Forced harmonic oscillator with damping and thermal effects

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Nonperturbative quantum-mechanical solutions of the forced harmonic oscillator with radiation-reaction damping are obtained from a previous analysis based on stochastic electrodynamics. The transition to excited states is shown to be to coherent states which follows the classical trajectory. The quantum Wigner distribution in phase space is constructed. All the results are extended to finite temperatures.

I. INTRODUCTION

In this paper we address ourselves to the quantum problem of a harmonically bounded charged particle in contact with a heat bath (blackbody radiation). Since the particle is charged we also have dissipation effects due to radiation reaction which is always present. We also include an external electromagnetic force with arbitrary time dependence in order to see how the excited states are generated by the external disturbance.

For simplicity we restrict the analysis to the one-dimensional case. The reader will find no difficulty in extending the results to three-dimensional motion.

We approach the above problem indirectly. We first analyze the same system within the framework of stochastic electrodynamics (SED) where the problem has a simple solution.[1-5] As we will see, this analysis will be useful for solving the problem within the quantum-mechanical (QM) context.

II. STOCHASTIC-ELECTRODYNAMICS APPROACH

The classical equation of motion in SED is

$$m\ddot{x}=-m\omega_0{}^2x-m\omega_0{}^2\tau\dot{x}+eE_x(t)-\frac{e}{c}\frac{\partial A_{\rm ext}}{\partial t}\ , \tag{1}$$

where $\tau=\frac{2}{3}e^2/mc^3$. The term proportional to $\dot{x}$ is an approximation of the radiation-reaction force,[6] $E_x(t)$ is the random electromagnetic field of SED, and $A_{\rm ext}(t)$, is the vector potential of an external deterministic force turned on at $t=0$. The effects of the magnetic random field and the space dependence of $E_x(t)$ and $A_{\rm ext}(t)$ have been neglected because the motion is nonrelativistic.[6]

The above linear equation has a simple solution. The trajectory $x(t)$ can be written as

$$x(t)=x_c(t)+x_f(t)\ , \tag{2}$$

where $x_c(t)$ is the deterministic part of $x(t)$ [obtained by putting $E_x=0$ in (1)] and $x_f(t)$ is the fluctuating part generated by the random fields.

The stationary statistical properties of x_f are well known.[1,3] We can consider x_f as a random walk with infinite steps and such that the ensemble average of x_f, denoted by $\langle x_f\rangle$, is zero, but the variance, at temperature T, is given by[1,3]

$$\langle x_f{}^2\rangle=\frac{\hbar}{2m\omega_0}\coth\left[\frac{\hbar\omega_0}{2kT}\right]=\langle p_f{}^2\rangle/m^2\omega_0{}^2\ , \tag{3}$$

where $p_f=m\dot{x}_f$.

According to the central-limit theorem the probability distribution $Q_T(x,t)$ in configuration space is given by the Gaussian function[1,3]

$$Q_T(x,t)=\frac{\exp\left[-\dfrac{(x-x_c)^2}{2\langle x_f{}^2\rangle}\right]}{(2\pi\langle x_f{}^2\rangle)^{1/2}} \tag{4}$$

valid for any temperature.

As was pointed out before,[1,3] $Q_T(x,t)$ coincides with the stationary quantum distribution of a harmonic oscillator at temperature T, when $A_{\rm ext}=0$ and after the transient, that is, when we can assume that $x_c=0$. In fact, for this system QM gives for the probability distribution the following expression:[1,3,7]

$$\frac{1}{Z}\sum_{n=0}^{\infty}|\phi_n(x)|^2\exp\left[-\frac{\hbar\omega_0}{kT}(\tfrac{1}{2}+n)\right]=\frac{\exp(-x^2/2\langle x_f{}^2\rangle)}{(2\pi\langle x_f{}^2\rangle)^{1/2}}\ , \tag{5}$$

where Z is the partition function and $\phi_n(x)$ are eigenfunctions of the unperturbed harmonic oscillator.

