Canonical formalism on a null surface: The scalar and the electromagnetic fields

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The Hamiltonian for the scalar and electromagnetic fields are set up on an outgoing null cone plus that portion of $\mathscr{I}^+$ which extends back to space-like infinity. The latter portion is just the energy radiated so that the Hamiltonian is the total energy, a constant of the motion. Because the formalism is set on a characteristic surface, the momenta must satisfy certain constraints in addition to the gauge constraints. These null-surface constraints form a second-class system in the nomenclature of Dirac. Therefore, they are eliminated from the theory by the construction of Dirac brackets. With the Dirac brackets, the Hamiltonian gives the once-integrated field equations for the dynamical field variables. The usual commutation relations for the field strengths restricted to the domain of integration for H are i times the Dirac brackets.

I. INTRODUCTION

The attempt to quantize the Einstein theory of gravity has been pursued actively for more than thirty years beginning with the important work of Dirac,[1,2] Bergmann,[3] and Schild and Pirani[4] and further developed by DeWitt[5] and the work of Arnowitt,[11] Deser, and Misner.[6] Since then the field has blossomed out in many directions. A review of the research can be found in the article by Isham in *Quantum Gravity 2*.[7] However, the program of quantization has not yet been successful for two principal reasons: (1) the intrinsic nonlinearities of the field equations make it difficult to isolate the independent degrees of freedom and, more importantly, (2) quantizing the geometry of space-time means that the stage on which physical processes take place is part of those processes.

With this paper we begin to study a different formulation of the canonical theory with the expectation that it will at least bring out the independent degrees of freedom. The idea is to use a null surface as the initial surface on which to set up the Hamiltonian. The motivation for doing so comes from the study of the gravitational radiation field in asymptotically flat space-times.[8–10] Bondi and his co-workers[8] show explicitly that the specification of the conformal two-geometry[11–13] for a family of two-surfaces on an outgoing null cone determines the metric on that cone up to a specification of the mass aspect and the dipole aspect. The latter quantities are specified only on the two-surface at null infinity and not all over the outgoing null cone itself. It is the shear of the outgoing null rays which determines the conformal two-geometry. And the shear has the further geometrical meaning of the (conformal) extrinsic curvature of the two-surfaces embedded in the null surface. Furthermore, the propagation of the geometry requires the specification of the rate of change of the shear[9,10] of the null rays at future null infinity. An integral over the square of this rate of change (which is positive definite) gives the gravitational energy radiated. It is clear, then, that it is the shear which can be identified with the independent degrees of freedom of the gravitational field.

This conclusion is reinforced by the analysis of Ashtekar[14] which elucidates the degrees of freedom on $\mathscr{I}^+$, future null infinity. $\mathscr{I}^+$ is a null surface with a universal structure[15] which is specified by the singular metric on $\mathscr{I}^+$ and the null vector n^μ which is tangent to the generators of $\mathscr{I}^+$. Since the space-time is asymptotically Minkowskian, this universal structure tells us nothing about the dynamical degrees of freedom. Ashtekar pointed out that the connection on $\mathscr{I}^+$ consists of a part which depends on the universal structure and a part which may be specified freely. He showed that the part which may be specified freely is related to the rate of change of the shear.

Thus, it is suggestive that an analysis which focuses attention on the conformal two-geometry will allow one to pick out the dynamical degrees of freedom of the gravitational field. This, in turn, may allow one to understand better the Hamiltonian of general relativity.

There are some additional problems, however, which require preliminary study. Because the field equations of general relativity are covariant under arbitrary diffeomorphisms, data set on an initial Cauchy surface do not have a unique propagation. This shows up in that four of the Einstein field equations do not have second time derivatives of field variables. These equations form constraints on the initial data. In the canonical formalism, these equations are constraint equations on the phase space whose points are (g_{mn},P^{mn}) the three-space metric and their canonical conjugates.[1–5] Linear functionals constructed from these constraints generate the symmetry transformations of theory.[15,16]

When the initial surface is a null surface, there are additional constraints because a null surface is a characteristic surface along which data is propagated. These additional constraints, which we shall refer to as *null-surface constraints,* occur in all theories with hyperbolic differential equations. In order to understand how to treat these constraints, we first study the massless scalar field in Minkowski space. In order to see how the null-surface constraints interact with constraints arising from a gauge group, the Maxwell theory is treated next. Treatment of the Einstein equations is left for a following paper.

The scalar field and the electromagnetic field have been studied in the infinite-momentum frame which makes use of null planes as the initial-value surface.[17–22] For the most part, these papers eliminate the constraints and work only with the independent field variables from the beginning. Therefore, they are not useful as models for the treatment of the gravitational field where the elimination of the constraints is a major problem. However, the work by Steinhardt[21] does work with the canonical formalism on a null plane and eliminates what we have referred to as null-surface constraints by use of the Dirac brackets[23,24] (Db). He defines the Db by explicitly inverting the matrix defined by the Poisson brackets (Pb) of the null-surface constraints which are second class in the nomenclature of Dirac.[23] The work in the following differs from that of Steinhardt in two important respects. First of all, we use null cones rather than null planes as our initial surface. Therefore, we require significantly different boundary conditions from those he uses. Second, we eliminate the second-class constraints by using the "starring process" of Bergmann and Komar.[25] This procedure seems simpler to use than that used by Steinhardt because it does not require the explicit inversion of the matrix of second-class constraints. The results we obtain for the Dirac brackets, however, are the same.

That one must pay strict attention to the boundary conditions in the definition of functional derivatives has been emphasized and utilized by Regge and Teitelboim.[22,26] Their work has been extremely important in helping us to carry out the research reported here.

II. MASSLESS SCALAR FIELD

The field equation for the massless scalar field is

$$(Jg^{\mu\nu}\phi_{,\nu})_{,\mu}=0\ , \tag{2.1}$$

where $g^{\mu\nu}$ $(\mu,\nu=0,\ldots,3)$ is the inverse of the metric tensor on Minkowski space $g_{\mu\nu}$, $J=\sqrt{-g}$, and the comma indicates ordinary differentiation. This equation is derivable from a variational principle by requiring that the action

$$S=\int_D Jg^{\mu\nu}\phi_{,\mu}\phi_{,\nu}d^4x$$

be stationary with fixed values on the boundary of the domain D. One usually assumes that the past boundary and the future boundary of D are spacelike Cauchy surfaces. Thus, one can define a Lagrangian as an integral over the past boundary Σ, for example,

$$L=\int_\Sigma \mathscr{L}\,d\tau_{(3)}\ .$$

From the Lagrangian, one can define momenta conjugate to the scalar field $\phi(x)\mid_\Sigma$ and then construct a Hamiltonian which will propagate the scalar field either into the future or into the past. The phase space consists of pairs of functions on Σ $(\phi(x),\pi(x))$, $\phi(x)$ being a scalar and $\pi(x)$ a scalar density with respect to diffeomorphisms of Σ onto itself. If r is the distance of x from the origin on Σ, then for large r, $\phi(x)\sim 1/r$ and $\pi(x)\sim 1/r^2$. Σ may be distorted into any other surface which is everywhere spacelike and none of the above discussion changes. However, in the limit that Σ is distorted into an outgoing null cone, the situation changes. In the conformal picture the initial surface is discontinuous on $\mathscr{I}^+$, future null infinity, and the Lagrangian breaks up into two integrals over the M-shaped surface shown in Fig. 1.

