Quantum mechanics as a classical theory

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The generator aspect of observables in classical mechanics leads naturally to a generalized classical mechanics, of which quantum mechanics is shown to be a particular case. Basic features of quantum mechanics follow, such as the identification of observables with self-adjoint operators, and canonical quantization rules. This point of view also gives a new insight on the geometry of quantum theory: Planck's constant is related for instance to the curvature of the quantum-mechanical space of states, and the uniqueness of quantum mechanics can be proved. Finally, the origin of the probabilistic interpretation is discussed.

I. INTRODUCTION

The motion of a quantum system is ruled by Schrödinger's equation $i\hbar d\,|\psi\rangle/dt=\hat{H}\,|\psi\rangle$ (we use standard notations and, in order to avoid confusion, we write a caret over operators).[1] Let $\{\,|\phi_k\rangle\}$ be an orthonormal basis of the Hilbert space, and λ_k the (complex) components of $|\psi\rangle$ in this basis: Schrödinger's equation reads then as a set of equations for the λ_k's.

Now decompose λ_k in real and imaginary parts, more precisely, set $\lambda_k=(x_k+ip_k)/\sqrt{2\hbar}$: We may write Schrödinger's equation as a set of equations for the x_k's and p_k's.

Then define the Hamiltonian function H as the mean value of $\hat{H}$, expressed in terms of the x_k's and p_k's. Assuming $|\psi\rangle$ is normalized, we have $H=\langle\psi\,|\,\hat{H}\,|\,\psi\rangle$. It is easy then to compute $\partial H/\partial p_k$ and $\partial H/\partial x_k$, and to show that Schrödinger's equation may be finally written as a set of Hamilton's equations,[2]

$$\frac{dx_k}{dt}=\frac{\partial H}{\partial p_k},\quad \frac{dp_k}{dt}=-\frac{\partial H}{\partial x_k}\ . \tag{1}$$

This is a striking result, since equations of motion in Hamiltonian form may be considered as the characteristic feature of classical mechanics.[3] Thus it suggests the possibility of a unified formulation for both classical and quantum mechanics. The purpose of this article is to exhibit such a unified formulation in an elementary way, and to explore some of its consequences.

The organization of the paper is as follows: In Sec. II, we propose a generalization of classical mechanics, which will prove able to include quantum mechanics as a particular case. In Sec. III, we investigate quantum mechanics from that point of view; our goal is to recover the essential features of the quantum-mechanical formalism in a natural and deductive way. Section IV is devoted to a more detailed study of the geometric structures involved in quantum mechanics, culminating with a geometrical interpretation of Planck's constant and a theorem on the uniqueness of quantum mechanics. Finally, in Sec. V, we discuss the origin of the usual probabilistic interpretation.

II. CLASSICAL MECHANICS AND ITS GENERALIZATION

A. Survey of classical mechanics

We refer for this introductory subsection to our previous paper;[4] more specific references are given below.

The evolution of a classical system takes place in its phase space: It is a space of even dimension, say $2n$, which is in fact the space of states of the system. In agreement with the experimental point of view, this leads to defining an observable as a real-valued regular function on that space.[5]

The space of states is provided with a Poisson bracket, i.e., with an operation $f,g\to\{f,g\}$ on the observables, which is linear, antisymmetric, nondegenerate, and satisfies the Jacobi identity and a derivation-like product formula.[6] One can prove[7] that there always exists (local) systems of coordinates on the space of states, say, (x_k,p_k), $k=1,\ldots,n$, such that $\{x_k,p_l\}=\delta_{kl}$, $\{x_k,x_l\}=\{p_k,p_l\}=0$, $k,l=1,\ldots,n$, where δ_{kl} is the Kronecker symbol, i.e., $\delta_{kl}=1$ for $k=l$, and $\delta_{kl}=0$ otherwise. Such coordinates are called canonical; they allow the Poisson bracket to be given the familiar form

$$\{f,g\}=\sum_k\left[\frac{\partial f}{\partial x_k}\frac{\partial g}{\partial p_k}-\frac{\partial f}{\partial p_k}\frac{\partial g}{\partial x_k}\right]. \tag{2}$$

The physical meaning of the Poisson bracket structure on the space of states is that the transformations of the states which do not modify the nature of the system, but merely correspond to a change of point of view, e.g., a rotation, a translation, a change of inertial frame, or the time evolution, preserve the Poisson bracket, i.e., are automorphisms of that structure. More precisely, let $\mathscr{G}$ be a transformation of the states; it induces naturally a transformation $f\to\mathscr{G}(f)$ of the observables. Then $\mathscr{G}$ is an automorphism of the space of states, provided with its Poisson bracket structure, if and only if, for any two observables f and g, we have

$$\mathscr{G}(\{f,g\})=\{\mathscr{G}(f),\mathscr{G}(g)\}\ . \tag{3}$$

A transformation $\mathscr{G}$ satisfying this condition is called canonical.[8]

The main result of classical mechanics may then be expressed as follows:[9] An infinitesimal transformation $\mathscr{G}$ (Ref. 10) of the space of states is an automorphism, i.e., is canonical, if and only if there exists some observable g such that, for any observable f, the transformation $f \rightarrow \mathscr{G}(f)$ reads $f \rightarrow f' = f + \{f,g\}\delta\alpha$, where $\delta\alpha$ is an infinitesimal real parameter. In terms of canonical coordinates, we get

$$\begin{aligned} x_k &\rightarrow x_k' = x_k + \frac{\partial g}{\partial p_k}\delta\alpha \ , \\ p_k &\rightarrow p_k' = p_k - \frac{\partial g}{\partial x_k}\delta\alpha \ . \end{aligned} \tag{4}$$

