatrix}=0 \ , \quad (6.1)$$

where $R(k)$ is a 2×2 regular matrix as $k\to 0$. The system (5.9) is more complicated, but we can perform a new change of variables in order to diagonalize the behavior as $k\to 0$:

$$\begin{aligned} n_{1+}&=\sqrt{J}h_1-\sqrt{J+1}h_2 \ , \\ n_{1-}&=\sqrt{J+1}h_1+\sqrt{J}h_3 \ , \\ n_{2+}&=\sqrt{J}h_2-\sqrt{J+1}h_4 \ , \\ n_{2-}&=\sqrt{J+1}h_2+\sqrt{J}h_4 \ . \end{aligned} \quad (6.2)$$

In terms of these new functions h_i $(i=1,\ldots,4)$, the system writes

$$\left[-\frac{d^2}{dk^2}+\begin{bmatrix} J(J-1)\mathbb{1} & 0 \\ 0 & (J+1)(J+2)\mathbb{1}\end{bmatrix}\frac{1}{k^2}+R(k)\right]h=0 \ , \quad (6.3)$$

where $R(k)$ is now a 4×4 matrix, regular as $k\to 0$, and h represents the four-component vector.

From the form (6.1) we see that there are regular and singular solutions as $k\to 0$,

$$n_i \propto k^{L+1}, \ k^{-L} \ .$$

Our problem of finding the eigenvalues and eigenfunctions simplifies considerably because all these equations satisfy the symmetry: $k\to -k$ (radial variable). It follows that the solutions split into even and odd functions of k. The same argument applies to the system (6.3) where we have the $k\to 0$ behavior

$$h_1,h_2 \propto k^{J+2}, \ k^{-(J+1)} \ ,$$
$$h_3,h_4 \propto k^{J}, \ k^{-(J-1)} \ .$$

To solve the equations we need to fix a number of initial conditions. Let us consider the case (6.3) to expose the method. Since we have four second-order differential equations, we need eight conditions. To choose the regular behavior as $k\to 0$, $h_1,h_2\sim k^{J+2}$, $h_3,h_4\sim k^J$ fixes four conditions. We are left with four conditions, to be fixed by the behavior as $k\to\infty$. Let us consider the four-vector functions behaving as $k\to 0$ like

$$\begin{aligned} h_m^{(i)}(k) &\propto k^{J+2}\delta_{im} \quad (i=1,2), \\ h_m^{(i)}(k) &\propto k^{J}\delta_{im} \quad (i=3,4) \ . \end{aligned} \quad (6.4)$$

The index i labels the vector, and m labels its components. It is clear that any solution f_m to the four differential equations can be written as a linear combination of functions of this type:

$$f_m(k)=\sum_i C_i h_m^{(i)}(k) \ , \quad (6.5)$$

where C_i $(i=1,\ldots,4)$ are independent of k. We solve then the system by the Runge-Kutta and Numerov methods for functions of the form (6.5) and we determine C_i by asking the solution $f_m(k)\to 0$ as $k\to\infty$. This gives the condition

$$\sum_i C_i h_m^{(i)}(k\to\infty)=0 \ . \quad (6.6)$$

The four free parameters left are the three independent coefficients C_i and the mass eigenvalue M. A solution $C_i\neq 0$ to (6.6) will exist if

$$\det[h_m^{(i)}(k\to\infty)]=0 \ , \quad (6.7)$$

where the determinant is understood for the matrix $h_m^{(i)}$, where m labels the component function, and i labels the initial conditions. We look for solutions $h_m^{(i)}$ such that (6.7) is satisfied. The position of the change in sign in the determinant fixes the mass eigenvalue M. Once we have the solutions satisfying (6.7) we go back to the equation (6.6) and we solve for C_i. This gives, from (6.5), the eigenfunctions.

We give the spectrum for two values of the energy scale $(\frac{4}{3}V_0{}^3)^{1/3}$: one that fits the ρ mass

$$(\tfrac{4}{3}V_0{}^3)^{1/3}=289 \text{ MeV} \quad (6.8)$$

and another that gives a better overall fit to the masses,

$$(\tfrac{4}{3}V_0^3)^{1/3}=247 \text{ MeV} . \tag{6.9}$$

We obtain the spectra of Table I. We have specified the different J^{PC} quantum numbers. Since we consider a single quark flavor one should compare these results with the isovector mesons, $\pi,\rho,\delta,A_1,B,A_2,\ldots$ to avoid the contribution of the anomaly for the isosinglets, a phenomenon beyond the scope of this paper. There are a number of interesting qualitative features in this spectrum. First, we get $M_\pi=0$ and M_ρ at the right scale, *without appealing to any spin-spin interaction,* merely as a consequence of the spontaneous breaking of chiral symmetry. Of course, the one-gluon-exchange short-distance interaction should also contribute to the spontaneous breaking and may improve the value of $\langle\bar\psi\psi\rangle$ and the ρ mass. We see also that there is a radially excited pion at the right scale, and two or three excited 1^{--} states in the 1—2-GeV region. The low mass of the first ρ' seems to point out to the old and controversial $\rho'(1250)$ seen in some e^+e^- experiments. The second one is roughly compatible with the conventional $\rho'(1600)$. These states are strong mixtures of a radial excitation and of an orbital $L=2$ excitation.

Another striking feature is the fine structure within the $L=1$ multiplet, which we get nonperturbatively. We obtain the same ordering as the one predicted by the L-S coupling of the Breit-Fermi one-gluon interaction:[19] $M(2^{++})>M(1^{+-})>M(1^{++})>M(0^{++})$. This feature is not surprising and comes from the vector character of the confining interaction that we have adopted. This ordering favors the old classification for the A_1 at a mass of 1060 MeV, instead of the currently accepted $A_1(1270)$, heavier than the $B(1235)$. Of course, one should not take our model's detailed predictions too seriously in view of its inability to predict a correct pion velocity (see later) and its too small value for $\langle\bar\psi\psi\rangle$. It is still worth noting that the classification with A_1 heavier than B also contradicts $\vec{L}\cdot\vec{S}$ coupling of one-gluon exchange as well as the ordering found in charmonium, where a Lorentz scalar potential is assumed to correct but not reverse the L-S ordering of one-gluon exchange. Our effective L-S splitting is also too large, and should be corrected by some other interaction, as we have suggested in the Introduction.

For the sake of completeness, let us give the quantities m_q, $\Delta\epsilon$, and $\langle\bar\psi\psi\rangle$ defined in Sec. II for the two values (6.8) and (6.9). We obtain, respectively,

$$m_q=142 \text{ MeV} ,$$

$$\Delta\epsilon=-28 \text{ MeV/fm}^3, \quad \langle\bar\psi\psi\rangle=-(140 \text{ MeV})^3$$

and

$$m_q=121 \text{ MeV} ,$$

$$\Delta\epsilon=-15 \text{ MeV/fm}^3, \quad \langle\bar\psi\psi\rangle=-(120 \text{ MeV})^3 .$$

VII. INSTABILITY OF THE CHIRAL-INVARIANT VACUUM AND ASYMPTOTIC SPECTRUM

It will be interesting to go back to the Salpeter equation and try to look for the form of the spectrum in the *chiral-invariant* "vacuum" (it is a stationary state, but not a minimum). We expect two phenomena: (1) a tachyon in the 0^{-+} or 0^{++} channels (corresponding to the π or the σ; because of chiral invariance both should be degenerate), expressing the instability of the invariant vacuum, and (2) a realization of chiral symmetry through parity doublets for the different states.

Let us first see that the 0^{-+} (π) and 0^{++} (σ) states are degenerate and have imaginary masses. From Eqs. (5.12) we get, for the chiral-invariant solution of the gap equation (2.20) $\varphi=0$ and for the $J=0$ channel

TABLE I. Meson spectrum in MeV for two values of the harmonic-oscillator strength: $(\frac{4}{3}V_0^3)^{1/3}=289$ MeV, which fits the ρ mass, and 247 MeV, which gives a better overall fit.

Oscillator strength (MeV)	J^{PC}	0^{-+}	0^{++}	1^{--}	1^{++}	1^{+-}	2^{++}	2^{--}	2^{-+}	3^{--}	3^{++}	3^{+-}	4^{++}	4^{--}	4^{-+}
289		0	941	778	1183	1348	1729	1772	1859	2202	2208	2269	2577	2580	2618
		1599	1858	1434	1988	2095	1877	2433	2502	2263	2805	2851	2618	3137	3172
		2341	2546	1740	2644	2727	2367	3019	3079	2791			3134		
		2981	3143	2185			2531			2860			3175		
				2433			2941								
				2817			3114								
				3033											
247		0	805	664	1012	1153	1479	1516	1590	1884	1888	1933	2205	2207	2240
		1368	1588	1227	1701	1793	1602	2081	2141	1936	2400	2440	2240	2684	2714
		2002	2178	1489	2262	2333	2025	2583	2635	2388			2681		
		2533	2689	1869			2165			2447			2718		
				2081			2518								
				2410			2664								
				2585											

$$\left[-\frac{d^2}{dk^2}+2E+\frac{1}{k^2}\right]v_l+\frac{1}{k^2}v_s=Mv_l\ ,$$
$$\left[-\frac{d^2}{dk^2}+2E+\frac{1}{k^2}\right]v_s+\frac{1}{k^2}v_l=-Mv_s\ . \tag{7.1}$$

Note that, since now $\varphi=0$, the fermion energy contains the infrared-singular self-energy

$$E(k)=k-\frac{1}{k^2} \tag{7.2}$$

in the units that we have adopted. Remember that this infrared singularity disappears for the broken solution, since $\cos\varphi(k)/k^2$ behaves like a constant for $k\rightarrow0$. Equations (7.1) correspond to the meson with $J^{PC}=0^{-+}$ quantum numbers. Let us now look for the equation for the $J^{PC}=0^{++}$ meson, corresponding to the σ. From Eqs. (5.14) we obtain for $J=0$ and $\varphi=0$ two coupled equations for n_{l+} and n_{s+} that are meaningless [since we see in (5.7) and (5.8) that only n_{iJJ-1} survive], and two coupled equations corresponding to the 0^{++} meson $J=0,L=1$:

$$\left[-\frac{d^2}{dk^2}+2E+\frac{1}{k^2}\right]n_{l-}-\frac{1}{k^2}n_{s-}=Mn_{l-}\ ,$$
$$\left[-\frac{d^2}{dk^2}+2E+\frac{1}{k^2}\right]n_{s-}-\frac{1}{k^2}n_{l-}=-Mn_{s-}\ . \tag{7.3}$$

With the change of variables $n_{l-}\rightarrow n_{l-}$, $n_{s-}\rightarrow -n_{s-}$, we see that these equations are identical to (7.1): the 0^{-+} (π) and 0^{++} (σ) mesons are degenerate.