Furthermore, as we are only interested in the radiation field without interaction with sources, we shall assume that ϕ goes to zero as rapidly as necessary in the limit of past or future timelike infinity.

To facilitate working on $\mathscr{I}^+$, we shall use in the physical space null spherical coordinates with the inversion of the radial coordinate $\rho=1/r$,

$$d\tilde{s}^2=\rho^{-2}(\rho^2du^2-2\,du\,d\rho-d\theta^2-\sin^2\theta\,d\phi^2)\ .$$

We find it particularly convenient to work with the conformally related compact space-time[16] for which

$$ds^2=\rho^2d\tilde{s}^2=\rho^2du^2-2\,du\,d\rho-d\theta^2-\sin^2\theta\,d\phi^2\ . \tag{2.2}$$

It is necessary to choose a timelike direction for propagation, $\partial/\partial t$. Although the results of interest do not depend on the choice, for simplicity we choose the direction for propagation to be

$$\partial/\partial t=\partial/\partial u\ . \tag{2.3a}$$

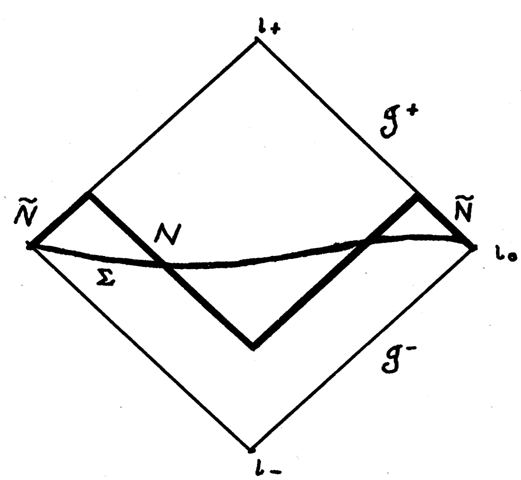

FIG. 1. The Cauchy surface Σ is replaced by the union of an outgoing null cone N and the section of $\mathscr{I}^+$ extending back to Σ at spacelike infinity. $M=N\cup\tilde{N}$.

However, because $\rho=0$ on N we parametrize incoming null surfaces by

$$v=u+2/\rho$$

with $u=\bar{u}$. Hence on the incoming surface, we have

$$\partial/\partial t=\partial/\partial v+\partial/\partial\bar{u} \ . \tag{2.3b}$$

We define

$$\psi=\rho^{-1}\phi=r\phi \tag{2.4}$$

so that ψ is finite on $\mathscr{I}^+$ and zero for $\rho\to\infty\,(r=0)$ (and goes to zero at timelike and spacelike infinity on $\mathscr{I}^+$). The Lagrangian on $N\cup\tilde{N}$ then takes the form ($u=u_0$ denotes the outgoing null surface, N)

$$\begin{aligned} L&=L_N+L_{\tilde{N}} \ , \\ L_N&=\int_N d\rho\int\sin\theta\, d\theta\, d\phi\tfrac{1}{2}[-2\dot{\psi}\psi_{,1}-\rho^2\psi_{,1}\psi_{,1} \\ &\qquad -q^{AB}\psi_{,A}\psi_{,B} \\ &\qquad -\psi^2-(2/\rho)\psi(\psi+\rho^2\psi_{,1})] \ , \\ L_{\tilde{N}}&=\int_{-\infty}^{u_0}du\int\sin\theta\, d\phi\, d\theta(\dot{\psi}\psi_{\bar{u}}-\psi_{\bar{u}}{}^2)^\sim \ . \end{aligned} \tag{2.5}$$

q^{AB} is the positive-definite inverse of the metric on the unit sphere. Derivation with respect to t is denoted by a dot and the subscript $\bar{u}$ indicates the in-surface derivative $\partial/\partial\bar{u}$. Functions with support on $\tilde{N}$ are indicated by the tilde while those without the tilde have their support on N. To simplify the appearance of equations, we shall place the tilde as a superscript outside a bracket rather than over each individual term within the brackets.

We define the momentum density conjugate to ψ by the functional derivative with respect to $\dot{\psi}$. Thus

$$\pi=-(1/\rho)(\rho\psi)_{,1}\sin\theta \ , \tag{2.6a}$$

$$\tilde{\pi}=\sin\theta\,\tilde{\psi}_{\bar{u}} \ . \tag{2.6b}$$

These relations are constraints on the phase space and give the constraint functional ($d\tau\equiv d\rho\, d\theta\, d\phi$, $d\tilde{\tau}\equiv d\bar{u}\, d\theta\, d\phi$)

$$\begin{aligned} C[\omega,\tilde{\omega}]&=\int_N\omega[\pi+(\sin\theta/\rho)(\rho\psi)_{,1}]d\tau \\ &\quad+\int_{\tilde{N}}\tilde{\omega}[\pi-\sin\theta\,\psi_u]^\sim d\tilde{\tau} \ . \end{aligned} \tag{2.7}$$

Since $\dot{\psi}$ is finite on $\mathscr{I}^+$ and vanishes for $\rho\to\infty$ and $u\to\pm\infty$, ω and $\tilde{\omega}$ have that behavior. Thus the weighting functions for the constraints $[\omega,\tilde{\omega}]$ may be restricted to functions which vanish for $\rho\to\infty$ and $u\to\pm\infty$, but may be otherwise finite. However, forming the variational derivation of $C[\omega,\omega]$ shows that we must have $\omega=-\tilde{\omega}$ on $N\cap\tilde{N}$.

The Hamiltonian becomes

$$\begin{aligned} H&=\int_N\{\alpha[\pi+(\sin\theta/\rho)(\rho\psi)_{,1}] \\ &\quad+\tfrac{1}{2}\sin\theta[\rho^2\psi_{,1}\psi_{,1}+q^{AB}\psi_{,A}\psi_{,B}+(\rho\psi^2)_{,1}]\}d\tau \\ &\quad+\int_{\tilde{N}}[\alpha(\pi-\sin\theta\,\psi_u)+\sin\theta\,\psi^2]^\sim d\tilde{\tau} \ . \end{aligned} \tag{2.8}$$

We see that on the constraint hypersurface, the contribution of the integral over $\tilde{N}$ is just the energy radiated. This contribution is needed in order to make H a constant of the motion. Differentiability of the Hamiltonian H requires that we identify $\tilde{\alpha}$ with $\tilde{\psi}_{\bar{u}}$ on $\tilde{N}$ and that $\alpha=\tilde{\alpha}$ on $N\cap\tilde{N}$. Since $\dot{\psi}$ and $\dot{\tilde{\psi}}$ are equal to α and $\tilde{\alpha}$ by the Hamiltonian field equations and on $\mathscr{I}^+$ $\tilde{\psi}_{\bar{u}}=\dot{\tilde{\psi}}$, these conditions are satisfied.