The observable g in (4) is called the generator of the transformation $\mathscr{G}$. Thus, there is in classical mechanics a correspondence between observables and infinitesimal automorphisms of the space of states. This correspondence is fundamental, since many important observables are defined in fact by the transformation they generate. For instance, the energy H, i.e., the Hamiltonian function, is defined as the generator of time evolution: Eqs. (4) simply reduce in that case to Hamilton's equations of motion (1).[11]

B. Generalized classical mechanics

The importance of the generator aspect of observables in classical mechanics is illustrated in Ref. 4. As is well known, that aspect also exists in quantum mechanics: This suggests retaining that aspect as the building block for a generalized classical mechanics which is to include quantum mechanics as a particular case. Since the correspondence between observables and infinitesimal automorphisms of the classical space of states rests on the properties of the Poisson bracket, we are led to assume the existence of some underlying Poisson bracket structure on the space of states of our generalized classical system. More precisely:[12]

(a) The space of states is provided with a structure whose physical meaning is that the transformations of the states which correspond to a change of point of view preserve that structure, i.e., are the automorphisms of the space of states.

(b) We assume that this structure intrinsically induces a Poisson bracket structure: *The space of states is thus a classical phase space, provided in the general case with some complementary structure.*

(c) The word "intrinsically" in (b) means precisely that the automorphisms of the space of states preserve the Poisson bracket, i.e., are canonical transformations. The converse is not true in general: Not every canonical transformation is an automorphism, i.e., also preserves the complementary structure. *We define then the observables as those real-valued regular functions of the state, whose canonical transformations they generate are automorphisms of the whole structure of the space of states.*

The usual classical mechanics is characterized by the fact that there is no complementary structure beside the Poisson bracket and, therefore, any real-valued function of the state is an observable. This is not true in the general case. It can be shown, however, that the set of observables is closed under addition, product by a scalar, and the Poisson bracket.[13] But the usual product of two observables, defined by the product of their values, no longer needs to be an observable.

Notice also that complications may arise in the general case with the definition of the states: Since not every real-valued regular function of the state is an observable, there may not be enough observables to distinguish between any two states. In such a case, these two states must of course be considered as identical: This gives rise to a redefinition of the space of states.[14]

III. QUANTUM MECHANICS

A. Poisson bracket structure

We now prove that quantum mechanics is a particular case of the generalized classical mechanics of Sec. II B, by showing the existence of an intrinsic Poisson bracket structure on the quantum-mechanical space of states.

Consider a quantum system, and let $\{ \mid \phi_k \rangle \}$ be an orthonormal basis of its Hilbert space of states. For any state vector $\mid \psi \rangle$, we may write

$$\mid \psi \rangle = \sum_k \lambda_k \mid \phi_k \rangle \ ,$$

where the λ_k's are complex numbers. Decompose then λ_k in real and imaginary parts; more precisely, set

$$\lambda_k = (x_k + ip_k)/\sqrt{2\hbar} \ . \tag{5}$$

As is well known, the states of the system are not the vectors themselves, but the rays of the Hilbert space, i.e., the nonzero vectors up to a multiplicative factor. In other words, a state may be represented by a normalized vector, which is then fixed up to a phase factor. We shall ignore for the moment these important features, which will be discussed in Sec. IV: Thus the x_k's and p_k's will be considered as real coordinates of the state, the normalization condition and the phase arbitrariness being treated as *ad hoc* additional requirements.

The result of the Introduction (Sec. I) suggests then that the x_k's and p_k's are canonical coordinates. Therefore, the Poisson bracket of two real-valued regular functions f and g of these variables will be defined by formula (2).

We now have to prove that this Poisson bracket is an intrinsic part of the quantum structure. In fact, we do not need to know precisely for the moment what this structure is: It will be sufficient to prove that this Poisson bracket is invariant under the automorphisms of quantum mechanics. Recall now that automorphisms represent changes of point of view: Hence the automorphisms of quantum mechanics are the unitary transformations of the Hilbert space.[15]

So, let U be a unitary transformation, represented by an operator $\hat{U}$,

$$\mid \psi \rangle \rightarrow \mid \psi' \rangle = \hat{U} \mid \psi \rangle \ .$$

The components λ_k of $|\psi\rangle$ in the orthonormal basis $\{|\phi_k\rangle\}$ change under U according to

$$\lambda_k \to \lambda_k' = \sum_l U_{kl}\lambda_l , \tag{6}$$

where $U_{kl}=\langle\phi_k|\hat{U}|\phi_l\rangle$. Let R_{kl} and I_{kl} be, respectively, the real and imaginary parts of U_{kl}, i.e., $U_{kl}=R_{kl}+iI_{kl}$. Replacing in (6) and making use of (5), we obtain, when we separate the real and imaginary parts, the transformation law of the x_k's and p_k's under U:

$$\begin{aligned} x_k \to x_k' &= \sum_l (R_{kl}x_l - I_{kl}p_l) , \\ p_k \to p_k' &= \sum_l (R_{kl}p_l + I_{kl}x_l) . \end{aligned} \tag{7}$$

From (2) and (7), we deduce

$$\begin{aligned} \{x_k',p_l'\} &= \{\sum_j (R_{kj}x_j - I_{kj}p_j), \sum_m (R_{lm}p_m + I_{lm}x_m)\} \\ &= \sum_{j,m} (R_{kj}R_{lm}\delta_{jm} + I_{kj}I_{lm}\delta_{jm}) \\ &= \sum_j (R_{kj}R_{lj} + I_{kj}I_{lj}) . \end{aligned}$$