Let us now see that $M^2<0$, i.e., that there are tachyons, indicating the instability of the chiral-invariant vacuum. Let us consider the form (5.2) of the pion equation ($L=0$):

$$\begin{bmatrix}0 & A\\ B & 0\end{bmatrix}\begin{bmatrix}w_1\\ w_2\end{bmatrix}=M\begin{bmatrix}w_1\\ w_2\end{bmatrix} \tag{7.4}$$

with

$$A=-\Delta_{\vec{k}}+2k-\frac{2}{k^2}\ ,$$
$$B=-\Delta_{\vec{k}}+2k\ , \tag{7.5}$$

and $w_1=v_1/k$, $w_2=v_2/k$.

The operator B is strictly positive; we can therefore define its inverse and we can write

$$w_1=(-\Delta_{\vec{k}}+2k)^{-1}Mw_2\ . \tag{7.6}$$

We have therefore

$$\left[-\Delta_{\vec{k}}+2k-\frac{2}{k^2}\right]w_1=(-\Delta_{\vec{k}}+2k)^{-1}M^2w_1 \tag{7.7}$$

or

$$(-\Delta_{\vec{k}}+2k)^{1/2}\left[-\Delta_{\vec{k}}+2k-\frac{2}{k^2}\right]\times(-\Delta_{\vec{k}}+2k)^{1/2}f=M^2f \tag{7.8}$$

with

$$f=(-\Delta_{\vec{k}}+2k)^{-1/2}w_1\ .$$

We will have a negative eigenvalue $M^2<0$ of this eigenvalue equation if we find a test function

$$w=(-\Delta_{\vec{k}}+2k)^{1/2}f$$

such that

$$\left\langle f\left|\left[-\Delta_{\vec{k}}+2k-\frac{2}{k^2}\right]\right|f\right\rangle<0\ . \tag{7.9}$$

We proved precisely in Ref. 15, when studying the instability of the chiral-invariant vacuum, that the quadratic form (7.9) can be made negative by conveniently choosing the test function. This can be seen immediately from the operator inequality

$$-\Delta_{\vec{k}}\geq\frac{1}{4k^2} \tag{7.10}$$

that means that we can approach as much as we want the right-hand side (RHS) by conveniently choosing the test function f. We are then led to the quadratic form

$$\left\langle f\left|\left[2k-\frac{3}{4k^2}\right]\right|f\right\rangle \tag{7.11}$$

that can be made negative as the two terms scale differently.[15] Therefore $M^2<0$, i.e., we have a chiral-degenerate set of tachyon states.

Let us now see that the rest of the spectrum appears in parity doublets. Consider the first equations (5.12) for $J=L,S=0$, for $\varphi=0$:

$$\left[-\frac{d^2}{dk^2}+2E+\frac{J(J+1)}{k^2}\right]v_l+\frac{1}{k^2}v_s=Mv_l\ ,$$
$$\left[-\frac{d^2}{dk^2}+2E+\frac{J(J+1)}{k^2}\right]v_s+\frac{1}{k^2}v_l=-Mv_s\ . \tag{7.12}$$

These states correspond to $P=(-1)^{J+1}$, $C=(-1)^J$ [0^{-+} (π), 1^{+-} (B), . . .].

The second set of equations (5.13) ($J=L,S=1$) becomes, for $\varphi=0$:

$$\left[-\frac{d^2}{dk^2}+2E+\frac{J(J+1)}{k^2}\right]m_l=Mm_l\ ,$$
$$\left[-\frac{d^2}{dk^2}+2E+\frac{J(J+1)}{k^2}\right]m_s=-Mm_s\ . \tag{7.13}$$

Their solutions correspond to the states $P=(-1)^{J+1}$, $C=(-1)^{J+1}$ ($A_1,\ldots$) .

Finally, the set (5.14) ($J=L\pm1$, $S=1$) becomes, for $\varphi=0$:

$$\left[-\frac{d^2}{dk^2}+2E+\frac{J(J+1)}{k^2}\right]n_{l+}=Mn_{l+}\ ,$$
$$\left[-\frac{d^2}{dk^2}+2E+\frac{J(J+1)}{k^2}\right]n_{s+}=-Mn_{s+}\ , \tag{7.14}$$

$$\left[-\frac{d^2}{dk^2}+2E+\frac{J(J+1)}{k^2}\right]n_{l-}-\frac{1}{k^2}n_{s-}=Mn_{l-}\,,$$

$$\left[-\frac{d^2}{dk^2}+2E+\frac{J(J+1)}{k^2}\right]n_{s-}-\frac{1}{k^2}n_{l-}=-Mn_{s-}\,. \tag{7.15}$$

The solutions of these equations correspond to states with $P=(-1)^J$, $C=(-1)^J$ $(\rho,A_2,\sigma,\ldots)$. Performing the change of variables $n_{l-}\to n_{l-}$, $n_{s-}\to -n_{s-}$, we see that the spectrum is degenerate in parity doublets: the two sets (7.12), (7.15) and (7.13), (7.14) correspond to opposite-parity states. The states $P=(-1)^{J+1}$, $C=(-1)^J$ [Eqs. (7.12)] are degenerate with the ones with $P=(-1)^J$, $C=(-1)^J$ [Eqs. (7.15)], and the states with $P=(-1)^{J+1}$, $C=(-1)^{J+1}$ [Eqs. (7.13)] are degenerate with the states with $P=(-1)^J$, $C=(-1)^J$, solutions of the system (7.14).

When we solve the Salpeter equation in the true vacuum we expect also to recover a near degeneracy between parity doublets for high masses, large compared to the scale of the spontaneous chiral-symmetry breaking. We find indeed that the splitting between parity doublets decreases as we go to high masses. We have not reached yet the asymptotic chiral-degenerate limit, as we see from the table, except for large angular momentum states.

VIII. AXIAL-VECTOR-CURRENT MATRIX ELEMENTS. PION VELOCITY

We have shown the existence of a Goldstone boson and we have found also the low-lying meson spectrum. We want here to go further and compute the matrix elements of the axial-vector current involving the pion. To do that we need to compute the BS wave function of the pion for center-of-mass momentum $\vec{p}\neq 0$ since the $\langle 0\,|\,j_5^\mu\,|\,\pi\rangle$ matrix elements depend on the momentum, being proportional to $f_\pi p_\mu$. The model is not covariant since we have adopted an instantaneous interaction. However, we will see, by making an expansion in powers of $\vec{p}$, that the model satisfies the expected properties of a theory with chiral invariance dynamically broken. We will see also in this section that we recover the expected dispersion law of the Goldstone boson

$$\omega(p)\underset{p\to 0}{\longrightarrow} cp\,, \tag{8.1}$$

where c is some constant (different from one in general as the model is not Lorentz covariant). We will moreover compute f_π and investigate how the conservation of the axial-vector current holds.

A. Wave function for the pion at nonvanishing momentum

We want to solve the Salpeter equation for the Goldstone boson at $\vec{p}\neq 0$. This equation (3.1) writes

$$H\left[\vec{k}+\frac{\vec{p}}{2}\right]\chi_{\vec{p}}(\vec{k})-\chi_{\vec{p}}(\vec{k})H\left[\vec{k}-\frac{\vec{p}}{2}\right]-\frac{4}{3}\frac{1}{(an)^3}\sum_{\vec{k}}\tilde{V}(\vec{k}-\vec{k}\,')\left[\chi_{\vec{p}}(\vec{k}\,')\Lambda_-\left[\vec{k}-\frac{\vec{p}}{2}\right]-\Lambda_-\left[\vec{k}+\frac{\vec{p}}{2}\right]\chi_{\vec{p}}(\vec{k}\,')\right]=\omega(p)\chi_{\vec{p}}(\vec{k}) \tag{8.2}$$

with $\chi_{\vec{p}}(\vec{k})$ satisfying the constraint

$$\Lambda_\pm\left[\vec{k}+\frac{\vec{p}}{2}\right]\chi_{\vec{p}}(\vec{k})=\chi_{\vec{p}}(\vec{k})\Lambda_\mp\left[\vec{k}-\frac{\vec{p}}{2}\right]. \tag{8.3}$$

We will make an expansion of the relevant quantities in powers of $\vec{p}$:

$$\begin{aligned}&\Lambda_\pm\left[\vec{k}+\frac{\vec{p}}{2}\right]=\Lambda_\pm^{(0)}(\vec{k})+\Lambda_\pm^{(1)}(\vec{k})+\cdots\,,\\ &H\left[\vec{k}+\frac{\vec{p}}{2}\right]=H^{(0)}(\vec{k})+H^{(1)}(\vec{k})+\cdots\,,\\ &\chi_{\vec{p}}(\vec{k})=\chi^{(0)}(\vec{k})+\chi^{(1)}(\vec{k})+\cdots\,,\\ &\omega(p)=\omega^{(1)}+\omega^{(2)}+\cdots\,,\end{aligned} \tag{8.4}$$

where the upper index means the order in $\vec{p}$. For the case of the Goldstone boson that we are considering, $\omega(p)$ begins at order one in p since the mass vanishes.