Forming the appropriate Poisson brackets, we find the canonical equations of motion,

$$\dot{\psi}=\alpha \ , \tag{2.9a}$$

$$\dot{\pi}=\sin\theta[\rho(\alpha/\rho)_{,1}+(\rho^2\psi_{,1})_{,1}]-\sin\theta(q^{AB}\psi_{,B})_{,A} \ , \tag{2.9b}$$

$$\dot{\tilde{\psi}}=\alpha \ , \tag{2.9c}$$

$$\dot{\tilde{\pi}}=-\sin\theta[\alpha_{\bar{u}}-2\psi_{\bar{u}\bar{u}}]^\sim \ , \tag{2.9d}$$

on N and $\tilde{N}$, respectively. On the constraint hypersurface these equations agree with the Lagrangian equations of motion on N and $\tilde{N}$. One finds that the Pb of the null constraints, Eq. (2.7), with the Hamiltonian, Eq. (2.8), vanishes provided $\alpha(\tilde{\alpha})$ satisfies the Lagrangian equations for ψ. Since α is identified with $\dot{\psi}$ by the canonical equations of motion, it follows that the equations of motion preserve the constraints.

It is also easy to show that the constraints form a second-class system in the terminology of Dirac.[23] That is, the Poisson brackets between constraints with different weighting functions in general do not vanish:

$$\begin{aligned} \{C[\omega,\tilde{\omega}],C[\nu,\tilde{\nu}]\}&=\int\sin\theta(\omega\nu_{,1}-\nu\omega_{,1})d\tau \\ &\quad+\int\sin\theta[\omega_{\bar{u}}\nu-\nu_{\bar{u}}\omega]^\sim d\tilde{\tau} \ . \end{aligned} \tag{2.10}$$

The existence of second-class constraints implies that the constraints involve canonical conjugate pairs of variables. In principle, the constraints should be solved for these pairs and the results substituted into the Hamiltonian. The phase space would thereby be reduced and within this reduced phase space there would be no constraints. The task is, in general, very difficult to carry out explicitly, but Dirac[24] has offered a prescription for modification of the Poisson brackets so that the constraints may be considered as strong relations. That is, the constraints would have an identically vanishing Dirac bracket with all variables.

This procedure has been extended by Steinhardt[21] to the situation of null constraints on a null plane. His method could be adapted to the null cones considered here. However, we find that it is easier to carry out the alternative construction of the Db given by Bergmann and Komar.[23] The idea is to define new field variables which differ from the old by a linear combination of the constraints themselves. The appropriate linear combination is determined by the requirement that the new field variables have vanishing Poisson brackets with the constraints. The Poisson brackets among the new variables are equal to the Dirac brackets among the old. With this identification we can use the constraints as strong relations.

Therefore, we introduce [$x=(\rho,\theta,\phi),x\in N$ and $x=(\bar{u},\theta,\phi),x\in\tilde{N}$]

$$\psi^*(x)=\psi(x)+\int_N K(x,x')\left[\pi(x')+\frac{\sin\theta'}{\rho'}[\rho'\psi(x')]_{,1'}\right]d\tau' ,$$

$$\tilde{\psi}^*(x)=\tilde{\psi}(x)+\int_{\tilde{N}}\tilde{K}(x,x')[\pi(x')-\sin\theta'\psi_{\bar{u}'}(x')]^{\sim}d\tilde{\tau}' . \tag{2.11}$$

From

$$\{\psi^*(x),C[\omega,\tilde{\omega}]\}=\{\tilde{\psi}^*(x),C[\omega,\tilde{\omega}]\}=0$$

we obtain

$$K(x,x')=-\tfrac{1}{2}S(\rho-\rho')\delta(\theta-\theta')\delta(\phi-\phi')/\sin\theta' , \tag{2.12a}$$

$$\tilde{K}(x,x')=-\tfrac{1}{2}S(\bar{u}'-\bar{u})\delta(\theta-\theta')\delta(\phi-\phi')/\sin\theta' , \tag{2.12b}$$

where

$$S(\rho)=\begin{cases}0, & \rho<0\\ 1, & \rho>0 .\end{cases} \tag{2.13}$$

Note that we have chosen the solution for $K(x,x')$ and $\tilde{K}(x,x')$, which as functions of x' for fixed x belong to the class of functions selected for the constraints. However, because $K(x,x')$ ($\tilde{K}(x,x')$) vanishes for $x\in\tilde{N}$ (N) the solution for $\psi^*(x)$ exists only for constraints constructed with weight functions which in fact vanish on $N\cap\tilde{N}$. Furthermore, because K ($\tilde{K}$) does not vanish on $N\cap\tilde{N}$, the variation of ψ^* ($\tilde{\psi}^*$) will have a boundary contribution which affects the definition of the functional derivative. In forming Poisson brackets we shall first consider the contribution coming from N ($\tilde{N}$) and then that coming from the boundary:

$$\{A,B\}=\left[\int_N d\tau+\int_N d\tilde{\tau}\right]\left[\frac{\delta A}{\delta\psi}\frac{\delta B}{\delta\pi}-\frac{\delta A}{\delta\pi}\frac{\delta B}{\delta\psi}\right]+\int_{N\cap\tilde{N}}\left[\frac{\delta a}{\delta\psi}\frac{\delta B}{\delta\pi}-\frac{\delta A}{\delta\pi}\frac{\delta B}{\delta\psi}\right]^{\sim}d\theta\, d\phi .$$

In the integrals over N and $\tilde{N}$ only the variation in the interior occurs while in the boundary term only the variations on the boundary occur. We then find for $(x,x')\in N$ or $\tilde{N}$

$$\{\psi(x_1),\psi(x_2)\}^*=\{\psi^*(x_1),\psi^*(x_2)\}=\tfrac{1}{2}\epsilon(\rho_1-\rho_2)\delta(\theta_1-\theta_2)/\sin\theta_1 , \tag{2.14a}$$

$$\{\tilde{\psi}(x_1),\tilde{\psi}(x_2)\}^*=[\{\psi^*(x_1),\psi^*(x_2)\}]^{\sim}=-\tfrac{1}{2}\epsilon(\rho_1-\rho_2)\delta(\theta_1-\theta_2)\delta(\phi_1-\phi_2)/\sin\theta_1 , \tag{2.14b}$$

$$\epsilon(\rho)=\begin{cases}-\tfrac{1}{2}, & \rho<0\\ \tfrac{1}{2}, & \rho>0 .\end{cases} \tag{2.15}$$