But, since U is unitary, we have $\hat{U}\hat{U}^\dagger=1$ (where the dagger means Hermitian conjugation), which gives (an asterisk denotes complex conjugation)

$$\begin{aligned} \delta_{kl} &= \langle\phi_k|\phi_l\rangle = \langle\phi_k|\hat{U}\hat{U}^\dagger|\phi_l\rangle \\ &= \sum_j U_{kj}U_{jl}^\dagger = \sum_j U_{kj}U_{lj}^* \\ &= \sum_j (R_{kj}+iI_{kj})(R_{lj}-iI_{lj}) \\ &= \sum_j (R_{kj}R_{lj}+I_{kj}I_{lj}) + (\text{imaginary part}) . \end{aligned}$$

Since δ_{kl} is real, the imaginary part is zero, hence $\{x_k',p_l'\}=\delta_{kl}$. In the same way, using again $\hat{U}\hat{U}^\dagger=1$ or $\hat{U}^\dagger\hat{U}=1$, we obtain $\{x_k',x_l'\}=\{p_k',p_l'\}=0$: The unitary transformation U preserves the Poisson brackets of the x_k's and p_k's. It is then a standard result[16] that, more generally, for any f and g,

$$U(\{f,g\}) = \{U(f),U(g)\} .$$

This means that U is a canonical transformation [formula (3)], and hence expresses the intrinsic character of the Poisson bracket we have defined. Quantum mechanics is thus a particular case of the generalized classical mechanics of Sec. II B.[17]

B. Observables in quantum mechanics

Since the set of quantum states is provided with an intrinsic Poisson bracket structure, we may apply to quantum mechanics the definition of observables given in Sec. II B: The observables are those real-valued regular functions of the state whose canonical transformations they generate are automorphisms of the whole quantum structure, i.e., are unitary transformations.

But an infinitesimal unitary transformation reads

$$\hat{U} = 1 - \frac{i}{\hbar}\hat{g}\delta\alpha , \tag{8}$$

where $\hat{g}$ is a self-adjoint operator: Thus there is in quantum mechanics some correspondence between observables and self-adjoint operators.

More precisely, the λ_k's transform under the unitary transformation (8) according to

$$\lambda_k \to \lambda_k' = \lambda_k - \frac{i}{\hbar}\sum_l g_{kl}\lambda_l\delta\alpha , \tag{9}$$

where $g_{kl}=\langle\phi_k|\hat{g}|\phi_l\rangle$. From the self-adjointness of $\hat{g}$, we have $g_{kl}^*=g_{lk}$, hence the decomposition of g_{kl} in real and imaginary parts may be written

$$g_{kl} = \frac{g_{kl}+g_{lk}}{2} + i\frac{g_{kl}-g_{lk}}{2i} . \tag{10}$$

Replacing in (9) and separating the real and imaginary parts we get, using (5),

$$\begin{aligned} x_k \to x_k' &= x_k + \sum_l \left[\frac{g_{kl}+g_{lk}}{2\hbar}p_l + \frac{g_{kl}-g_{lk}}{2i\hbar}x_l\right]\delta\alpha , \\ p_k \to p_k' &= p_k - \sum_l \left[\frac{g_{kl}+g_{lk}}{2\hbar}x_l - \frac{g_{kl}-g_{lk}}{2i\hbar}p_l\right]\delta\alpha . \end{aligned} \tag{11}$$

Consider now the function $\langle\psi|\hat{g}|\psi\rangle$. We have

$$\begin{aligned} \langle\psi|\hat{g}|\psi\rangle &= \sum_{k,l} g_{kl}\lambda_k^*\lambda_l \\ &= \sum_{k,l} g_{kl}\frac{x_k-ip_k}{\sqrt{2\hbar}}\frac{x_l+ip_l}{\sqrt{2\hbar}} \\ &= \sum_{k,l} g_{kl}\frac{(x_kx_l+p_kp_l)+i(x_kp_l-p_kx_l)}{2\hbar} . \end{aligned}$$

Hence,

$$\begin{aligned} \frac{\partial}{\partial p_k}\langle\psi|\hat{g}|\psi\rangle &= \sum_l \left[\frac{g_{kl}+g_{lk}}{2\hbar}p_l + \frac{g_{kl}-g_{lk}}{2i\hbar}x_l\right] , \\ \frac{\partial}{\partial x_k}\langle\psi|\hat{g}|\psi\rangle &= \sum_l \left[\frac{g_{kl}+g_{lk}}{2\hbar}x_l - \frac{g_{kl}-g_{lk}}{2i\hbar}p_l\right] . \end{aligned} \tag{12}$$

Comparing with (11) and with the transformation law (4) of canonical coordinates under the canonical transformation generated by an observable g, we see that, up to an additive constant,[18]

$$g(x_k,p_k) = \langle\psi|\hat{g}|\psi\rangle . \tag{13}$$

This is the desired relation between observables and self-adjoint operators. Because of the phase arbitrariness, the operator $\hat{g}$ in (8) is defined up to an additive constant: Thus, it is possible to fix that constant so that (13) holds exactly. Since $|\psi\rangle$ is normalized, we recover thus, with a new interpretation, the formula for the mean value of a quantum observable. Said another way, what we have shown in this subsection is that *a real-valued regular function g of the state is an observable if and only if there exists some self-adjoint operator $\hat{g}$ such that (13) holds.*[19]

C. Algebra of quantum observables

It follows from formula (13) that the correspondence between the observable g and the self-adjoint operator $\hat{g}$ is linear and also, because $|\psi\rangle$ is normalized, that $\hat{1}$ is the identity operator. Thus the addition and product by a scalar of observables are represented by the same operations on operators. But, as anticipated in Sec. II B, the product of two observables is not in general an observable: That operation is not relevant in quantum mechanics. We shall now express the Poisson bracket in terms of operators.