At zero order in $\vec{p}$ we obtain the equations that we have considered before:

$$[H^{(0)}(\vec{k}),\chi^{(0)}(\vec{k})]-\frac{4}{3}\frac{1}{(an)^3}\sum_{\vec{k}'}\tilde{V}(\vec{k}-\vec{k}\,')[\chi^{(0)}(\vec{k}\,'),\Lambda_-^{(0)}(\vec{k})]=0\,, \tag{8.5}$$

$$\Lambda_\pm^{(0)}(\vec{k})\chi^{(0)}(\vec{k})\Lambda_\pm^{(0)}(\vec{k})=0\,. \tag{8.6}$$

The solution is, as we have seen, proportional to

$$\chi^{(0)}(\vec{k})=\sin\varphi(k)i\gamma_5\gamma_0\,. \tag{8.7}$$

At first order in $\vec{p}$ we get

$$[H^{(0)}(\vec{k}),\chi^{(1)}(\vec{k})]-\frac{4}{3}\frac{1}{(an)^3}\sum_{\vec{k}'}\tilde{V}(\vec{k}-\vec{k}\,')[\chi^{(1)}(\vec{k}\,'),\Lambda_-^{(0)}(\vec{k})]+\{H^{(1)}(\vec{k}),\chi^{(0)}(\vec{k})\}+\frac{4}{3}\frac{1}{(an)^3}\sum_{\vec{k}'}\tilde{V}(\vec{k}-\vec{k}\,')\{\chi^{(0)}(\vec{k}\,'),\Lambda_-^{(1)}(\vec{k})\}=\omega^{(1)}\chi^{(0)}(\vec{k}) \tag{8.8}$$

with the condition

$$[\Lambda_\pm^{(1)}(\vec{k}),\chi^{(0)}(\vec{k})]=\chi^{(1)}(\vec{k})\Lambda_\mp^{(0)}(\vec{k})-\Lambda_\pm^{(0)}(\vec{k})\chi^{(1)}(\vec{k}) . \tag{8.9}$$

In (8.8), { , } means anticommutator. This equation simplifies considerably, since $\chi^{(0)}\sim\rho_2$ anticommutes with $\rho_3=\beta$ and $\rho_1\vec{\sigma}=\vec{\alpha}$, and $\Lambda_\pm^{(1)}$ and $H^{(1)}$ can only contain these matrices. We get, therefore,

$$[H^{(0)}(\vec{k}),\chi^{(1)}(\vec{k})] -\frac{4}{3}\frac{1}{(an)^3}\sum_{\vec{k}'}\tilde{V}(\vec{k}-\vec{k}')[\chi^{(1)}(\vec{k}'),\Lambda_-^{(0)}(\vec{k})] =\omega^{(1)}\chi^{(0)}(\vec{k}) \tag{8.10}$$

and the constraint can be written, since $\Lambda_\pm^{(0)}=\frac{1}{2}(1\pm H/E)$, as

$$\{H^{(0)}(\vec{k}),\chi^{(1)}(\vec{k})\}=-2E[\Lambda_+^{(1)}(\vec{k}),\chi^{(0)}(\vec{k})] . \tag{8.11}$$

Let us first find a solution for the constraint (8.9). The solution will be equal to the general solution of the homogeneous equation

$$\{H^{(0)}(\vec{k}),\chi^{(1)}(\vec{k})\}=0 \tag{8.12}$$

plus a particular solution of the inhomogeneous equation (8.11). It is easy to see that the homogeneous equation can be written

$$\Lambda_\pm^{(0)}(\vec{k})\chi^{(1)}(\vec{k})\Lambda_\pm^{(0)}(\vec{k})=0 \tag{8.13}$$

and its general solution is, as we have seen in Sec. IV,

$$\chi^{(1)}(\vec{k})=L_1\left[\rho_1-\frac{B}{A}\rho_3\vec{\sigma}\cdot\hat{k}\right]+L_2\rho_2 +\vec{N}_1\left[\rho_1\vec{\sigma}-\frac{B}{A}\rho_3\hat{k}\right]+\vec{N}_2\left[\rho_2\vec{\sigma}+\frac{B}{A}\vec{\sigma}\times\hat{k}\right] . \tag{8.14}$$

Let us now look for a particular solution of the inhomogeneous equation. Let us first compute $\Lambda_+^{(1)}(\vec{k})$, the first order in $\vec{p}$ in the expansion of $\Lambda_+(\vec{k}+\vec{p}/2)$:

$$\Lambda_+^{(1)}(\vec{k})=\frac{1}{4}\left\{(\hat{k}\cdot\vec{p})\left[\frac{A}{E}\right]'\rho_3+\left[(\hat{k}\cdot\vec{p})\left[\frac{B}{E}\right]'\hat{k} -\frac{1}{k}\left[\frac{B}{E}\right](\vec{p}\times\hat{k})\times\hat{k}\right]\rho_1\vec{\sigma}\right\} . \tag{8.15}$$

We get, therefore, for the commutator in the RHS of (8.11)

$$[\Lambda_+^{(1)},\chi^{(0)}]=\frac{1}{2}\frac{A}{E}\left\{(\hat{k}\cdot\vec{p})\left[\frac{A}{E}\right]'\rho_1 -\left[(\hat{k}\cdot\vec{p})\left[\frac{B}{E}\right]'\hat{k} -\frac{1}{k}\frac{B}{E}(\vec{p}\times\hat{k})\times\hat{k}\right]\rho_3\vec{\sigma}\right\} . \tag{8.16}$$

Let us try the particular form $\chi^{(1)}(\vec{k})=\vec{\sigma}\cdot\vec{u}(\vec{k})$. We obtain

$$\frac{1}{2E}\{H^{(0)}(\vec{k}),\chi^{(1)}(\vec{k})\}=\left[\frac{A}{E}\right]\vec{u}\cdot\rho_3\vec{\sigma}+\left[\frac{B}{E}\right](\vec{u}\cdot\hat{k})\rho_1 . \tag{8.17}$$

It is easy to verify that $\chi^{(1)}=\vec{u}\cdot\vec{\sigma}$ satisfies (8.11) with

$$\vec{u}=\frac{1}{2}\left[(\hat{k}\cdot\vec{p})\left[\frac{B}{E}\right]'\hat{k}-\frac{1}{k}\left[\frac{B}{E}\right](\vec{p}\times\hat{k})\times\hat{k}\right] . \tag{8.18}$$

The general solution to the constraint will then be

$$\chi^{(1)}(\vec{k})=L_1\left[\rho_1-\frac{B}{A}\rho_3\vec{\sigma}\cdot\hat{k}\right]+L_2\rho_2 +\vec{N}_1\cdot\left[\rho_1\vec{\sigma}-\frac{B}{A}\rho_3\hat{k}\right] +\vec{N}_2\cdot\left[\rho_2\vec{\sigma}+\frac{B}{A}(\vec{\sigma}\times\hat{k})\right]+\vec{\sigma}\cdot\vec{u} \tag{8.19}$$

with $\vec{u}$ given by (8.18).

Let us now impose the Salpeter equation (8.10) to this expression. We need to compute the commutators $[\chi^{(1)}(\vec{k}'),\Lambda_-^{(0)}(\vec{k})]$ and $[H^{(0)}(\vec{k}),\chi^{(1)}(\vec{k})]$ and take into account the linear independence of the Dirac matrices, along the same lines as in Sec. IV. We obtain $L_2=\vec{N}_1=0$. Furthermore, since $\vec{\sigma}$ appears in the second term of the bracket multiplying $\vec{N}_2$, it will be coupled to $\vec{u}$. L_1 and $\vec{N}_2$ satisfy the equations

$$\frac{2E^2(k)}{A(k)}L_1(\vec{k})-\frac{4}{3}\frac{1}{(an)^3}\sum_{\vec{k}'}\tilde{V}(\vec{k}-\vec{k}')\frac{A(k)A(k')+B(k)B(k')(\hat{k}\cdot\hat{k}')}{E(k)E(k')}\frac{E(k')}{A(k')}L_1(\vec{k}')=\frac{A(k)}{E(k)}\omega^{(1)} , \tag{8.20}$$

$$-\frac{2}{A(k)}\{A^2(k)\vec{N}_2(\vec{k})-B^2(k)[\vec{N}_2(\vec{k})\times\hat{k}]\times\hat{k}\} +\frac{4}{3}\frac{1}{(an)^3}\sum_{\vec{k}'}\tilde{V}(\vec{k}-\vec{k}')\frac{E(k')}{A(k')}\left[\frac{A(k)A(k')}{E(k)E(k')}\vec{N}_2(\vec{k}')-\frac{B(k)B(k')}{E(k)E(k')}[\vec{N}_2(\vec{k}')\times\hat{k}']\times\hat{k}\right] =\left[2B(k)\vec{u}(\vec{k})-\frac{B(k)}{E(k)}\frac{4}{3}\frac{1}{(an)^3}\sum_{\vec{k}'}\tilde{V}(\vec{k}-\vec{k}')\vec{u}(\vec{k}')\right]\times\hat{k} . \tag{8.21}$$

It is convenient to parametrize L_1 and $\vec{N}_2$ in the following form:

$$L_1=\frac{A}{E}\frac{f_1}{k}\omega^{(1)},\quad \vec{N}_2=\frac{A}{E}\frac{(f_2-\frac{1}{2})}{k}(\vec{p}\times\hat{k})\ . \tag{8.22}$$

Then, in the case of the harmonic oscillator, both equations [(8.20) and (8.21)] are satisfied provided the functions f_1 and f_2 are solutions of the differential equations

$$\left[-\frac{d^2}{dk^2}+2k\cos\varphi\right]f_1=k\sin\varphi\ , \tag{8.23}$$

$$\left[-\frac{d^2}{dk^2}+2k\cos\varphi+\frac{2\sin^2\varphi}{k^2}\right]f_2=\frac{\sin^2\varphi}{k^2}\ . \tag{8.24}$$

In terms of these functions, the pion wave function up to first order in $\vec{p}$ then writes

$$\chi_{\vec{p}}(\vec{k})=\frac{A}{E}\rho_2+\omega^{(1)}\frac{A}{E}\frac{f_1}{k}\left[\rho_1-\frac{B}{A}\rho_3\vec{\sigma}\cdot\hat{k}\right]+\frac{A}{E}\frac{1}{k}(f_2-\tfrac{1}{2})(\vec{p}\times\hat{k})\cdot\rho_2\vec{\sigma}+\left[\frac{1}{2}\left[\frac{B}{E}\right]'(\hat{k}\cdot\vec{p})\hat{k}-\frac{B}{E}\frac{f_2}{k}(\vec{p}\times\hat{k})\times\hat{k}\right]\cdot\vec{\sigma}\ . \tag{8.25}$$

In Sec. VIII D we will compute $\omega^{(1)}$, for the moment undetermined. As we have seen in Sec. III, the zeroth-order pion wave function (8.7) is not normalizable. The perturbed wave function can, on the contrary, be normalized. Using the expression (8.25) in (3.3) we obtain, since Λ_- enters only to order $\Lambda_-^{(0)}$ ($\chi^{(0)}$ anticommutes with $\Lambda_-^{(1)}$):

$$\mathrm{Tr}[\chi_{\vec{p}},\Lambda_-^{(0)}]\chi_{\vec{p}}^{\dagger}=24\,\mathrm{Im}\left[\frac{E}{A}L_1\right]^*L_2\ ,$$

where the factor three comes from color. We get therefore,

$$\|\chi_\pi\|^2\equiv\int\frac{d\vec{k}}{(2\pi)^3}\mathrm{Tr}[\chi_{\vec{p}}(\vec{k}),\Lambda_-^{(0)}(\vec{k})]\chi_{\vec{p}}^{\dagger}(\vec{k})$$
$$=24\omega^{(1)}\int\frac{d\vec{k}}{(2\pi)^3}\frac{f_1(k)}{k}\sin\varphi(k)\ . \tag{8.26}$$