Because only the constraints with weight functions which vanish on $N\cap N$ may be considered to be strong, we cannot eliminate the constraints from the Hamiltonian because in general $\alpha=\tilde{\alpha}\neq 0$ on $N\cap\tilde{N}$. The Poisson brackets of $\psi^*(\tilde{\psi}^*)$ with such constraints will contribute only a term on $N\cap\tilde{N}$. Thus

$$\{\psi^*(x),C[\alpha,\tilde{\alpha}]\}=(\alpha/2)(0,\theta,\phi)=(\tilde{\alpha}/2)(u_0,\theta,\phi) . \tag{2.16}$$

Furthermore, if one constructs $\pi^*(x)$ in the same manner, one can show that

$$\{\psi^*(x),\pi^*(x')\}=-(\sin\theta'/\rho')[(\partial/\partial\rho')\rho'\{\psi^*(x),\psi^*(x')\}] .$$

Therefore, in all functionals except the constraint functional, one can treat the constraints as strong relations and thereby substitute for $\pi(x)$ from the constraints. To get the correct equations of motion, it is necessary to keep the constraint in the Hamiltonian.

From Eqs. (2.8), (2.14), and (2.16) we find field equations for ψ^* to be

$$\{\psi^*(x),H\}^*=\dot{\psi}^*(x)=-\tfrac{1}{4}\int_0^{\rho}[(\rho'^2\psi_{,1'})_{,1'}+(\sin\theta)^{-1}(\sin\theta\, q^{AB}\psi_{,B})_{,A}]d\rho'+\tfrac{1}{2}\alpha(0,\theta,\phi)$$
$$+\tfrac{1}{4}\int_{\rho}^{\infty}[(\rho'^2\psi_{,1'})_{,1'}+(\sin\theta)^{-1}(\sin\theta\, q^{AB}\psi_{,B})_{,A}]d\rho' \quad (x\in N) , \tag{2.17a}$$

$$[\{\psi(x),H\}^*]=\dot{\tilde{\psi}}=\tilde{\psi}_{\bar{u}} \quad (x\in\tilde{N}) . \tag{2.17b}$$

Equations (2.17) are just the once-integrated Lagrangian equations (2.1) on N and $\tilde{N}$. Note that in the absence of sources, $\alpha(0,\theta,\phi)$ is not independent of the data on N. Indeed, we find

$$\alpha(0,\theta,\phi)=\tfrac{1}{2}(\rho^2\psi\,|_{\rho=0})-\tfrac{1}{2}\int_0^{\infty}(\sin\theta)^{-1}(\sin\theta q^{AB}\psi_{,B})_{,A}d\rho . \tag{2.18}$$

Looking at the Dirac brackets for $\psi(x)$, the transition to the quantized field theory appears to be direct. The quantum field theory for a massless scalar field $\phi(x)$ leads to the commutation relations

$$[\phi(x),\phi(y)]=iD(x-y)\ ,$$
$$D(x-y)=(1/2\pi)\epsilon(x^0-y^0)\delta((x-y)^2)\ ,$$

for x and y any pair of points in Minkowski space. On N and $\tilde{N}$ these relations reduce to

$$[\phi(x),\phi(y)]=i\{\phi(x),\phi(y)\}^*\ .$$

However, there remains the interesting question of what role, if any, the constraints in the Hamiltonian will play.

III. ELECTROMAGNETIC THEORY

The electromagnetic field differs in two important respects from the scalar field: (1) it is a vector field, and (2) it has the gauge group as a local symmetry. Both aspects are of particular importance to us in preparation for the study of the gravitational field. In this section we shall focus on the new properties and pass over those parts which are similar to those discussed in the previous section.

First of all, the electromagnetic field is conformally invariant and therefore the Lagrangian has the same form in the nonphysical space as in the physical space-time,

$$L=-\tfrac{1}{4}\int_N F_{\mu\nu}F^{\mu\nu}\sin\theta\, d\tau_{(3)} -\tfrac{1}{4}\int_{\tilde{N}}\tilde{F}_{\mu\nu}\tilde{F}^{\mu\nu}\sin\theta\, d\tilde{\tau}_{(3)}\ . \quad (3.1)$$

However, in order to have better control over the asymptotic behavior of the field, it is useful to introduce a null tetrad in the nonphysical space and to work with the tetrad components. Therefore, we define

$$\begin{aligned} &k^0=du,\quad k_0=\partial_u+\tfrac{1}{2}\rho^2\partial_\rho\ ,\\ &k^1=\tfrac{1}{2}\rho^2du-d\rho,\quad k_1=-\partial\rho\ ,\\ &\sqrt{2}k^2=\sqrt{2}\bar{k}^3=d\theta+i\sin\theta\, d\phi\ ,\\ &\sqrt{2}k_2=\sqrt{2}k_3=\partial_\theta-(i/\sin\theta)\partial_\phi\ . \end{aligned} \quad (3.2)$$

The tetrad components are then given by

$$(1)\quad A_a=A_\mu k_a{}^\mu\quad (a=0,\ldots,3)$$

and differentiation is indicated by a slash or double slash as follows:

$$\begin{aligned} &k_a{}^\mu\nabla_\mu\phi=\phi_{|a}\ ,\\ &k_b{}^\nu k_a{}^\mu\nabla_\mu V_\nu=V_{b\,||a}=V_{b\,|a}-\gamma_b{}^c{}_a V_c\ . \end{aligned} \quad (3.3)$$

For the tetrad chosen, the only rotation coefficients different from zero are

$$\gamma_{010}=-\gamma_{100}=\rho,\quad \gamma_{233}=\gamma_{322}=\cot\theta/\sqrt{2}\ . \quad (3.4)$$

Raising and lowering of tetrad indices is carried out with the matrix

$$\eta_{ab}=\eta^{ab}=\begin{bmatrix}0&1&0&0\\1&0&0&0\\0&0&0&-1\\0&0&-1&0\end{bmatrix}\ . \quad (3.5)$$

One can show that the tetrad components have the following behavior:

For $\rho\to 0$ $(r\to\infty)$	For $\rho\to\infty$ $(r\to 0)$
$A_0=O(\rho)$	$A_0=O(1)$
$A_2=O(1)$	$A_1=O(1/\rho^2)$
$A_B=O(1)$	$A_B=O(1/\rho)$

(3.6)

Variations of these quantities are assumed to have the same behavior. As for the scalar field, all relevant quantities and derivatives vanish in the limit of spacelike infinity—$\rho\to 0$, $\mu\to-\infty$ and timelike infinity, $u\to+\infty$.