Let g be an observable, and $\hat{g}$ the corresponding self-adjoint operator. From (5) and (10) we have

$$\mathrm{Re}(g_{kl}\lambda_l)=\frac{g_{kl}+g_{lk}}{2\sqrt{2\hbar}}x_l-\frac{g_{kl}-g_{lk}}{2i\sqrt{2\hbar}}p_l\ ,$$

$$\mathrm{Im}(g_{kl}\lambda_l)=\frac{g_{kl}+g_{lk}}{2\sqrt{2\hbar}}p_l+\frac{g_{kl}-g_{lk}}{2i\sqrt{2\hbar}}x_l\ ,$$

where Re and Im mean, respectively, the real and imaginary parts. Comparing with (12) and making use of (13), we get

$$\frac{\partial g}{\partial x_k}=\left[\frac{2}{\hbar}\right]^{1/2}\mathrm{Re}\left[\sum_l g_{kl}\lambda_l\right],$$

$$\frac{\partial g}{\partial p_k}=\left[\frac{2}{\hbar}\right]^{1/2}\mathrm{Im}\left[\sum_l g_{kl}\lambda_l\right].$$

This allows us to write

$$\begin{aligned}\{f,g\}&=\sum_k\left[\frac{\partial f}{\partial x_k}\frac{\partial g}{\partial p_k}-\frac{\partial f}{\partial p_k}\frac{\partial g}{\partial x_k}\right]\\
&=\frac{2}{\hbar}\sum_k\left[\mathrm{Re}\left[\sum_l f_{kl}\lambda_l\right]\mathrm{Im}\left[\sum_j g_{kj}\lambda_j\right]-\mathrm{Re}\left[\sum_j g_{kj}\lambda_j\right]\mathrm{Im}\left[\sum_l f_{kl}\lambda_l\right]\right]\\
&=\frac{2}{\hbar}\sum_k\frac{1}{2i}\left[\left[\sum_l f_{kl}\lambda_l\right]^*\left[\sum_j g_{kj}\lambda_j\right]-(\text{complex conjugate})\right]\\
&=\frac{1}{i\hbar}\sum_{k,l,j}(f_{lk}\lambda_l^*g_{kj}\lambda_j-f_{kl}\lambda_l g_{jk}\lambda_j^*)\\
&=\frac{1}{i\hbar}\sum_{l,j}[(\hat{f}\hat{g})_{lj}\lambda_l^*\lambda_j-(\hat{g}\hat{f})_{jl}\lambda_j^*\lambda_l]\\
&=\frac{1}{i\hbar}\sum_{l,j}[\hat{f},\hat{g}]_{lj}\lambda_l^*\lambda_j\\
&=\langle\psi|[\hat{f},\hat{g}]/i\hbar|\psi\rangle\ ,\end{aligned}$$

where $[\hat{f},\hat{g}]=\hat{f}\hat{g}-\hat{g}\hat{f}$ is the commutator. This proves that the operator version of the Poisson bracket is

$$\widehat{\{f,g\}}=[\hat{f},\hat{g}]/i\hbar\ . \tag{14}$$

Thus, $[\ ,\ ]/i\hbar$ is not only the quantum analog of the Poisson bracket: *It is in fact a true Poisson bracket.* This result strongly supports the so-called "canonical quantization rules."

IV. GEOMETRY OF QUANTUM MECHANICS

A. Normalization and phase arbitrariness

Up to now, the structure of the quantum-mechanical space of states has not been explicit: It has been present only through its automorphisms, i.e., unitary transformations, and through the requirements of normalization and phase arbitrariness. We now study these two requirements.

The normalization condition means that part of the complementary structure of the space of states (in the sense of Sec. II B) consists of a constraint. More precisely, using (5), we have

$$\langle\psi|\psi\rangle=\sum_k|\lambda_k|^2=\sum_k\frac{x_k^2+p_k^2}{2\hbar}\ ,$$

and the normalization condition $\langle\psi|\psi\rangle=1$ is

$$\sum_k(x_k^2+p_k^2)=2\hbar\ . \tag{15}$$

Let g be an observable. Since (15) is part of the structure of the space of states, it must be preserved under the infinitesimal automorphism generated by g. Now, using (4), we obtain the transformation law

$$\sum_k (x_k{}^2+p_k{}^2) \to \sum_k (x_k'{}^2+p_k'{}^2)$$
$$= \sum_k (x_k{}^2+p_k{}^2) + 2\sum_k \left[\frac{\partial g}{\partial p_k}x_k - \frac{\partial g}{\partial x_k}p_k\right]\delta\alpha \ .$$

Thus, g may be an observable only if

$$\sum_k \left[\frac{\partial g}{\partial p_k}x_k - \frac{\partial g}{\partial x_k}p_k\right] = 0 \ . \tag{16}$$

This condition means the following: Consider the infinitesimal phase transformation

$$|\psi\rangle \to |\psi'\rangle = \exp(i\delta\alpha)|\psi\rangle \ , \tag{17}$$

i.e.,

$$\lambda_k \to \lambda_k' = \exp(i\delta\alpha)\lambda_k = (1+i\delta\alpha)\lambda_k \ .$$