B. Pion decay constant

Once we have the pion wave function for $\vec{p}\neq 0$ we can compute the axial-vector-current matrix element

$$\langle 0|j_5{}^0|\pi\rangle=\sqrt{2}f_\pi\omega\ , \tag{8.27}$$

where the normalization of the pion state $\sqrt{2\omega}\chi_\pi/\|\chi_\pi\|$ is given by (3.11). We obtain, since $j_5^0=\psi^\dagger\gamma_5\psi=\psi^\dagger\rho_1\psi$,

$$\langle 0|j_5{}^0|\pi\rangle=\frac{3}{\|\chi_\pi\|}\int\frac{d\vec{k}}{(2\pi)^3}\mathrm{Tr}[\rho_1\chi_\pi(\vec{k})]$$
$$=\frac{12\omega^{(1)}\sqrt{2\omega}}{\|\chi_\pi\|}\int\frac{d\vec{k}}{(2\pi)^3}\frac{f_1(k)}{k}\sin\varphi(k)\ , \tag{8.28}$$

where the factor three comes from color. We get therefore from (8.26)

$$f_\pi=\left[6\int\frac{d\vec{k}}{(2\pi)^3}\frac{f_1(k)}{k}\sin\varphi(k)\right]^{1/2}\ . \tag{8.29}$$

$f_1(k)$ satisfies the differential equation (8.23) in the case of the harmonic oscillator or, more generally, from (8.20) and (8.22), $f_1(k)/k$ satisfies the integral equation

$$2E(k)\frac{f_1(k)}{k}=\sin\varphi(k)+\frac{4}{3}\frac{1}{(an)^3}\sum_{\vec{k}'}\tilde{V}(\vec{k}-\vec{k}')[\sin\varphi(k)\sin\varphi(k')+\cos\varphi(k)\cos\varphi(k')(\hat{k}\cdot\hat{k}')]\frac{f(k')}{k'}\ . \tag{8.30}$$

$f_1(k)/k$ is just the function $g(k)$ of Adler and Davis,[16] who follow the analysis of Govaerts, Mandula, and Weyers;[21] we agree with them for the expression of f_π. We get, however, a larger numerical value than Adler and Davis, perhaps because they use the linear instead of the harmonic-oscillator potential. We obtain $f_\pi=20$ MeV for $(\frac{4}{3}V_0{}^3)^{1/3}=247$ MeV, a factor two bigger than Adler and Davis. We discuss our result in the conclusion.

C. Dynamical chiral-symmetry breaking with a current mass

Up to now we have assumed chiral symmetry to be exact. To obtain in our model some familiar formulas of current algebra, let us introduce a small current mass, i.e., a term $m\bar{\psi}\psi$ in the Hamiltonian, and see how the formalism is modified. We have to make the replacement $\vec{\alpha}\cdot\vec{k}\to\vec{\alpha}\cdot\vec{k}+\beta m$ everywhere. For example, the gap equation is now

$$H(\vec{k})=\vec{\alpha}\cdot\vec{k}+\beta m$$
$$+\frac{4}{3}\frac{1}{(an)^3}\frac{1}{2}\sum_{\vec{k}'}\tilde{V}(\vec{k}-\vec{k}')[1-2\Lambda_-(\vec{k}')]\ , \tag{8.31}$$

$$[\Lambda_-(\vec{k}),H(\vec{k})]=0\ , \tag{8.32}$$

and we get now the coupled integral equations

$$A(k)=m+\frac{1}{2}\frac{4}{3}\frac{1}{(an)^3}\sum_{\vec{k}'}\tilde{V}(\vec{k}-\vec{k}')\sin\varphi(k') ,$$
$$(8.33)$$
$$B(k)=k+\frac{1}{2}\frac{4}{3}\frac{1}{(an)^3}\sum_{\vec{k}'}\tilde{V}(\vec{k}-\vec{k}')\cos\varphi(k')(\hat{k}\cdot\hat{k}') ,$$

with the same conditions (2.16). The single integral equation for $\varphi(k)$ now becomes

$$\frac{1}{2}\frac{4}{3}\frac{1}{(an)^3}\sum_{\vec{k}'}\tilde{V}(\vec{k}-\vec{k}')[\sin\varphi(k')\cos\varphi(k) - \cos\varphi(k')\sin\varphi(k)(\hat{k}\cdot\hat{k}')] = k\sin\varphi(k)-m\cos\varphi(k) . \quad (8.34)$$

In the harmonic-oscillator case, the gap equation now becomes

$$\tfrac{4}{3}V_0^3(k^2\varphi')'=2k^3\sin\varphi-2mk^2\cos\varphi-\tfrac{4}{3}V_0^3\sin 2\varphi . \quad (8.35)$$

It is not difficult to see that the Salpeter equation for the pion [Eqs. (5.3) for $L=0$] remains of the same form

$$\left[-\frac{d^2}{dk^2}+2k\cos\varphi\right]v_1=Mv_2 ,$$
$$(8.36)$$
$$\left[-\frac{d^2}{dk^2}+2E\right]v_2=Mv_1 ,$$

but now φ is the solution of (8.35) and $E(k)$ is now the fermion energy in the massive case. At order $O(m^0)$, the solution to the Salpeter equation is $M=0$, $v_2^{(0)}=k\sin\varphi(k)$, as we know. It is useful to call $Mv_1=F_1$, $v_2=F_2$. We get the equations

$$\left[-\frac{d^2}{dk^2}+2k\cos\varphi\right]F_1=F_2 ,$$
$$(8.37)$$
$$\left[-\frac{d^2}{dk^2}+2E\right]F_2=M^2F_1 .$$

We can write for F_2 the expansion

$$F_2=F_2^{(0)}+O(m)=k\sin\varphi+O(m) . \quad (8.38)$$

Therefore, the function F_1 will have an expansion in m,

$$F_1=F_1^{(0)}+O(m)=f_1+O(m) , \quad (8.39)$$

f_1 being solution of Eq. (8.23).

Let us multiply the second equation (8.37) by $k\sin\varphi$ and integrate

$$\int_0^\infty dk\,k\sin\varphi\left[-\frac{d^2}{dk^2}+2E\right]F_2=M^2\int_0^\infty dk\,k\sin\varphi F_1 .$$

Integrating by parts and making use of the gap equation (8.35) under the form (in our usual units)

$$2Ek\sin\varphi=2mk+\frac{d^2}{dk^2}(k\sin\varphi)$$

we get

$$2m\int_0^\infty dk\,kF_2=M^2\int_0^\infty dk\,k\sin\varphi F_1 . \quad (8.40)$$

Using now the expansions (8.38) and (8.39) we obtain, keeping the lowest order,

$$2m\int_0^\infty dk\,k^2\sin\varphi=M^2\int_0^\infty dk\,k\sin\varphi f_1+O(m^2) . \quad (8.41)$$

Comparing to our expression for f_π (8.29) and $\langle\bar{\psi}\psi\rangle$ (2.23) we obtain the well-known expression

$$-2m\langle\bar{\psi}\psi\rangle=M_\pi^2f_\pi^2+O(m^2) . \quad (8.42)$$

Here f_π corresponds to the empirical value $f_\pi=95$ MeV, consistent with our definition.

D. Conservation of the axial-vector current; pion velocity

The coupling f_π that enters in the preceding relations corresponds to the definition in terms of the time component of the axial-vector current j_5^0. Our model is not covariant; we do not expect therefore this value of f_π to be equal to the one defined in terms of the spatial components of the current, $\vec{j}_5$. We will see nevertheless that the axial-vector current is conserved in the chiral limit $m\to 0$. We will prove indeed that $p^\mu\langle 0|j_5^\mu|\pi\rangle=0$. To see this, let us consider the Salpeter equation for any state $\chi_{\vec{p}}(\vec{k})$ in motion, Eq. (3.1) and the gap equation under the form

$$-\frac{4}{3}\frac{1}{(an)^3}\sum_{\vec{k}'}\tilde{V}(\vec{k}-\vec{k}')\Lambda_-\left[\vec{k}-\frac{\vec{p}}{2}\right] = H\left[\vec{k}'-\frac{\vec{p}}{2}\right]-\vec{\alpha}\cdot\left[\vec{k}'-\frac{\vec{p}}{2}\right]-\beta m -\frac{1}{2}\times\frac{4}{3}\frac{1}{(an)^3}\sum_{\vec{k}}\tilde{V}(\vec{k}-\vec{k}') . \quad (8.43)$$

Using this relation and integrating the Salpeter equation over $\vec{k}$, we obtain

$$\sum_{\vec{k}}\left[[\vec{\alpha}\cdot\vec{k}+\beta m,\chi_{\vec{p}}(\vec{k})]+\left\{\vec{\alpha}\cdot\frac{\vec{p}}{2},\chi_{\vec{p}}(\vec{k})\right\}\right] = \sum_{\vec{k}}\omega(p)\chi_{\vec{p}}(\vec{k}) . \quad (8.44)$$

Multiplying on the left by γ_5 and taking the trace we get, *for any state* $\chi_{\vec{p}}(\vec{k})$,

$$\vec{p}\cdot\mathrm{Tr}\left[\vec{\sigma}\sum_{\vec{k}}\chi_{\vec{p}}(\vec{k})\right]+2m\,\mathrm{Tr}\left[\gamma_5\beta\sum_{\vec{k}}\chi_{\vec{p}}(\vec{k})\right] =\omega(p)\mathrm{Tr}\left[\gamma_5\sum_{\vec{k}}\chi_{\vec{p}}(\vec{k})\right] . \quad (8.45)$$

Defining the pion decay constants for the time $f_\pi^{(t)}$ and spatial $f_\pi^{(s)}$ components,

$$\sum_{\vec{k}} \mathrm{Tr}[\vec{\sigma}\chi_{\vec{p}}(\vec{k})] \sim \langle 0 | \vec{j}_5 | \pi \rangle = \sqrt{2} f_\pi^{(s)} \vec{p} ,$$
$$\sum_{\vec{k}} \mathrm{Tr}[\gamma_5 \chi_{\vec{p}}(\vec{k})] \sim \langle 0 | j_5{}^0 | \pi \rangle = \sqrt{2} f_\pi^{(t)} \omega(p) . \qquad (8.46)$$

In (8.45), $\mathrm{Tr}[\gamma_5 \beta \chi_{\vec{p}}(\vec{k})]$ will get a nonzero contribution only from the pion wave function at $\vec{p}=0$, proportional to $\langle \bar{\psi}\psi \rangle$ up to a factor $(||\chi_\pi||/\sqrt{2\omega})^{-1}$, which is itself proportional to $f_\pi^{(t)}$, as we have seen in Sec. VIII B. We obtain therefore

$$\vec{p}\,^2 f_\pi^{(s)} - \frac{2m\langle \bar{\psi}\psi \rangle}{f_\pi^{(t)}} = \omega^2(p) f_\pi^{(t)} \qquad (8.47)$$

which reduces to (8.42) for $\vec{p}=0$. When we make $m=0$, this expression reflects the conservation of the axial-vector current.