The direction of propagation $\partial/\partial t$ is given by

$$\partial/\partial t=k_0+\tfrac{1}{2}\rho^2 k_1 \quad (3.7)$$

and all physical components are to be expressed in terms of vectors which are in the direction of propagation and those which are tangent to the surfaces N and $\tilde{N}$. In particular, since k_1 lies in N, we write on N

$$k_0=\partial/\partial t-\tfrac{1}{2}\rho^2 k_1; \quad (3.8a)$$

on the other hand, k_0 lies in $\tilde{N}$ so we write

$$k_1=(2/\rho^2)[(\partial/\partial t)-k_0] \quad (3.8b)$$

on $\tilde{N}$. The space-time derivatives have to be broken up in the same way. Therefore, on N we have

$$A_0=A_t-\tfrac{1}{2}\rho^2 A_1\ , \quad (3.9a)$$

$$F_{01}=A_{t\,|1}-\dot{A}_1\ , \quad (3.9b)$$

$$F_{0A}=A_{t\,|A}-\dot{A}_A-\tfrac{1}{2}\rho^2(A_{1\,|A}-A_{A\,|1})\ , \quad (3.9c)$$

$$F_{1A}=A_{1\,|A}-A_{A\,|1}\ , \quad (3.9d)$$

while on $\tilde{N}$ we have

$$\tilde{A}_1=(2/\rho^2)(\dot{A}_t-A_0)^\sim\ , \quad (3.10a)$$

$$\tilde{F}_{01}=(2/\rho^2)[\dot{A}_0-A_{t\,|0}]^\sim\ , \quad (3.10b)$$

$$\tilde{F}_{1A}=(2/\rho^2)[A_{t\,|A}-\dot{A}_A-A_{0\,|A}+A_{A\,|0}]^\sim\ , \quad (3.10c)$$

$$\tilde{F}_{0A}=[A_{0\,|A}-A_{A\,|0}]^\sim\ . \quad (3.10d)$$

In these equations we have indicated the component in the direction of $\partial/\partial t$ by the subscript t and differentiation with respect to time (t) by a dot. The Lagrangian now takes the form

$$L=-\tfrac{1}{2}\int_N\{-(A_{t\,|1}-\dot{A}_1)^2+2\eta^{AB}F_{1B}[A_{t\,|A}-\dot{A}_A-\tfrac{1}{2}\rho^2(A_{1\,|A}-A_{A\,|1})]+\tfrac{1}{2}F_{AB}F^{AB}\}\sin\theta\,d\tau_{(3)}$$
$$-\int_{\tilde{N}}\{[-(1/\rho^2)(A_0-A_{t\,|0})^2-\eta^{AB}F_{0B}(A_A+F_{0A})]^\sim\sin\theta\}d\tau_{(3)}\,. \quad (3.11)$$

Any terms whose variation does not have a finite limit on $\tilde{N}$ have been dropped.

We find for the momenta ($x\in N$)

$$\delta L/\delta\dot{A}_t\equiv\pi=0\,,$$
$$\delta L/\delta\dot{A}_1\equiv\pi^1=\sin\theta(A_1-A_{t\,|1})\,, \quad (3.12)$$
$$\delta L/\delta\dot{A}_A\equiv\pi^A=\sin\theta\eta^{AB}F_{1B}\,,$$

and ($x\in N$)

$$\delta L/\delta\dot{\tilde{A}}_t\equiv\pi=0\,,$$
$$\delta L/\delta\dot{\tilde{A}}_0\equiv\pi^0=(2/\rho^2)\sin\theta(\dot{A}_0-A_{t\,|0})^\sim\,, \quad (3.13)$$
$$\delta L/\delta\dot{\tilde{A}}_A\equiv\pi^A=\sin\theta\eta^{AB}\tilde{F}_{0B}\,.$$

The Hamiltonian takes the form

$$H=\int_N\{\alpha\pi+\alpha_A(\pi^A-\sin\theta\eta^{AB}F_{1B})+\tfrac{1}{2}\sin\theta(\pi^1)^2-A_t(\pi^1{}_{|1}+\pi^A{}_{|A})-\tfrac{1}{2}\rho^2\sin\theta\eta^{AB}F_{1A}F_{1B}$$
$$+\tfrac{1}{4}\sin\theta F_{AB}F^{AB}\}d\tau_{(3)}+\int_{\tilde{N}}[\alpha\pi+\alpha_A(\pi^A-\sin\theta\eta^{AB}F_{0B})-A_t(\pi^0{}_{|0}+\pi^A{}_{|A}-\eta^{AB}\sin\theta F_{0B}F_{0A})]^\sim d\tilde{\tau}_{(3)}\,. \quad (3.14)$$

A total divergence

$$\{(A_t\pi^1)_{|1}+(A_t\pi^A)_{|A}\}$$

has been subtracted from the integrand on N and $\{(A_t\pi^0)_{|0}+(A_t\pi^A)_{|A}\}^\sim$ on $\tilde{N}$. The functions α, α_A, $\tilde{\alpha}$, $\tilde{\alpha}_A$ are weighting functions for the constraints and take the values $\dot{A}_t$, $\dot{A}_A-A_{t\,|A}$ on N and $\tilde{N}$, respectively, when the canonical field equations are satisfied. Therefore, they satisfy the same boundary conditions as these quantities. Differentiability of the Hamiltonian requires that $\alpha_A=\tilde{\alpha}_A$ on $N\cap\tilde{N}$, which is to be expected from the considerations of the scalar field.

The null-surface constraints now take the form

$$C[\omega_A,\tilde{\omega}_A]=\int_N\omega_A(\pi^A-\sin\theta\eta^{AB}F_{1B})d\tau_{(3)}$$
$$+\int_{\tilde{N}}\tilde{\omega}_A[\pi^A-\sin\theta\eta^{AB}F_{0B}]^\sim d\tilde{\tau}_{(3)}\,, \quad (3.15)$$

while the gauge constraints are

$$G_0[\xi,\tilde{\xi}]=\int_N\xi\pi\,d\tau_{(3)}+\int_{\tilde{N}}\tilde{\xi}\tilde{\pi}\,d\tilde{\tau}_{(3)}\,, \quad (3.16a)$$

$$G_1[\sigma,\tilde{\sigma}]=\int_N\sigma(\pi^1{}_{|1}+\pi^A{}_{|A})d\tau_{(3)}$$
$$+\int_{\tilde{N}}\tilde{\sigma}[\pi^0{}_{|0}+\pi^A{}_{|A}]^\sim d\tau_{(3)}\,. \quad (3.16b)$$

The weight functions $(\omega_A,\tilde{\omega}_A)$ belong to the same class of functions as $(\alpha_A,\tilde{\alpha}_A)$. G_0 generates the gauge change in A_t which should vanish on $\mathscr{I}^+$. Therefore, ξ and $\tilde{\xi}$ vanish on $\tilde{N}$. On the other hand, G_1 generates the gauge transformations of A_1, A_A, $\tilde{A}_0$, and $\tilde{A}_A$. Therefore, σ and $\tilde{\sigma}$ are finite on $\tilde{N}$, but $\tilde{\sigma}_{|0}=0$. Since $\pi^1=-\tilde{\pi}^0$ on $N\cap\tilde{N}$, differentiability of $G_1[\sigma,\tilde{\sigma}]$ requires that $\sigma=\tilde{\sigma}$ on $N\cap\tilde{N}$. This is an important consistency condition if we are to maintain $A_A=\tilde{A}_A$ on $N\cap\tilde{N}$.