Under this transformation, the x_k's and p_k's transform according to

$$x_k \to x_k' = x_k - p_k\delta\alpha \ ,$$
$$p_k \to p_k' = p_k + x_k\delta\alpha \ .$$

Hence, to first order in $\delta\alpha$,

$$g(x_k',p_k') = g(x_k - p_k\delta\alpha,\ p_k + x_k\delta\alpha)$$
$$= g(x_k,p_k) + \sum_k \left[\frac{\partial g}{\partial p_k}x_k - \frac{\partial g}{\partial x_k}p_k\right]\delta\alpha \ .$$

Equation (16) is thus equivalent to $g(x_k',p_k') = g(x_k,p_k)$: The value of an observable is invariant under the infinitesimal transformation (17), hence, under any finite phase transformation. In other words, any observable has the same value on state vectors which differ from a phase factor. Therefore, two such state vectors must be considered as representations of the same state: This is just the requirement of phase arbitrariness.

Of course this requirement could have been deduced directly, starting from the characterization of observables obtained in Sec. III B [formula (13)]. The interesting point in the above derivation is that it shows clearly that the phase arbitrariness is related to the normalization condition.[20] This derivation also shows that the normalization condition is not the whole complementary structure of the space of states, since not any function satisfying (16) is an observable: For instance, the product of two functions satisfying (16) has the same property, whereas we know that the product of two observables is not an observable in general. Thus there exists a remaining structure on the space of states, which we shall now show to be a Riemannian metric.

B. Intrinsic formulation

The normalization condition and its consequence, the phase arbitrariness, means that the true space of states is not the Hilbert space itself, but its space of rays, which can be identified with what mathematicians call a complex projective space.

Suppose the Hilbert space has finite (complex) dimension n. As a real space, it has then even dimension $2n$. The normalization condition reduces this dimension by 1. Because of the phase arbitrariness, the dimension is again reduced by 1. Thus the complex projective space also has even dimension, a necessary condition for itself to carry a Poisson bracket structure. The question then naturally arises whether it is possible to work directly on the complex projective space, getting rid of normalization and phase arbitrariness.

The answer is positive: All the results of Sec. III, mainly formulas (13) and (14), may be recovered as well by working on the projective space only. Details are given in Ref. 21: we just present here a brief outline.

The automorphisms of the (true) space of states, i.e., of the complex projective space, are the transformations of that space which are induced by the unitary transformations of the Hilbert space. But the unitary transformations may be defined as those transformations which preserve the Hermitian scalar product $\langle \ | \ \rangle$ (Ref. 22). Thus we may say that the structure of the space of states is induced by that scalar product.

Because of the Hermiticity, the imaginary part of the scalar product is antisymmetric. As part of the geometry of the complex projective space, it may be considered as an antisymmetric covariant tensor ω,

$$\omega = \sum_{\alpha,\beta} \omega_{\alpha\beta}\, dx^\alpha \otimes dx^\beta \ ,$$

where the x^α's are coordinates on the complex projective space. Alternatively, ω can be identified with a Poisson bracket structure, through the formula

$$\{f,g\} = \sum_{\alpha,\beta} \omega^{\alpha\beta}\frac{\partial f}{\partial x^\alpha}\frac{\partial g}{\partial x^\beta} \ ,$$

where $\omega^{\alpha\beta}$ is the inverse of $\omega_{\alpha\beta}$.[23] This is the starting point for a derivation of formulas (13) and (14) within the intrinsic approach.

The complementary structure of the space of states is then clearly the real part of the scalar product. It is a symmetric covariant tensor g,

$$g = \sum_{\alpha,\beta} g_{\alpha\beta}\, dx^\alpha \otimes dx^\beta \ ;$$

more precisely, g is shown to be a Riemannian metric. Formula (15) suggests then a simple geometrical interpretation of Planck's constant, which is confirmed by calculation: Up to numerical factors, it is the squared radius of curvature of the space of states, i.e., the inverse of the Riemannian scalar curvature.

C. Uniqueness of quantum mechanics

We just saw that quantum mechanics is a generalized classical mechanics, in which the complementary structure of the space of states is a Riemannian metric. Conversely, one may ask whether any such generalized mechanics necessarily coincides with quantum mechanics.

The answer is positive, under the additional assumptions that the space of states is connected, simply connected, homogeneous, isotropic, and that its scalar curvature (which is constant by homogeneity) is strictly positive.[24] We just give here an idea of the proof.[25]

Because of homogeneity and isotropy, the Riemannian and Poisson bracket structures of the space of states are not independent: One can prove that they are, respectively, something like the real and imaginary parts of a complex Riemannian metric; more precisely, the space of states is what mathematicians call a Kählerian manifold.

The desired result follows then from the fact that any connected, simply connected, homogeneous, and isotropic Kählerian manifold with strictly positive scalar curvature is a complex projective space.[26] The theorem invoked here is the Kählerian analog of that intuitive result of Riemannian geometry which states that, under suitable additional requirements (mainly of symmetry) a Riemannian manifold with strictly positive scalar curvature is a sphere.

Thus we may conclude that the only difference between classical and quantum mechanics lies in the existence, in the quantum case, of a Riemannian structure on the space of states. In other words, there exists between quantum states a distance which is intrinsic, i.e., invariant under changes of point of view, whereas there is only the weaker notion of neighborhood between classical states. It is not surprising that Planck's constant is related to the specific feature of quantum mechanics, i.e., to the Riemannian metric.