We see also from this relation that, for $m \to 0$, we get the typical dispersion law for a Goldstone boson:

$$\omega(p) \underset{p\to 0}{\longrightarrow} cp , \qquad (8.48)$$

where $c = (f_\pi^{(s)}/f_\pi^{(t)})^{1/2}$. To compute the pion velocity c we need $f_\pi^{(s)}$ besides $f_\pi^{(t)}$. Using the pion wave function in motion (8.25) we find

$$\sum_{\vec{k}} \mathrm{Tr}[\vec{\sigma}\chi_{\vec{p}}(\vec{k})] = \left\{ \frac{2}{\pi^2} \int_0^\infty k^2 dk \left[\frac{1}{2} \left[\frac{B}{E} \right]' + 2 \left[\frac{B}{E} \right] \frac{f_2}{k} \right] \right\} \vec{p} . \qquad (8.49)$$

In the case of the harmonic oscillator, f_2 is the solution of the differential equation (8.24). We obtain for c,

$$c = \left[\frac{f_\pi^{(s)}}{f_\pi^{(t)}} \right]^{1/2} = \left\{ \frac{\int_0^\infty k^2 dk \left[(\cos\varphi)' + 4\cos\varphi \frac{f_2}{k} \right]}{6\int_0^\infty k^2 dk \frac{f_1}{k} \sin\varphi} \right\}^{1/2} . \qquad (8.50)$$

Solving Eqs. (8.23) and (8.24) (imposing convergence as $k \to \infty$), we obtain

$$c \cong 3.1 .$$

We do not attach a particular significance to the fact that the pion velocity is larger than 1 because the model is not covariant. This gives a possible hint on the problem that we have found concerning the value of $f_\pi^{(t)}$. It is possible that the noncovariance of the model could be at the origin of this small value ($f_\pi^{(s)}$ is nine times bigger), although the addition of short-distance attraction by one-gluon exchange could also enhance $f_\pi^{(t)}$.

Let us conclude with a comment on the decay coupling of excited pions. Since relation (8.47) is completely general, let us see what happens for a radially excited pion π' with mass $M_{\pi'} \neq 0$ in the limit $m=0$. We have, in this limit,

$$\vec{p}\,^2 f_{\pi'}^{(s)} = \omega^2(p) f_{\pi'}^{(t)} . \qquad (8.51)$$

It is clear that if π' is at rest, $\vec{p}=0$, we have

$$M_{\pi'}{}^2 f_{\pi'}^{(t)} = 0 ,$$

i.e., $f_{\pi'}^{(t)}=0$. From this result, taking $\vec{p}\neq 0$, it also follows that $f_{\pi'}^{(s)}=0$. We see that the matrix elements $\langle 0 | j_\mu^5 | \pi' \rangle = 0$. This is consistent with the view that the corrections to PCAC (partial conservation of axial-vector current) due to poles of radial excitations of the pion can at most be of the order of the explicit chiral-symmetry breaking, as has been emphasized.[23]

IX. CONCLUSION

We have solved a quark model with chiral symmetry in the Nambu-Goldstone mode. Although the harmonic oscillator can only roughly approximate the long-distance part of the q-$\bar{q}$ interaction, and we have neglected the short-distance piece, the spectrum that we obtain is interesting as a rough and *qualitative* description of light mesons in a case of dynamical breaking of chiral symmetry. We get a spectrum that is very different from the naive SU(6) limit, as well as from the pattern of chiral doublets around the invariant vacuum. *Without appealing to any spin-spin interaction,* we find, besides a Goldstone pion $M_\pi=0$, a ρ in the right mass range. We obtain also a radially excited π' at the correct scale and two vector excitations (mixing of orbital $L=2$ and radial) in the 1–2-GeV region. The ρ is somewhat light in the best overall fit, $M_\rho \cong 664$ MeV, but we must keep in mind that this large "hyperfine" splitting $M_\rho - M_\pi = M_\rho$ has been obtained by the mechanism of dynamical breaking of chirality alone, without appealing to short-distance spin-spin interactions. Taking into account those and explicit chiral-symmetry breaking should improve the π and ρ masses. Our light ρ', $M_{\rho'} \cong 1227$ MeV, seems to support the old controversial 1^{--} state in this region. The other ρ' should be associated with the well established state $\rho'(1600)$. These states are a strong mixture of radial and orbital $L=2$ excitations. Moreover, we obtain a radially excited π' at the right mass, close to the observed $\pi'(1300)$. Notice that the position of the first ρ', and the ordering $M_{\rho'} < M_{\pi'}$ could be unstable ; it will be sensitive to spin-spin perturbations.

We obtain a large effective L-S splitting for the $L=1$ mesons, with the ordering

$$M(2^{++}) > M(1^{+-}) > M(1^{++}) > M(0^{++}) ,$$

familiar from the one-gluon-exchange perturbative contribution. This ordering is a general feature of our scheme, we expect it as we adopt a confining *Lorentz-vector* potential. Our model cannot pretend to give a detailed description of reality due to noncovariance of its potential and the absence of short-distance interaction. However, since the A_1 classification has known a long period of uncertainty, we feel useful to notice that our predicted ordering, if taken seriously, would favor the old assignment for the A_1 at 1060 MeV, and not the new one at 1270 MeV, slightly heavier than the $B(1235)$, in contrast with the charmonium $L=1$ states, where the ordering is that of the $\vec{L}\cdot\vec{S}$ coupling of one-gluon exchange, somewhat corrected (but not reversed) by the opposite-sign $\vec{L}\cdot\vec{S}$ coupling due to an *ad hoc* Lorentz-scalar potential (we do not consider it since it would violate chiral invariance). Our fine splitting is also too large and we would also need a new *Lorentz-scalar* contribution to the long-distance interaction that could give a fine structure of the opposite sign. It is indeed a serious problem to keep to chiral invariance and avoid too large L-S splittings. However, an effective Lorentz-scalar piece (not present in the original Hamiltonian for massless quarks since it would break explicitly chiral invariance) could come from high-order QCD graphs in the *broken* vacuum, where chirality is no longer a symmetry. Another possible origin could be the diamagnetic interaction between quarks due to the gluon condensate.[24]

Although the model is not covariant, all the relations of current algebra are satisfied as long as chirality is concerned. We get $f_{\pi'}=0$ for any radially excited pion in the chiral limit, as expected. However, f_π turns out to be too small for the value of the scale that fits qualitatively the spectrum and $\langle\bar\psi\psi\rangle$. This drawback could come from the noncovariance. Also, as emphasized by Adler and Davis, the short-distance gluon interaction could improve f_π, as this is a quantity testing the short-distance part of the pion wave function.

Finally, let us emphasize that the result $f_{\pi'}=0$, where π' is any radially excited pion, shows, at the level of the wave functions, a situation radically different from the nonrelativistic quark model concerning the wave functions at the origin $\psi(0)$. It is a very dramatic constraint from dynamical breaking of chiral symmetry to get $f_{\pi'}=0$ for any radially excited state. The detailed study of the wave functions that accomplish this fact is now in progress.

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APPENDIX A: FERMION PROPAGATOR

Let us write down the fermion propagator in the broken theory:

$$\begin{aligned} D_{\alpha_1\alpha_2}(\vec{x}_1-\vec{x}_2,t_1-t_2) &= \langle T(\psi_{\alpha_1}(\vec{x}_1,t_1)\psi^\dagger_{\alpha_2}(\vec{x}_2,t_2))\rangle \\ &= \frac{1}{n^3}\sum_{\vec{k}} e^{i\vec{k}\cdot(\vec{x}_1-\vec{x}_2)} D_{\alpha_1\alpha_2}(\vec{k},t_1-t_2) \end{aligned} \quad (A1)$$

with

$$D_{\alpha_1\alpha_2}(\vec{k},t_1-t_2)=\int_{-\infty}^{+\infty}\frac{d\omega}{2\pi}e^{-i\omega(t_1-t_2)}\tilde{D}_{\alpha_1\alpha_2}(\vec{k},\omega)\,. \quad (A2)$$

The temporal evolution of $\psi(\vec{x},t)$ will be determined by the equation

$$\frac{d}{dr}\psi(\vec{x},t)=i[:H_2:,\psi(\vec{x},t)]\,, \quad (A3)$$

where $:H_2:$ is the bilinear piece of the Hamiltonian in (2.3), normal ordered relative to the new broken vacuum. From this equation we obtain for the temporal evolution of the fermion field:

$$i\frac{d}{dt}\tilde\psi(\vec{k},t)=H(\vec{k})\tilde\psi(\vec{k},t)\,, \quad (A4)$$

$$H(\vec{k})=\vec\alpha\cdot\vec{k}+\frac{4}{3}\times\frac{1}{2}\frac{1}{(an)^3}\sum_{\vec{k}'}\tilde V(\vec{k}-\vec{k}')[1-2\Lambda_-(\vec{k}')]$$

with the condition

$$[H(\vec{k}),\Lambda_-(\vec{k})]=0 \quad (A5)$$

and $[\Lambda_-(\vec{k})]^2=\Lambda_-(\vec{k})$. The Fourier-transformed field in (A4) is defined by

$$\psi(\vec{k},t)=\frac{1}{n^{3/2}}\sum_{\vec{x}}\psi(\vec{x},t)e^{-i\vec{k}\cdot\vec{x}}\,. \quad (A6)$$

From these equations one obtains then the fermion propagator

$$D_{\alpha_1\alpha_2}(\vec{k},t_1-t_2)=\langle T(\tilde\psi_{\alpha_1}(\vec{k},t_1)\tilde\psi_{\alpha_2}(\vec{k},t_2))\rangle = \begin{cases}[e^{-i(t_1-t_2)H(\vec{k})}\Lambda_+(\vec{k})]_{\alpha_1\alpha_2} & (t_1>t_2)\,,\\ -[e^{-i(t_1-t_2)H(\vec{k})}\Lambda_-(\vec{k})]_{\alpha_1\alpha_2} & (t_1<t_2)\,,\end{cases} \quad (A7)$$

i.e., finally,

$$\tilde D_{\alpha_1\alpha_2}(\vec{k},\omega)=i\left[\frac{1}{\omega-H(\vec{k})+i\epsilon}\Lambda_+(\vec{k})+\frac{1}{\omega-H(\vec{k})-i\epsilon}\Lambda_-(\vec{k})\right]_{\alpha_1\alpha_2}, \quad (A8)$$

or, in a more compact form,

$$\tilde D_{\alpha_1\alpha_2}(\vec{k},\omega)=i\left[\frac{1}{\omega-H(\vec{k})+i\epsilon\Lambda(\vec{k})}\right]_{\alpha_1\alpha_2}, \quad (A9)$$

with $\Lambda=\Lambda_+-\Lambda_-$. All these quantities refer to the broken theory, where $\Lambda_\pm(\vec{k})$ and $H(\vec{k})$ are given by (2.9) and (2.14) and $\varphi(k)$, $A(k)$, and $B(k)$ satisfy the gap equation (2.15) and (2.16).