Adding the divergence to the Hamiltonian density had the effect of separating the gauge constraints from the null-surface constraints. The gauge constraints are first class, whereas the null-surface constraints, as before, form a second-class system which are to be removed by the introduction of the Dirac brackets.

The canonical equations of motion are the following.

(i) on N:

$$\dot{A}_t=\alpha\,,$$
$$\dot{A}_1=(\sin\theta)^{-1}\pi^1+A_{t\,|1}\,,$$
$$\dot{A}_A=\alpha_A+A_{t\,|A}\,,$$
$$\dot{\pi}=\pi^1{}_{|1}+\pi^A{}_{|A}=0\,, \quad (3.17)$$
$$\dot{\pi}^1=-[\sin\theta\eta^{AB}(\alpha_B+\rho^2F_{|B})]\,,$$
$$\dot{\pi}^A=[\sin\theta\eta^{AB}(\alpha_B+\rho^2F_{|B})]-\sin\theta F^{AB}{}_{||B}\,.$$

(ii) on $\tilde{N}$:

$$\begin{aligned}
&\dot{\tilde{A}}_t=\tilde{\alpha}\ ,\\
&\dot{\tilde{A}}_0=\tilde{A}_{t\,|0}\ ,\\
&\dot{\tilde{A}}_A=\tilde{\alpha}_A+\tilde{A}_{t\,|A}\ ,\\
&\dot{\tilde{\pi}}=\tilde{\pi}^0{}_{|0}+\tilde{\pi}^A{}_{|A}=0\ ,\\
&\dot{\tilde{\pi}}^0=-[\sin\theta\eta^{AB}(\alpha_B+2F_{0B})]^{\sim}{}_{|A}\ ,\\
&\dot{\tilde{\pi}}^A=[\sin\theta\eta^{AB}(\alpha_B+2F_{0B})]^{\sim}{}_{|0}\ .
\end{aligned} \tag{3.18}$$

Together with the constraints, these equations are equivalent to the Lagrangian equations of motion. The constraint $G_1[\sigma,\tilde{\sigma}]$ follows from the requirement that $\dot{\pi}=0$ and no further conditions are required to propagate G_1. The propagation of the null-surface constraints requires that α_A ($\tilde{\alpha}_A$) satisfy the Lagrangian equations for $A_A-A_{t/A}$. This requirement is consistent with the canonical equation of motion as was true for the scalar field.

We may proceed immediately to the construction of the starred quantities and the Dirac brackets. To simplify the writing we introduce

$$\mathscr{C}^B(x)=\pi^B(x)-\sin\theta\eta^{BC}(A_{1\,|C}-A_{C\,|1})(x)\ , \tag{3.19a}$$

$$\tilde{\mathscr{C}}^B(x)=\tilde{\pi}^B(x)-\sin\theta\eta^{BC}(A_{0\,|C}-A_{C\,|0})^{\sim}(x)\ . \tag{3.19b}$$

We define

$$A^*{}_A(x)=A_A(x)+\int_N K_{AB}(x,x)\mathscr{C}^B(x')d\tau+\int_{\tilde{N}}\tilde{K}_{AB}(x,x')\tilde{\mathscr{C}}^B(x')d\tilde{\tau}\ , \tag{3.20a}$$

$$\pi^{A*}(x)=\pi^A(x)+\int_N P^A{}_B(x,x')\mathscr{C}^B(x')d\tau+\int_{\tilde{N}}\tilde{P}^A{}_B(x,x')\tilde{\mathscr{C}}^B(x')d\tilde{\tau}\ , \tag{3.20b}$$

$$\pi^{1*}(x)=\pi^1(x)+\int_N P_B(x,x')\mathscr{C}^B(x')d\tau,\quad x\in N\text{ only}\ , \tag{3.20c}$$

$$\tilde{\pi}^{0*}(x)=\tilde{\pi}^0(x)+\int\tilde{P}_B(x,x')\tilde{\mathscr{C}}^B(x')d\tau,\quad x\in\tilde{N}\text{ only}\ . \tag{3.20d}$$

For $x\in N$ the kernels with the tilde vanish and for $x\in\tilde{N}$ those without the tilde vanish. The remaining variables do not require starring to have vanishing Poisson brackets with the constraints.

The kernels are determined by the conditions

$$\text{(i)}\ \ \{0^*(x),\ C[\tilde{\omega}_A,\omega_A]\}=0\ , \tag{3.21}$$

where $0^*(x)$ represents the variables defined in (3.20), and (ii) for fixed x, the kernels should belong to the same class of functions as the $(\alpha_A,\tilde{\alpha}_A)$. As in the scalar case, a solution of Eqs. (3.21) exists only if the class of functions $(\omega_A,\tilde{\omega}_A)$ is restricted to those which vanish on $N\cap\tilde{N}$. We find then

$$K_{AB}(x,x')\sin\theta'=\eta_{AB}(\tfrac{1}{2})\epsilon(\rho-\rho')\delta(\theta-\theta')\delta(\phi-\phi')\ , \tag{3.22a}$$

$$\tilde{K}_{AB}(x,x')\sin\theta'=(\tfrac{1}{2})\eta_{AB}\epsilon(\bar{u}'-\bar{u})\delta(\theta-\theta')\delta(\phi-\phi')\ , \tag{3.22b}$$

$$P^A{}_B(x,x')=-\tfrac{1}{2}\delta^A{}_B\delta^{(3)}(x-x')\ , \tag{3.22c}$$

$$\tilde{P}^A{}_B(x,x')=-\tfrac{1}{2}\delta^A{}_B\delta^{(3)}(x-x')\ , \tag{3.22d}$$

$$P_B(x,x')=\tfrac{1}{2}S(\rho-\rho')[\delta(\theta-\theta')\delta(\phi-\phi')]_{|B'}\ , \tag{3.22e}$$

$$\tilde{P}_B(x,x')=\tfrac{1}{2}S(\bar{u}'-\bar{u})[\delta(\theta-\theta')\delta(\phi-\phi')]_{|B'}\ . \tag{3.22f}$$