Notice also that the linear character of quantum mechanics, despite its practical importance, may be considered as an accident: It is only a consequence of the fact that a complex projective space happens to be the space of rays of some Hilbert space.

V. PROBABILISTIC INTERPRETATION

A. Nonseparability and indeterminism

Let g be an observable, and $\hat{g}$ the corresponding self-adjoint operator. From (13), the value of g when the system is in the state described by the normalized vector $|\psi\rangle$ is $\langle\psi|\hat{g}|\psi\rangle$. The point of view of Sec. II B suggests that this value has the same objective character as in ordinary classical mechanics. We shall now explain the origin of the usual probabilistic interpretation of quantum mechanics, according to which $\langle\psi|\hat{g}|\psi\rangle$ is only the mean value of measurements of g (Ref. 27).

Consider an ensemble of two systems. A pair of states of these systems defines a state of the ensemble. Therefore, the Cartesian product of the spaces of states of the two systems is included in the space of states of the ensemble.

In classical mechanics, the Poisson bracket structures on the spaces of states of the two systems induce naturally a Poisson bracket structure on their Cartesian product.[28] Hence this Cartesian product may be taken as the space of states of the ensemble, i.e., the inclusion is in fact an equality. This means that any state of the ensemble can be decomposed in states of the two systems.

The situation is very different in quantum mechanics, because then the spaces of states are complex projective spaces. The Cartesian product of two such spaces is not itself a complex projective space and, therefore, the inclusion is now strict. This means that there exist states of the ensemble which are not decomposable in states of the two systems, a feature of quantum mechanics called nonseparability.

More precisely, it is well known that the Hilbert space of the ensemble is the tensorial product of the Hilbert spaces of the two systems: Let $|\psi\rangle$ and $|\chi\rangle$ be the state vectors of the two systems. Then the state vector of the ensemble is the tensorial product $|\psi\rangle\otimes|\chi\rangle$. Any state vector of the ensemble can be written as a linear combination of such tensorial products, and is not decomposable unless it is itself a tensorial product.

A particular case is that in which one of the systems is a measuring apparatus operating on the other system. Let $|\psi\rangle$ and $|\chi_0\rangle$ be, respectively, the states of the system and of the apparatus before the measurement. The state of the ensemble is then $|\psi\rangle\otimes|\chi_0\rangle$, but the state after the measurement, say, $U(|\psi\rangle\otimes|\chi_0\rangle)$, takes in general the form

$$U(|\psi\rangle\otimes|\chi_0\rangle)=\sum_{k,l}\lambda_{kl}\,|\psi_k\rangle\otimes|\chi_l\rangle\ ,$$

i.e., is not decomposable in a state of the measured system and a state of the apparatus. Thus the notion of a state of the apparatus after the measurement is not defined in general. This shows that the nonseparability of quantum mechanics implies some kind of indeterminism in the results of measurements.

This analysis leads to the following definition: We shall say that an observable g of the system has a determined value in the state $|\psi\rangle$ if and only if the state of the ensemble after a measurement of g is decomposable, i.e., takes the form

$$U(|\psi\rangle\otimes|\chi_0\rangle)=|\psi'\rangle\otimes|\chi'\rangle\ .$$

B. Deterministic measurements

It seems natural to impose on the transformation U, which describes the measurement of g, the following requirements:

(a) U represents the time evolution of the ensemble during the measurement process. Hence it is a unitary transformation, with operator $\hat{U}$.

(b) An ideal measurement should disturb the system as little as possible. This leads us to impose that U preserve the value of g. Since U is unitary, this reads, with $|\Phi\rangle$ = state of the ensemble,

$$\begin{aligned}\langle\Phi|\hat{g}|\Phi\rangle&=\langle\hat{U}\Phi|\hat{g}|\hat{U}\Phi\rangle\\&=\langle\Phi|\hat{U}^{\dagger}\hat{g}\hat{U}|\Phi\rangle=\langle\Phi|\hat{U}^{-1}\hat{g}\hat{U}|\Phi\rangle\end{aligned}$$

for any $|\Phi\rangle$, hence $\hat{g}=\hat{U}^{-1}\hat{g}\hat{U}$, i.e., $[\hat{U},\hat{g}]=0$.

(c) A good apparatus should discriminate between distinct values of g: If $|\psi_1\rangle$ and $|\psi_2\rangle$ are states of the system in which g has determined and distinct values, i.e.,

$$\hat{U}|\psi_1\rangle\otimes|\chi_0\rangle=|\psi_1'\rangle\otimes|\chi_1'\rangle\ ,$$

$$\hat{U}|\psi_2\rangle\otimes|\chi_0\rangle=|\psi_2'\rangle\otimes|\chi_2'\rangle\ ,$$

then $|\chi'_1\rangle$ and $|\chi'_2\rangle$ represent distinct states of the apparatus, and are therefore linearly independent.

For the sake of simplicity, we now assume that $\hat{g}$ has a purely discrete and nondegenerate spectrum. Let g_k and $|\phi_k\rangle$ be, respectively, its eigenvalues and normalized eigenvectors: $\hat{g}|\phi_k\rangle = g_k|\phi_k\rangle$, and $\{|\phi_k\rangle\}$ is an orthonormal basis of the Hilbert space of the system.