APPENDIX B: BETHE-SALPETER EQUATION

1. Inhomogeneous BS equation

We are looking for fermion-antifermion bound states in the new broken vacuum, but before writing the homogeneous BS equation for the bound states, it will be useful to consider the more general inhomogeneous BS equation for the irreducible four-particle Green's function

$$G_{\alpha_1\alpha_2\alpha_1'\alpha_2'}(\vec{k}_1,\vec{k}_2,\vec{k}'_1,\vec{k}'_2;\omega_1,\omega_2,\omega_1',\omega_2')=\int dt_1dt_2dt_1'dt_2'\exp(i\omega_1t_1-i\omega_2t_2-i\omega_1't_1'+i\omega_2't_2')$$
$$\times[\langle T(\psi_{\alpha_1}(\vec{k}_1,t_1)\psi^\dagger_{\alpha_2}(\vec{k}_2,t_2)\psi^\dagger_{\alpha_1'}(\vec{k}'_1,t_1')\psi_{\alpha_2'}(\vec{k}'_2,t_2'))\rangle$$
$$-\langle T(\psi_{\alpha_1}(\vec{k}_1,t_1)\psi^\dagger_{\alpha_2}(\vec{k}_2,t_2))\rangle\langle T(\psi^\dagger_{\alpha_1'}(\vec{k}'_1,t_1')\psi_{\alpha_2'}(\vec{k}'_2,t_2'))\rangle] . \quad \text{(B1)}$$

To write the BS equation we can omit s-channel exchange because our interaction is color octet and we are restricting to color singlet mesons. We will use the following Feynman rules. For the fermion propagator

$$2\pi\delta(\omega-\omega')\delta_{\vec{k}\vec{k}'}[\tilde{D}(\vec{k},\omega)]_{\alpha\alpha'} , \quad \text{(B2a)}$$

where $\tilde{D}(\vec{k},\omega)$ is given by (A9). Vector exchange:

$$-i2\pi\delta(\omega_1+\omega_2-\omega_1'-\omega_2')\delta_{\vec{k}_1+\vec{k}_2,\vec{k}'_1+\vec{k}'_2}\tfrac{4}{3}\tilde{V}(\vec{k}_1-\vec{k}_2)\sum_a\left[\frac{\lambda^a}{2}\right]_{ii'}\left[\frac{\lambda^a}{2}\right]_{jj'} . \quad \text{(B2b)}$$

$\vec{k}_1$ and $-\vec{k}_2$ (ω_1 and $-\omega_2$) refer, respectively, to quark and antiquark momenta (energies). Defining new variables $\vec{p},\vec{k},E,\omega$:

$$\vec{p}=\vec{k}_1-\vec{k}_2,\quad \omega=\omega_1-\omega_2,\quad \vec{k}=\frac{\vec{k}_1+\vec{k}_2}{2},\quad E=\frac{\omega_1+\omega_2}{2} . \quad \text{(B3)}$$

$\vec{p}$ and ω will be the center-of-mass momentum and energy, and $\vec{k},E$ the relative variables. The Green's function can be written in terms of a reduced function

$$G_{\alpha_1\alpha_2\alpha_1'\alpha_2'}(\vec{k}_1,\vec{k}_2,\vec{k}'_1,\vec{k}'_2;\omega_1,\omega_2,\omega_1',\omega_2')=2\pi\delta(\omega-\omega')\delta_{\vec{p},\vec{p}'}G_{\alpha_1\alpha_2\alpha_1'\alpha_2'}(\omega,\vec{p};E,\vec{k},E',\vec{k}') \quad \text{(B4)}$$

and the BS equation then reads

$$G_{\alpha_1\alpha_2\alpha_1'\alpha_2'}(\omega,\vec{p};E,\vec{k},E',\vec{k}')=2\pi\delta(E-E')\delta_{\vec{k},\vec{k}'}[\tilde{D}(\omega_1,\vec{k}_1)]_{\alpha_1\alpha_1'}[\tilde{D}(\omega_2,\vec{k}_2)]_{\alpha_2\alpha_2'}$$
$$+\tfrac{4}{3}i\int\frac{dE''}{2\pi}\frac{1}{(an)^3}\sum_{\vec{k}''}\sum_{\alpha_1''\alpha_2''}\{[\tilde{D}(\omega_1,\vec{k}_1)]_{\alpha_1\alpha_1''}[\tilde{D}(\omega_2,\vec{k}_2)]_{\alpha_2''\alpha_2}\tilde{V}(\vec{k}-\vec{k}'')$$
$$\times G_{\alpha_1''\alpha_2''\alpha_1'\alpha_2'}(\omega,\vec{p};E'',\vec{k}'',E',\vec{k}')\} . \quad \text{(B5)}$$

2. Homogeneous BS equation for the bound states

Let us rewrite the reduced Green's function (B4) in terms of time-ordered vacuum expectation values. Calling $t_1-t_2=t$, $t_1'-t_2'=t'$, $T=(t_1+t_2)/2$ we obtain

$$\delta_{\vec{p},\vec{p}'}G_{\alpha_1\alpha_2\alpha_1'\alpha_2'}(\omega,\vec{p};E,\vec{k},E',\vec{k}')=\int dt\,dt'dT e^{i\omega T}e^{i(Et-E't')}\left[\left\langle T\left[\psi_{\alpha_1}\left[T+\frac{t}{2},\vec{k}_1\right]\psi^\dagger_{\alpha_2}\left[T-\frac{t}{2},\vec{k}_2\right]\right.\right.\right.$$
$$\left.\left.\times\psi^\dagger_{\alpha_1'}\left[\frac{t'}{2},\vec{k}'_1\right]\psi_{\alpha_2'}\left[-\frac{t'}{2},\vec{k}'_2\right]\right]\right\rangle$$
$$\left.-\text{ vacuum contribution}\right] . \quad \text{(B6)}$$

The singularities in E are related to the large-T behavior. We have for $T\gg0$

$$\left\langle T\left[\psi_1\left[T+\frac{t}{2}\right]\psi_2^\dagger\left[T-\frac{t}{2}\right]\psi_{1'}^\dagger\left[\frac{t'}{2}\right]\psi_{2'}\left[-\frac{t'}{2}\right]\right]\right\rangle$$
$$=\sum_n\left\langle 0\left|T\left[\psi_1\left[T+\frac{t}{2}\right]\psi_2^\dagger\left[T-\frac{t}{2}\right]\right]\right|n\right\rangle\left\langle n\left|T\left[\psi_{1'}^\dagger\left[\frac{t'}{2}\right]\psi_{2'}\left[-\frac{t'}{2}\right]\right]\right|0\right\rangle \quad \text{(B7)}$$

and for $T\ll0$

$$\left\langle T\left[\psi_1\left[T+\frac{t}{2}\right]\psi_2^\dagger\left[T-\frac{t}{2}\right]\psi_{1'}^\dagger\left[\frac{t'}{2}\right]\psi_{2'}\left[-\frac{t'}{2}\right]\right]\right\rangle$$
$$=\sum_n\left\langle 0\left|T\left[\psi_{2'}\left[-\frac{t'}{2}\right]\psi_{1'}^\dagger\left[\frac{t'}{2}\right]\right]\right|n\right\rangle\left\langle n\left|T\left[\psi_2^\dagger\left[T-\frac{t}{2}\right]\psi_1\left[T+\frac{t}{2}\right]\right]\right|0\right\rangle \quad \text{(B8)}$$

and we have, on the other hand,

$$\left\langle 0\left|T\left[\psi_1\left[T+\frac{t}{2}\right]\psi_2^\dagger\left[T-\frac{t}{2}\right]\right]\right|n\right\rangle=e^{-i\omega_n T}\left\langle 0\left|T\left[\psi_1\left[\frac{t}{2}\right]\psi_2^\dagger\left[-\frac{t}{2}\right]\right]\right|n\right\rangle$$
$$\left\langle n\left|T\left[\psi_2^\dagger\left[T-\frac{t}{2}\right]\psi_1\left[T+\frac{t}{2}\right]\right]\right|0\right\rangle=e^{i\omega_n T}\left\langle n\left|T\left[\psi_2^\dagger\left[-\frac{t}{2}\right]\psi_1\left[\frac{t}{2}\right]\right]\right|0\right\rangle . \quad \text{(B9)}$$

If we now define the bound-state amplitudes by

$$\delta_{\vec{p},\vec{p}_n}\chi^{(n)}_{\alpha_1\alpha_2}(E,\vec{k})=\int dt\, e^{iEt}\left\langle 0\left|T\left[\psi_{\alpha_1}\left[\frac{t}{2},\vec{k}_1\right]\psi_{\alpha_2}^\dagger\left[-\frac{t}{2},\vec{k}_2\right]\right]\right|n\right\rangle , \quad \text{(B10)}$$

$$\delta_{\vec{p},\vec{p}_n}\widetilde{\chi}^{(n)}_{\alpha_1\alpha_2}(E,\vec{k})=\int dt\, e^{-iEt}\left\langle n\left|T\left[\psi_{\alpha_1}^\dagger\left[\frac{t}{2},\vec{k}_1\right]\psi_{\alpha_2}\left[-\frac{t}{2},\vec{k}_2\right]\right]\right|0\right\rangle , \quad \text{(B11)}$$

we obtain, integrating over T,

$$G_{\alpha_1\alpha_2\alpha_1'\alpha_2'}(\omega,\vec{p};E,\vec{k},E',\vec{k}')=i\sum_n\left[\delta_{\vec{p},\vec{p}_n}\frac{\chi^{(n)}_{\alpha_1\alpha_2}(E,\vec{k})\widetilde{\chi}^{(n)}_{\alpha_1'\alpha_2'}(E',\vec{k}')}{\omega-\omega_n+i\epsilon}-\delta_{\vec{p},-\vec{p}_n}\frac{\widetilde{\chi}^{(n)}_{\alpha_1\alpha_2}(E,\vec{k})\chi^{(n)}_{\alpha_2'\alpha_1'}(E',\vec{k}')}{\omega+\omega_n-i\epsilon}\right] . \quad \text{(B12)}$$

The equation satisfied by the four-fermion Green's function $G(\omega)$ (B6) can be written symbolically as

$$G(\omega)=G^{(0)}(\omega)+G^{(0)}(\omega)K(\omega)G(\omega) . \quad \text{(B13)}$$

We have assumed that $G(\omega)$ has a pole at

$$G(\omega)=\frac{\chi^{(n)}\widetilde{\chi}^{(n)}}{\omega-\omega_n}+(\text{regular terms at }\omega=\omega_n) . \quad \text{(B14)}$$

The bound-state amplitude satisfies therefore the homogeneous BS equation

$$\chi^{(n)}=G^{(0)}(\omega)K(\omega_n)\chi^{(n)} \quad \text{(B15)}$$

or, more precisely,

$$\chi^{(n)}(E,\vec{k})=(-i)\int\frac{d\Omega}{2\pi}\frac{1}{(an)^3}\frac{4}{3}\sum_{\vec{k}}\widetilde{V}(\vec{k}-\vec{k}')\widetilde{D}\left[\vec{k}+\frac{\vec{p}_n}{2},E+\frac{\Omega}{2}\right]\chi^{(n)}(\Omega,\vec{k}')\widetilde{D}\left[\vec{k}-\frac{\vec{p}_n}{2},E-\frac{\Omega}{2}\right] . \quad \text{(B16)}$$

This equation determines also the possible values of ω_n. We see also that for any solution ω_n, p_n, $\chi^{(n)}_{\alpha_1\alpha_2}(E,\vec{k})$ there is another solution $\widetilde{\chi}^{(n)}_{\alpha_1\alpha_2}(E,\vec{k})$ for $-\omega_n$, $-\vec{p}_n$, as we see from (B12).