As before the Dirac brackets are defined to be the Poisson brackets among the starred quantities. Therefore, we have

$$\{A_A(x),A_B(x')\}^*=\tfrac{1}{2}\eta_{AB}[\epsilon(\rho'-\rho)\delta(\theta-\theta')\delta(\phi-\phi')]/\sin\theta\ , \tag{3.23a}$$

$$\{A_A(x),\pi^B(x')\}^*=\tfrac{1}{2}\delta^B{}_A\delta^3(x-x')\ , \tag{3.23b}$$

$$\{A_A(x),\pi^1(x')\}^*=\tfrac{1}{2}\epsilon(\rho-\rho')[\delta(\theta-\theta')\delta(\phi-\rho')]_{|A'}\ , \tag{3.23c}$$

$$\{\pi^A(x),\pi^B(x')\}^*=\tfrac{1}{2}\eta^{AB}[\delta^{(3)}(x-x')]_{|1'}\ , \tag{3.23d}$$

$$\{\pi^A(x),\pi^1(x')\}^*=-\tfrac{1}{2}\eta^{AC}[\sin\theta'\delta^{(3)}(x-x')]_{|C'}\ , \tag{3.23e}$$

$$\{\pi^1(x),\pi^1(x')\}=\tfrac{1}{2}\epsilon(\rho-\rho')\eta^{AB}\{[\delta(\theta-\theta')\delta(\phi-\phi')]_{|B'}\sin\theta'\}_{|A'} \tag{3.23f}$$

on the N surface and,

$$[\{A_A(x),A_B(x')\}^*]^{\sim}=\tfrac{1}{2}\eta_{AB}[\epsilon(u-u')\delta(\theta-\theta')\delta(\phi-\phi')]/\sin\theta\ , \tag{3.23g}$$

$$[\{A_A(x),\pi^B(x')\}^*]^{\sim}=\tfrac{1}{2}\delta^B{}_A\delta^{(3)}(x-x')\ , \tag{3.23h}$$

$$[\{A_A(x),\pi^B(x')\}^*]^\sim=\tfrac{1}{2}\epsilon(u'-u)[\delta(\theta-\theta')\delta(\phi-\phi')]_{|A'}\ , \tag{3.23i}$$

$$[\{\pi^A(x),\pi^B(x')\}^*]^\sim=\tfrac{1}{2}\eta^{AB}[\delta^{(3)}(x-x')]_{|0'}\ , \tag{3.23j}$$

$$[\{\pi^A(x),\pi^0(x')\}^*]^\sim=-\tfrac{1}{2}\eta^{AC}[\sin\theta\,\delta^{(3)}(x-x)]_{|C'}\ , \tag{3.23k}$$

$$[\{\pi^0(x),\pi^0(x')\}^*]^\sim=\tfrac{1}{2}\epsilon(u'-u)\{[\delta(\theta-\theta')\delta(\phi-\phi')]_{|B'}\sin\theta'\}_{|A'}\eta^{AB} \tag{3.23l}$$

on the $\widetilde{N}$ surface. The Db (a)—(f) are consistent with those obtained by Steinhardt[21] using his extension of the method of Dirac.[24] The brackets with constraints defined with functions $(\alpha,\widetilde{\alpha})$ which are finite on $N\cap\widetilde{N}$ are

$$\begin{aligned}\{A_A(x),C[\alpha_A,\widetilde{\alpha}_A]\}^*&=\tfrac{1}{2}\alpha_A(0,\theta,\phi),\quad x\in N\\ &=\tfrac{1}{2}\widetilde{\alpha}_A(u_0,\theta,\phi),\quad x\in\widetilde{N}\end{aligned} \tag{3.24a}$$

$$\{\pi^1(x),C[\alpha_A,\widetilde{\alpha}_A]\}^*=-\tfrac{1}{2}\eta^{AC}\alpha_{C\,||A}\sin\theta\ , \tag{3.24b}$$

$$\{\widetilde{\pi}^0(x),C[\alpha_A,\widetilde{\alpha}_A\}^*=-\tfrac{1}{2}\eta^{AC}\widetilde{\alpha}_{C\,||A}\sin\theta\ . \tag{3.24c}$$

In Eqs. (3.23) we could have omitted $\pi^A(x)$ from consideration and used the constraints to substitute for $\pi^A(x)$ everywhere except in the constraint appearing in the Hamiltonian. However, there is a considerable savings in computation by working directly with $\pi^A(x)$ as it appears in the gauge constraints as well as in the Hamiltonian.

Therefore, the Hamiltonian itself may be used in the form given in Eq. (3.14). Using the Dirac brackets, Eqs. (3.23), we find for the canonical field equations on N:

$$\{A_t(x),H\}^*=\dot{A}_t=\alpha\ , \tag{3.25a}$$

$$\{\pi(x),H\}^*=\dot{\pi}=-(\pi_{|1}+\pi^A{}_{|A})=0\ , \tag{3.25b}$$

$$\{A_1(x),H\}^*=\dot{A}_1(x)=(\sin\theta)^{-1}\pi'+A_{t\,|1}\ , \tag{3.25c}$$

$$\{\pi^1(x),H\}^*=\dot{\pi}^1(x)=\tfrac{1}{4}\left[\int_0^\rho+\int_\infty^\rho\right]d\rho'(\{\sin\theta[(\pi^1/\sin\theta)+A_{t\,|1}]_{|B}\}_{|A}\eta^{AB})+\tfrac{1}{2}[\sin\theta(A_{t\,|A}\eta^{AC}-\rho^2\pi^C/\sin\theta)]_{|C}\ , \tag{3.25d}$$

$$\{A_A(x),H\}^*=A_A(x)=-\tfrac{1}{4}\left[\int_0^\rho+\int_\infty^\rho\right]d\rho\{A_{1|A}-\eta_{AB}F^{BC}{}_{||C}\}+\tfrac{1}{2}[A_{t\,|A}=\rho^2(A_{1|t}-A_{A\,|1})]+\tfrac{1}{2}\alpha_A(0,\theta,\phi)\ , \tag{3.25e}$$

and on $\widetilde{N}$:

$$[\{A_t(x),H\}^*]^\sim=\dot{A}_t(x)=\alpha(x)\ , \tag{3.26a}$$

$$[\{\pi(x),H\}^*]^\sim=\dot{\widetilde{\pi}}(x)=[\pi^0{}_{|0}+\pi^A{}_{|A}]^\sim=0\ , \tag{3.26b}$$

$$[\{A_0(x),H\}^*]^\sim=\dot{\widetilde{A}}_0(x)=A_{t\,|0}(x)\ , \tag{3.26c}$$

$$[\{\pi^0(x),H\}^*]^\sim=\dot{\widetilde{\pi}}^0=-\pi^A{}_{|A}\ , \tag{3.26d}$$

$$[\{A_A(x),H\}^*]^\sim=\dot{\widetilde{A}}_A(x)=\widetilde{A}_{A\,|0}(x)\ . \tag{3.26e}$$

We note that Eqs. (3.25) and (3.26), with $\alpha_A(0,\theta,\phi)=\dot{A}_A(0,\theta,\phi)$, are the once-integrated Lagrangian equations for $A_A(x)$. In the absence of sources $\alpha_A(0,\theta,\phi)$ is determined by the field on N.

$$\alpha_A(0,\theta,\phi)=\tfrac{1}{2}\int_0^\infty d\rho[A_{1\,|A}-\eta_{AB}F^{BC}{}_{||C}]\ . \tag{3.27}$$

On the other hand, on $\widetilde{N}$, $\widetilde{\alpha}_A=\dot{A}_A=A_{A\,|0}$ which, of course, is data given on $\mathscr{I}^+$, the only condition being that $\widetilde{\alpha}_A(u_0,\theta,\phi)=\alpha_A(0,\theta,\phi)$. This data is required in order to determine the solution in the past.