From condition (b), it follows that

$$\begin{aligned}\hat{g}\hat{U}|\phi_k\rangle\otimes|\chi_0\rangle &= \hat{U}\hat{g}|\phi_k\rangle\otimes|\chi_0\rangle\\ &= \hat{U}g_k|\phi_k\rangle\otimes|\chi_0\rangle\\ &= g_k\hat{U}|\phi_k\rangle\otimes|\chi_0\rangle\ .\end{aligned}$$

$\hat{U}|\phi_k\rangle\otimes|\chi_0\rangle$ is thus an eigenvector of $\hat{g}$, with eigenvalue g_k. Since the spectrum of $\hat{g}$ is nondegenerate, this implies the existence of states $|\chi_k\rangle$ of the apparatus such that

$$\hat{U}|\phi_k\rangle\otimes|\chi_0\rangle = |\phi_k\rangle\otimes|\chi_k\rangle\ .$$

Thus g has in the state $|\phi_k\rangle$ the determined value

$$\langle\phi_k|\hat{g}|\phi_k\rangle = g_k\ .$$

Now let $|\psi\rangle$ be any state of the system. We may write

$$|\psi\rangle = \sum_k \lambda_k|\phi_k\rangle\ .$$

Since U is unitary [condition (a)], $\hat{U}$ is a linear operator, hence

$$\begin{aligned}\hat{U}|\psi\rangle\otimes|\chi_0\rangle &= \hat{U}\sum_k\lambda_k|\phi_k\rangle\otimes|\chi_0\rangle\\ &= \sum_k\lambda_k\hat{U}|\phi_k\rangle\otimes|\chi_0\rangle\\ &= \sum_k\lambda_k|\phi_k\rangle\otimes|\chi_k\rangle\ .\end{aligned}\tag{18}$$

Suppose g has a determined value in state $|\psi\rangle$. Then $\hat{U}|\psi\rangle\otimes|\chi_0\rangle$ takes the form $|\psi'\rangle\otimes|\chi'\rangle$, i.e.,

$$\sum_k\lambda_k|\phi_k\rangle\otimes|\chi_k\rangle = |\psi'\rangle\otimes|\chi'\rangle\ .$$

Take the (partial) scalar product with $|\phi_l\rangle$. Making use of $\langle\phi_l|\phi_k\rangle = \delta_{kl}$, we get

$$\lambda_l|\chi_l\rangle = \langle\phi_l|\psi'\rangle|\chi'\rangle\ ,$$

i.e., whenever λ_l is nonzero, $|\chi_l\rangle$ is proportional to the fixed vector $|\chi'\rangle$. But, from condition (c), the $|\chi_k\rangle$'s are linearly independent, hence this implies that all λ_l's are zero except one ($|\psi\rangle$ being nonzero). Therefore, $|\psi\rangle$ is one of the $|\phi_k\rangle$'s, i.e., it is one of the eigenvectors of $\hat{g}$.

Thus we have shown that g has a determined value in state $|\psi\rangle$ if and only if $|\psi\rangle$ is an eigenvector of $\hat{g}$, the eigenvalue being the value of g.

C. Nondeterministic measurements

The vector $|\chi_k\rangle$ represents the state of the apparatus when the result of the measurement is the determined value g_k. In the general case, where the state $|\psi\rangle$ of the system is not an eigenvector of $\hat{g}$, the state of the ensemble after the measurement is given by (18), and the state of the apparatus alone is no longer defined.

To understand what this means, consider an observable f of the apparatus, and let $\hat{f}$ be the corresponding self-adjoint operator. The value of f after the measurement, say, $\langle f\rangle$, is

$$\begin{aligned}\langle f\rangle &= \sum_{k,l}\lambda_l^*\lambda_k(\langle\phi_l|\otimes\langle\chi_l|)\hat{f}(|\phi_k\rangle\otimes|\chi_k\rangle)\\ &= \sum_{k,l}\lambda_l^*\lambda_k\langle\phi_l|\phi_k\rangle\langle\chi_l|\hat{f}|\chi_k\rangle\\ &= \sum_{k,l}\lambda_l^*\lambda_k\delta_{kl}\langle\chi_l|\hat{f}|\chi_k\rangle\\ &= \sum_k|\lambda_k|^2\langle\chi_k|\hat{f}|\chi_k\rangle\ .\end{aligned}$$

$\langle\chi_k|\hat{f}|\chi_k\rangle$ is the value of f when the apparatus is in the state $|\chi_k\rangle$. Thus this formula shows that, after the measurement, the apparatus may be described as a statistical mixture of the states $|\chi_k\rangle$, with probabilities $|\lambda_k|^2$. In that sense, we may say that the result of the measurement of g is g_k with probability $|\lambda_k|^2$, in agreement with the usual probabilistic interpretation of quantum mechanics. In particular, since

$$\sum_k|\lambda_k|^2 g_k = \langle\psi|\hat{g}|\psi\rangle\ ,$$

what we called the (objective) value of g appears experimentally as the mean value of measurements of g.

VI. CONCLUDING REMARKS

Most textbooks introduce quantum mechanics in an axiomatic way, and insist on its departure from classical mechanics whose foundations, considered as intuitive, do not need to be questioned.

We have shown in this paper how a deeper study of classical mechanics (Poisson bracket structure and the generator aspect of observables) results in a natural generalization of the classical scheme, which allows us to deduce the quantum axioms, including those dealing with the probabilistic interpretation: Quantum mechanics may be considered as a classical theory, in which a distance between states is defined, the natural unit of distance being fixed by the magnitude of Planck's constant.

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[1]All features of quantum mechanics needed in this paper can be found, for instance, in A. Messiah, *Quantum Mechanics* (Wiley, New York, 1958).