Let us now look for the relation between χ and $\widetilde{\chi}$. Consider the simplifying notations

$$h(E)=\chi^{(n)}_{\alpha_1\alpha_2}(E,\vec{k})\delta_{\vec{p},\vec{p}_n} ,$$
$$\widetilde{h}(E)=\widetilde{\chi}^{(n)}_{\alpha_1\alpha_2}(E,\vec{k})\delta_{\vec{p},\vec{p}_n} ,$$
$$f(t)=\left\langle 0\left|\psi_{\alpha_1}\left[\frac{t}{2},\vec{k}_1\right]\psi_{\alpha_2}^\dagger\left[-\frac{t}{2},\vec{k}_2\right]\right|n\right\rangle ,$$
$$g(t)=\left\langle 0\left|\psi_{\alpha_2}^\dagger\left[-\frac{t}{2},\vec{k}_2\right]\psi_{\alpha_1}\left[\frac{t}{2},\vec{k}_1\right]\right|n\right\rangle . \quad \text{(B17)}$$

It is then easy to see that Eqs. (B10) and (B11) can be written as

$$h(\omega)=\int_0^\infty dt\, e^{i\omega t}f(t)+\int_{-\infty}^0 dt\, e^{i\omega t}g(t) ,$$
$$\widetilde{h}(\omega)=-\int_{-\infty}^0 dt\, e^{-i\omega t}f^*(t)-\int_0^\infty dt\, e^{-i\omega t}g^*(t) , \quad \text{(B18)}$$

which can be written in terms of

$$\widetilde{f}(\omega)=\int dr\, f(t)e^{i\omega t},\quad \widetilde{g}(\omega)=\int dt\, g(t)e^{i\omega t} \quad \text{(B19)}$$

in the form

$$h(\omega)=i\int\frac{d\omega'}{2\pi}\left[\frac{f(\omega')}{\omega-\omega'+i\epsilon}-\frac{g(\omega')}{\omega-\omega'-i\epsilon}\right] ,$$
$$\widetilde{h}(\omega)=-i\int\frac{d\omega'}{2\pi}\left[\frac{f^*(\omega')}{\omega-\omega'+i\epsilon}-\frac{g^*(\omega')}{\omega-\omega'-i\epsilon}\right] . \quad \text{(B20)}$$

We see then that $\tilde{\chi}^{(n)}_{\alpha_1\alpha_2}(E,\vec{k})$ is equal to $\chi^{(n)*}_{\alpha_1\alpha_2}(E,\vec{k})$ *up to the* $i\epsilon$ *prescription* $(\epsilon\to-\epsilon)$:

$$\tilde{\chi}^{(n)}_{\alpha_1\alpha_2}(E,\vec{k})=\chi^{(n)*}_{\alpha_1\alpha_2}(E,\vec{k})\,|_{\epsilon\to-\epsilon}\,. \tag{B21}$$

3. Salpeter equation

Since the potential is instantaneous, we can integrate over the relative energies in our equations. Let us call

$$G_{\alpha_1\alpha_2\alpha_1'\alpha_2'}(\omega,\vec{p};\vec{k},\vec{k}\,')=\int\frac{dE}{2\pi}\frac{dE'}{2\pi}G_{\alpha_1\alpha_2\alpha_1'\alpha_2'}(\omega,\vec{p};E,\vec{k},E',\vec{k}\,')\,,$$

$$\chi^{(n)}_{\alpha_1\alpha_2}(\vec{k})=\int\frac{dE}{2\pi}\chi^{(n)}_{\alpha_1\alpha_2}(E,\vec{k})\,, \tag{B22}$$

$$\tilde{\chi}^{(n)}_{\alpha_1\alpha_2}(\vec{k})=\int\frac{dE}{2\pi}\tilde{\chi}^{(n)}_{\alpha_1\alpha_2}(E,\vec{k})\,.$$

We obtain, from (B5), for the inhomogeneous equations,

$$G_{\alpha_1\alpha_2\alpha_1'\alpha_2'}(\omega,\vec{p};\vec{k},\vec{k}\,')=-i\delta_{\vec{k},\vec{k}\,'}\left\{\frac{[\Lambda+(\vec{k}_1)]_{\alpha_1\alpha_1'}[\Lambda_-(\vec{k}_2)]_{\alpha_2'\alpha_2}}{\omega-E(\vec{k}_1)-E(\vec{k}_2)+i\epsilon}-\frac{[\Lambda_-(\vec{k}_1)]_{\alpha_1\alpha_1'}[\Lambda_+(\vec{k}_2)]_{\alpha_2'\alpha_2}}{\omega+E(\vec{k}_1)+E(\vec{k}_2)-i\epsilon}\right\}$$
$$+i\frac{4}{3}\frac{1}{(an)^3}\sum_{\vec{k}\,''}\sum_{\alpha_1''\alpha_2''}i\tilde{V}(\vec{k}-\vec{k}\,'')\left\{\frac{[\Lambda_+(\vec{k}_1)]_{\alpha_1\alpha_1''}[\Lambda_-(\vec{k}_2)]_{\alpha_2''\alpha_2}}{\omega-E(\vec{k}_1)-E(\vec{k}_2)+i\epsilon}-\frac{[\Lambda_-(\vec{k}_1)]_{\alpha_1\alpha_1''}[\Lambda_+(\vec{k}_2)]_{\alpha_2''\alpha_2}}{\omega+E(\vec{k}_1)+E(\vec{k}_2)-i\epsilon}\right\}$$
$$\times G_{\alpha_1''\alpha_2''\alpha_1'\alpha_2'}(\omega,\vec{p};\vec{k},\vec{k}\,') \tag{B23}$$

and from (B16), for the homogeneous equation,

$$\chi^{(n)}_{\alpha_1\alpha_2}(\vec{k})=i\frac{4}{3}\frac{1}{(an)^3}\sum_{\vec{k}\,'}i\tilde{V}(\vec{k}-\vec{k}\,')\left\{\frac{\left[\Lambda_+\left[\vec{k}+\frac{\vec{p}_n}{2}\right]\chi^{(n)}(\vec{k}\,')\Lambda_-\left[\vec{k}-\frac{\vec{p}_n}{2}\right]\right]_{\alpha_1\alpha_2}}{\omega-E\left[\vec{k}+\frac{\vec{p}_n}{2}\right]-E\left[\vec{k}-\frac{\vec{p}_n}{2}\right]+i\epsilon}-\frac{\left[\Lambda_-\left[\vec{k}+\frac{\vec{p}_n}{2}\right]\chi^{(n)}(\vec{k}\,')\Lambda_+\left[\vec{k}-\frac{\vec{p}_n}{2}\right]\right]_{\alpha_1\alpha_2}}{\omega+E\left[\vec{k}+\frac{\vec{p}_n}{2}\right]+E\left[\vec{k}-\frac{\vec{p}_n}{2}\right]-i\epsilon}\right\}. \tag{B24}$$

The product of the Dirac matrices $\Lambda_+,\Lambda_-,\chi^{(n)}$ is understood in (B24).

Let us call $(\sigma_1,\sigma_2=\pm)$

$$\Lambda^{\sigma_1\sigma_2}_{\alpha_1\alpha_2\alpha_1'\alpha_2'}=\left[\Lambda_{\sigma_1}\left[\vec{k}+\frac{\vec{p}}{2}\right]\right]_{\alpha_1\alpha_1'}\left[\Lambda_{\sigma_2}\left[\vec{k}-\frac{\vec{p}}{2}\right]\right]_{\alpha_2'\alpha_2}. \tag{B25}$$

From the preceding equations we see that

$$\Lambda^{++}G=\Lambda^{--}G=0,\quad \Lambda^{++}\chi=\Lambda^{--}\chi=0\,. \tag{B26}$$

Therefore, all the physics happens in the null space of Λ^{++} and Λ^{--}. We can write finally from this last observation the homogeneous equation in the form

$$\left[E\left[\vec{k}+\frac{\vec{p}_n}{2}\right]+E\left[\vec{k}-\frac{\vec{p}_n}{2}\right]\right]\chi^{(n)}(\vec{k})-\frac{4}{3}\frac{1}{(an)^3}\sum_{\vec{k}\,'}\tilde{V}(\vec{k}-\vec{k}\,')\left[\Lambda_+\left[\vec{k}+\frac{\vec{p}_n}{2}\right]\chi^{(n)}(\vec{k}\,')\Lambda_-\left[\vec{k}-\frac{\vec{p}_n}{2}\right]+\Lambda_-\left[\vec{k}+\frac{\vec{p}_n}{2}\right]\chi^{(n)}(\vec{k}\,')\Lambda_+\left[\vec{k}-\frac{\vec{p}_n}{2}\right]\right]$$
$$=\omega(p_n)\left[\Lambda_+\left[\vec{k}+\frac{\vec{p}_n}{2}\right]\chi^{(n)}(\vec{k})\Lambda_-\left[\vec{k}-\frac{\vec{p}_n}{2}\right]-\Lambda_-\left[\vec{k}+\frac{\vec{p}_n}{2}\right]\chi^{(n)}(\vec{k})\Lambda_+\left[\vec{k}-\frac{\vec{p}_n}{2}\right]\right] \tag{B27}$$

with the condition, following from (B26),

$$\Lambda_+\left[\vec{k}+\frac{\vec{p}_n}{2}\right]\chi^{(n)}(\vec{k})\Lambda_+\left[\vec{k}-\frac{\vec{p}_n}{2}\right]=\Lambda_-\left[\vec{k}+\frac{\vec{p}_n}{2}\right]\chi^{(n)}(\vec{k})\Lambda_-\left[\vec{k}-\frac{\vec{p}_n}{2}\right]=0\ . \tag{B28}$$