From Eq. (3.25c) we substitute A_1 into the integrand on the right-hand side of (3.25d) and then use (3.25e) to obtain

$$\dot{\pi}^1=-[\sin\theta\eta^{AB}(A_A-A_{t\,|A})+\rho^2\pi^B]_{|B}\ ,$$

which agrees with the corresponding equation in Eqs. (3.17).

Thus we have demonstrated that by the introduction of the Dirac brackets the momenta $\pi^A(x)$ are effectively eliminated from the theory. One may also argue that since A_A appears only in the form F_{1A}, F_{0A}, and F_{AB}, any further developments toward a quantum theory should proceed only with these gauge-invariant quantities. To be sure A_t appears undifferentiated in the Hamiltonian, but there it is the coefficient of the gauge constraint. This, together with its equation of motion in Eqs. (3.25) and (3.26), indicate that it does not play a role in the dynamics of the field. Furthermore, up to now the gauge constraints, Eqs. (3.16), have not been taken into account. These may be used to determine π^1 and $\tilde{\pi}^0$ in terms of π^A and $\tilde{\pi}^A$. A_1 and $\tilde{A}_0$ are then fixed by Eqs. (3.25c) and (3.26c). This leaves only A_A or π^A as the independent dynamical degrees of freedom on $N \cup \tilde{N}$. However, the results we have obtained for the Dirac brackets do not depend on any choice of gauge, although the propagation equations do. We need not be concerned about this point here.

The passage to a quantum theory is made by identifying the commutators of variables with i times the Dirac brackets. In the usual formulation of the quantum electromagnetic field, the commutation relations among field variables are gauge dependent.[27] However, the commutation relations among gauge-invariant quantities can be shown to reduce on N and $\tilde{N}$ to those we would obtain with the Dirac brackets in Eqs. (3.25) and (3.26).

IV. CONCLUSION

We have shown that for a massless scalar field and for the electromagnetic field a Hamiltonian formalism can be constructed on a null cone extending out to $\mathscr{I}^+$, null infinity, provided one includes in the domain of definition of the Hamiltonian that portion of $\mathscr{I}^+$ which extends back to i^0, spacelike infinity. The null-surface constraints, which arise because the null surface is a characteristic surface, can be eliminated by introduction of the Dirac brackets. In a transition to a quantum theory the Dirac brackets go over to commutators which are consistent with the usual results. The quantum theory in this formalism will not be pursued because this study has been initiated in order to understand problems which may arise in the canonical formulation of the gravitational field on a null surface. This study is also underway by d'Inverno and Smallwood.[12] However, to set the gravitational equations on a null surface requires an algebraic coordinate condition, $g^{00}=0$. In the electromagnetic field the corresponding gauge condition in the electromagnetic field is $A_1=0$. The introduction of this gauge condition into the Lagrangian leads to some interesting results which will be described in a subsequent paper.

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[1]P. A. M. Dirac, Can. J. Math. **2**, 129 (1950).
[2]P. A. M. Dirac, Can. J. Math. **3**, 1 (1951).
[3]P. G. Bergmann, Phys. Rev. **75**, 680 (1949).
[4]F. A. E. Pirani and A. Schild, Phys. Rev. **79**, 986 (1950).
[5]B. S. DeWitt, in *Gravitation: An Introduction to Current Research,* edited by L. Witten (Wiley, New York, 1962).
[6]R. Arnowitt, S. Deser, and C. Misner (Ref. 5).
[7]*Quantum Gravity 2,* edited by C. Isham, R. Penrose, and D. Sciama (Clarendon, Oxford, 1981).
[8]H. Bondi, M. G. J. van der Burg, and A. W. K. Metzner, Proc. R. Soc. London **A269**, 21 (1962).
[9]R. K. Sachs, Proc. R. Soc. London **A270**, 103 (1962).
[10]E. T. Newman and R. Penrose, J. Math. Phys. **3**, 566 (1962).
[11]R. A. d'Inverno and J. Stachel, J. Math. Phys. **19**, 2447 (1978).
[12]R. A. d'Inverno and J. Smallwood, Phys. Rev. D **22**, 1233 (1980).
[13]J. Smallwood, J. Math. Phys. **24**, 599 (1983).
[14]A. Ashtekar, J. Math. Phys. **22**, 2885 (1981).
[15]R. Geroch, in *Asymptotic Structure of Space-Time,* edited by F. P. Esposito and L. Witten (Plenum, New York, 1977).
[16]R. Penrose, Proc. R. Soc. London **A284**, 159 (1965).
[17]F. Rohrlich, Acta Phys. Austriaca **32**, 87 (1970).
[18]F. Rohrlich, Acta Phys. Austriaca, Suppl. VIII, 277 (1971).
[19]J. B. Kogut and D. E. Soper, Phys. Rev. D **1**, 2901 (1970).
[20]J. D. Bjorken, J. B. Kogut, and D. E. Soper, Phys. Rev. D **3**, 1382 (1971).
[21]P. J. Steinhardt, Ann. Phys. (N.Y.) **128**, 425 (1980).
[22]A. J. Hanson, T. Regge, and C. Teitelboim, *Constrained Hamiltonian Systems* (Academia Nazionale dei Lincei, Rome, 1976).
[23]T. Regge and C. Teitelboim, Ann. Phys. (N.Y.) **88**, 286 (1974).
[24]P. A. M. Dirac, Proc. R. Soc. London **A246**, 333 (1958).
[25]P. A. M. Dirac, Phys. Rev. **114**, 924 (1959).
[26]P. G. Bergmann and A. Komar, Phys. Rev. Lett. **4**, 432 (1960).
[27]C. Itzykson and J.-B. Zuber, *Quantum Field Theory* (McGraw-Hill, New York, 1980).
[28]J. N. Goldberg, Report No. ARL 70-0003, 1970 (unpublished).
[29]R. Motill, Ph.D. thesis, Syracuse University, 1974.
[30]L. Eimers, Ph.D. thesis, Syracuse University, 1970.