[2]The calculation is carried out in more detail and in a broader context in Sec. III B. Notice that the Hamiltonian function H is quadratic in the x_k's and p_k's. To reduce it to a simple form, take for $\{|\phi_k\rangle\}$ a proper basis of $\hat{H}$. Then $H=\sum E_k|\lambda_k|^2=\sum E_k(p_k{}^2+x_k{}^2)/2\hbar$, where the E_k's are the energy levels. H appears thus as a sum of Hamiltonian functions of harmonic oscillators with pulsations $\omega_k = E_k/\hbar$.

[3]Hamilton's equations express the existence of an underlying Poisson bracket (or symplectic) structure. See Sec. II A and

references therein.

[4]A. Heslot, Am. J. Phys. **51**, 1096 (1983).

[5]More precisely, the space of states is a smooth manifold, and regular means smooth. Notice that we exclude the case of explicitly time-dependent observables.

[6]Such a structure is called symplectic by mathematicians. The even dimension of the space of states is a necessary condition for that structure to exist. For details on the Poisson bracket, see, for instance, R. J. Finkelstein, *Nonrelativistic Mechanics* (Benjamin, New York, 1973), Chap. 1.

[7]This is the Darboux theorem. See for a proof V. I. Arnold, *Mathematical Methods in Classical Mechanics* (Mir, Moskow, 1975) [Also published as Springer Graduate Texts in Mathematics, No. 60 (Springer, New York, 1978)], Chap. 8.

[8]We exclude the case of time-dependent canonical transformations. For a proof of the equivalence with the usual definition of canonical transformations (also called symplectomorphisms), see Arnold (Ref. 7).

[9]See, for instance, Finkelstein (Ref. 6), Chap. 3.

[10]The reason for this restriction to infinitesimal transformations is that the set of canonical transformations is a Lie group: Up to topological details, such a group is characterized by its Lie algebra, i.e., by its infinitesimal elements.

[11]Other important examples are the components of the linear (angular) momentum, defined as the generators of translations (rotations).

[12]A. Heslot, in *Dynamical Systems and Microphysics: Geometry and Mechanics,* edited by A. Avez, A. Blaquière, and A. Marzollo (Academic, New York, 1982).

[13]This is a consequence of the fact that the set of observables is a Lie algebra under the Poisson bracket, closely related to the Lie algebra of the group of automorphisms of the space of states. See Heslot (Ref. 12).

[14]This happens, for instance, in the Hamiltonian description of the classical electromagnetic field, starting from a Lagrangian description in terms of potentials, with no choice of gauge. The conjugate variables of the potentials obey, then, constraint relations, which play the role of a complementary structure on the space of states. Our definition identifies observables with gauge-invariant quantities, and states which differ from a gauge transformation are thus equivalent. See for details, A. Heslot, Thèse, de troisième cycle, Université Paris VI, 1979.

[15]We do not consider antiunitary transformations which describe, for instance, time reversal as true automorphisms. The situation is very similar to that of ordinary classical mechanics, where time reversal changes the sign of the Poisson bracket, and hence is not a true canonical transformation.

[16]See, for instance, Arnold (Ref. 7).

[17]The tool used in Sec. III A, i.e., decomposition in real and imaginary parts, "explains" the occurrence of complex numbers in quantum mechanics: They ensure the even real dimension of the space of states which is necessary for it to carry a Poisson bracket structure.

[18]The result of the Introduction (Sec. I) is recovered with $g = H$.

[19]An observable is thus necessarily quadratic in the x_k's and p_k's: For instance, these canonical variables are not themselves observables. Compared to classical mechanics, there are very few observables in quantum mechanics. This explains why the description of most physical systems requires an infinite-dimensional Hilbert space.

[20]The phase arbitrariness is thus very similar to the gauge invariance of classical electromagnetism. See Ref. 14.

[21]Heslot (Ref. 12). Concerning formula (14), see also V. Cantoni, Rend. Acad. Nat. Lincei **62**, 628 (1977). This formula also occurs in D. J. Rowe, A. Ryman, and G. Rosensteel, Phys. Rev. A **22**, 2362 (1980).

[22]The linearity of unitary transformations need not be assumed: It is a consequence of the fact that they preserve $\langle \ | \ \rangle$.

[23]In mathematical terms, ω is a nondegenerate closed two-form, and defines what is properly called a symplectic structure (mathematicians usually consider the Poisson bracket as a derived notion).

[24]It can be shown that a classical phase space is necessarily homogeneous and isotropic: See Y. Hatakeyama, Tôhoku Math. J. **18**, 338 (1966): Our assumptions of homogeneity and isotropy are natural extensions to the general case. The assumption of positiveness of the curvature is a physical requirement on Planck's constant. The connexity and simple connexity are technical requirements.

[25]See for the detailed proof, A. Heslot, C. R. Acad. Sci. Paris **298**, 95 (1984). This result has been obtained independently by V. Cantoni (private communication).

[26]See, for instance, S. Kobayashi and K. Nomizu, *Foundations of Differential Geometry* (Wiley, New York, 1963), Vol. 2, Chap. 9. The result holds at least when the space of states has finite dimension. Extension to the more realistic case of infinite dimension is now under investigation.

[27]Heslot (Ref. 14).

[28]Let $\{\ ,\ \}_1$ and $\{\ ,\ \}_2$ be the Poisson brackets on the two spaces of states. Then the Poisson bracket on their Cartesian product is $\{\ ,\ \} = \{\ ,\ \}_1 + \{\ ,\ \}_2$.