This equation is the Salpeter equation, which we can also write in a simpler form, using the one-fermion Hamiltonian $H(\vec{k})$:

$$H\left[\vec{k}+\frac{\vec{p}_n}{2}\right]\chi^{(n)}(\vec{k})-\chi^{(n)}(\vec{k})H\left[\vec{k}-\frac{\vec{p}_n}{2}\right]-\frac{4}{3}\frac{1}{(an)^3}\sum_{\vec{k}'}\tilde{V}(\vec{k}-\vec{k}')\left[\chi^{(n)}(\vec{k}')\Lambda_-\left[\vec{k}-\frac{\vec{p}_n}{2}\right]-\Lambda_-\left[\vec{k}+\frac{\vec{p}_n}{2}\right]\chi^{(n)}(\vec{k}')\right]=\omega(p_n)\chi^{(n)}(\vec{k})\ . \tag{B29}$$

APPENDIX C: NORMALIZATION OF THE BS WAVE FUNCTION

The equation satisfied by the bound-state amplitude, being homogeneous, cannot determine the normalization. This normalization must be fixed by the inhomogeneous equation. The method is the following, as sketched by Llewellyn Smith.[17] Equation (B13) can be written in the form

$$\{[G^{(0)}(\omega)]^{-1}-K(\omega)\}G(\omega)=\mathbb{1} \tag{C1}$$

and we therefore get

$$G(\omega)[G^{(0)}(\omega)^{-1}-K(\omega)]G(\omega)=G(\omega)\ . \tag{C2}$$

Using now (B14) we make equal the residues at the poles in both sides of this equation, and we have

$$\tilde{\chi}^{(n)}\frac{d}{d\omega}[G^{(0)}(\omega)^{-1}-K(\omega)]\,|_{\omega=\omega_n}\chi^{(n)}=1\ . \tag{C3}$$

We must now apply this equation to $G^{(0)}(\ )$ in equation (B23). In the space (B26),

$$G^{(0)}_{\alpha_1\alpha_2\alpha_1'\alpha_2'}(\omega,\vec{p};\vec{k})=i\left\{\frac{\left[\Lambda_+\left[\vec{k}+\frac{\vec{p}}{2}\right]\right]_{\alpha_1\alpha_1'}\left[\Lambda_-\left[\vec{k}-\frac{\vec{p}}{2}\right]\right]_{\alpha_2'\alpha_2}}{\omega-E\left[\vec{k}+\frac{\vec{p}}{2}\right]-E\left[\vec{k}-\frac{\vec{p}}{2}\right]+i\epsilon}-\frac{\left[\Lambda_-\left[\vec{k}+\frac{\vec{p}}{2}\right]\right]_{\alpha_1\alpha_1'}\left[\Lambda_+\left[\vec{k}-\frac{\vec{p}}{2}\right]\right]_{\alpha_2'\alpha_2}}{\omega+E\left[\vec{k}+\frac{\vec{p}}{2}\right]+E\left[\vec{k}-\frac{\vec{p}}{2}\right]+i\epsilon}\right\} \tag{C4}$$

admits an inverse

$$G^{(0)-1}_{\alpha_1\alpha_1'\alpha_2\alpha_2'}(\omega,\vec{p};\vec{k})=-i\left\{\left[\omega-E\left[\vec{k}+\frac{\vec{p}}{2}\right]-E\left[\vec{k}-\frac{\vec{p}}{2}\right]\right]\Lambda^{+-}_{\alpha_1\alpha_2\alpha_1'\alpha_2'}(\omega,\vec{p};\vec{k})-\left[\omega+E\left[\vec{k}+\frac{\vec{p}}{2}\right]+E\left[\vec{k}-\frac{\vec{p}}{2}\right]\right]\Lambda^{-+}_{\alpha_1\alpha_2\alpha_1'\alpha_2'}(\omega,\vec{p};\vec{k})\right\}\ . \tag{C5}$$

One has, in fact,

$$\sum_{\alpha_1''\alpha_2''}G^{(0)-1}_{\alpha_1\alpha_2\alpha_1''\alpha_2''}(\omega,\vec{p};\vec{k})G^{(0)}_{\alpha_1''\alpha_2''\alpha_1'\alpha_2'}(\omega,\vec{p};\vec{k})=\Lambda^{+-}_{\alpha_1\alpha_2\alpha_1'\alpha_2'}(\omega,\vec{p};\vec{k})+\Lambda^{-+}_{\alpha_1\alpha_2\alpha_1'\alpha_2'}(\omega,\vec{p};\vec{k})\ . \tag{C6}$$

We need to compute

$$\frac{d}{d\omega}G^{(0)-1}(\omega,\vec{p};\vec{k})=-i[\Lambda^{+-}(\omega,\vec{p};\vec{k})-\Lambda^{-+}(\omega,\vec{p};\vec{k})],\quad \frac{d}{d\omega}K=0\ . \tag{C7}$$

We obtain, therefore, from (C3) and (C7), the condition of normalization

$$\frac{1}{(an)^3}\sum_{\vec{k}}\sum_{\substack{\alpha_1\alpha_2\\ \alpha_1'\alpha_2'}}\chi^{(n)*}_{\alpha_1\alpha_2}(\vec{k})\left\{\left[\Lambda_+\left[\vec{k}+\frac{\vec{p}_n}{2}\right]\right]_{\alpha_1\alpha_1'}\left[\Lambda_-\left[\vec{k}-\frac{\vec{p}_n}{2}\right]\right]_{\alpha_2'\alpha_2}-\left[\Lambda_-\left[\vec{k}+\frac{\vec{p}_n}{2}\right]\right]_{\alpha_1\alpha_1'}\left[\Lambda_+\left[\vec{k}-\frac{\vec{p}_n}{2}\right]\right]_{\alpha_2'\alpha_2}\right\}\chi^{(n)}_{\alpha_1'\alpha_2'}(\vec{k})=1 \tag{C8}$$

which can also be written

$$\frac{1}{(an)^3}\sum_{\vec{k}}\mathrm{Tr}\left[\Lambda_+\left[\vec{k}+\frac{\vec{p}_n}{2}\right]\chi^{(n)}(\vec{k})\Lambda_-\left[\vec{k}-\frac{\vec{p}_n}{2}\right]\chi^{(n)\dagger}(\vec{k})-\Lambda_-\left[\vec{k}+\frac{\vec{p}_n}{2}\right]\chi^{(n)}(\vec{k})\Lambda_+\left[\vec{k}-\frac{\vec{p}_n}{2}\right]\chi^{(n)\dagger}(\vec{k})\right]=1 \tag{C9}$$

or, using (B28)

$$\frac{1}{(an)^3}\sum_{\vec{k}}\mathrm{Tr}\left[\chi^{(n)}(\vec{k})\Lambda_-\left[\vec{k}-\frac{\vec{p}_n}{2}\right]\chi^{(n)\dagger}(\vec{k})-\Lambda_-\left[\vec{k}+\frac{\vec{p}_n}{2}\right]\chi^{(n)}(\vec{k})\chi^{(n)\dagger}(\vec{k})\right]=1\ . \tag{C10}$$

APPENDIX D: VECTOR SPHERICAL HARMONICS

Let us first expand $\vec{N}(\vec{k})$ in a spherical base:

$$N(\vec{k})=\sum_q(-1)^q N_q(\vec{k})\vec{e}_{-q}\ , \tag{D1}$$

$$N_q(\vec{k})=\sum_{LM}N_{qL}(k)Y_L^M(\hat{k})\ , \tag{D2}$$

$$Y_L^M(\hat{k})\vec{e}_{-q}=\sum_{JM}\langle L,1;M,-q\,|\,JM\rangle\vec{Y}_{JLM}(\hat{k})\ . \tag{D3}$$

We need a number of formulas involving $\vec{Y}_{JLM}(\hat{k})$:

$$\vec{Y}_{JML}(\hat{k})\cdot\hat{k}=\langle L,1;0,0\,|\,J,0\rangle\left[\frac{2L+1}{2J+1}\right]^{1/2}Y_J^M(\hat{k})\ , \tag{D4}$$

$$\frac{i}{\sqrt{2}}[\vec{Y}_{JLM}(\hat{k})\times\hat{k}]=\sqrt{3}\sum_{L'}(-1)^{L'+J}\begin{Bmatrix}L' & J & 1\\ 1 & 1 & L\end{Bmatrix}\vec{Y}_{JL'M}(\hat{k})[(2L+1)^{1/2}\delta_{L',L+1}-\sqrt{L}\,\delta_{L',L-1}]\ , \tag{D5}$$

$$[\vec{Y}_{JLM}(\hat{k})\cdot\hat{k}]\hat{k}=\left[\frac{2L+1}{2J+1}\right]^{1/2}\langle L,1;0,0\,|\,J,0\rangle\sum_{L'}\left[\frac{2L'+1}{2J+1}\right]^{1/2}\langle L',1;0,0\,|\,J,0\rangle\vec{Y}_{JL'M}(\hat{k})\ , \tag{D6}$$

$$\begin{aligned}[\vec{Y}_{JLM}(\hat{k})\times\hat{k}]\times\hat{k}=6\Bigg[&[(L+1)(L+2)]^{1/2}\begin{Bmatrix}L+1 & J & 1\\ 1 & 1 & L\end{Bmatrix}\begin{Bmatrix}L+2 & J & 1\\ 1 & 1 & L+1\end{Bmatrix}\vec{Y}_{JL+2M}(\hat{k})\\ &+[L(L-1)]^{1/2}\begin{Bmatrix}L-1 & J & 1\\ 1 & 1 & L\end{Bmatrix}\begin{Bmatrix}L-2 & J & 1\\ 1 & 1 & L-1\end{Bmatrix}\vec{Y}_{JL-2M}(\hat{k})\\ &-L\begin{Bmatrix}L-1 & J & 1\\ 1 & 1 & L\end{Bmatrix}\begin{Bmatrix}L & J & 1\\ 1 & 1 & L-1\end{Bmatrix}\vec{Y}_{JLM}(\hat{k})\\ &-(L+1)\begin{Bmatrix}L+1 & J & 1\\ 1 & 1 & L\end{Bmatrix}\begin{Bmatrix}L & J & 1\\ 1 & 1 & L+1\end{Bmatrix}\vec{Y}_{JLM}(\hat{k})\Bigg]\ .\end{aligned} \tag{D7}$$

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