When $T=0$, but $A_{\rm ext}\neq 0$, we have

$$Q_0(x,t)=\frac{\exp\left[-\dfrac{(x-x_c)^2m\omega_0}{\hbar}\right]}{(\pi\hbar/m\omega_0)^{1/2}}=|\phi_0(x-x_c)|^2\ , \tag{6}$$

which is valid for any time $t>0$.

The above expression will guide us to the solution of the quantum problem at $T=0$.

III. QUANTUM-MECHANICAL APPROACH

In this case the complete (dissipation included) Schrödinger equation is

$$i\hbar\frac{\partial\psi}{\partial t}=\left[\frac{1}{2m}\left[-i\hbar\frac{\partial}{\partial x}-\frac{e}{c}A\right]^2+\tfrac{1}{2}m\omega_0^2x^2\right]\psi\ , \qquad (7)$$

where $A(t)=A_{\rm ext}(t)+A_{\rm rad}(t)$ and $A_{\rm rad}(t)$ is the radiation-reaction vector potential (classically speaking, since we are not considering quantized electromagnetic fields). The precise meaning of $A_{\rm rad}(t)$ will be clarified below when we explicitly connect $A_{\rm rad}(t)$ with the function $x_c(t)$ defined in (2). It will turn out that $A_{\rm rad}(t)$ will not depend on $\dot{x}$ so that (7) can be described as a Schrödinger equation.

The previous result (6) of SED suggests that we look for a solution of (7) in the form

$$\psi(x,t)=\phi_0(x-x_c)\exp\left[\frac{i}{\hbar}\left[p_c+\frac{e}{c}A\right]x-\frac{i}{\hbar}g\right]\ , \qquad (8)$$

where x_c is the same as before, $p_c\equiv m\dot{x}_c$, and $g(t)$ is a function to be determined by substituting $\psi(x,t)$ into (7). After a short calculation we find that (8) satisfies (7) only if

$$m\ddot{x}_c=-m\omega_0^2x_c-\frac{e}{c}\frac{\partial}{\partial t}(A_{\rm ext}+A_{\rm rad}) \qquad (9)$$

and

$$g(t)=\frac{\hbar\omega_0}{2}t+\int_0^t dt'\left[\frac{p_c^2(t')}{2m}-\frac{m\omega_0^2x_c^2(t')}{2}\right]\ . \qquad (10)$$

Equation (9) is the Abraham-Lorentz equation (1) in the absence of the random field. Therefore dissipation is included in our QM approach. If we approximate the radiation-reaction force by $-m\omega_0^2\tau\dot{x}_c$ [note that here $x_c(t)\neq x$, which is the actual position of the particle], then $(e/c)A_{\rm rad}(t)=m\omega_0^2\tau x_c(t)$ and Eq. (9) has a general solution:

$$x_c(t)=\left[\left[\frac{m\beta x_0+2p_0}{2m\omega_1}\right]\sin(\omega_1t)+x_0\cos(\omega_1t)\right]\exp\left[-\frac{\beta t}{2}\right]+\frac{e}{mc\omega_1}\int_0^t d\xi\frac{\partial A_{\rm ext}(\xi)}{\partial\xi}\sin[\omega_1(t-\xi)]\exp\left[-\frac{\beta}{2}(t-\xi)\right]\ . \qquad (11)$$

Here $\beta=m\omega_0^2\tau$, $\omega_1^2=\omega_0^2-\beta^2/4$, and x_0 and p_0 are free parameters representing the initial position and kinetic momentum, respectively, of a particle following the classical trajectory $x_c(t)$.

For each pair of parameters x_0 and p_0 we can construct functions $\psi_{x_0,p_0}(x,t)$ which are different exact solutions of the Schrödinger equation (7).

These functions are usually called coherent states[8] of the harmonic oscillator and can be expanded using the basis $\phi_n(x)$ as

$$\psi_{x_0,p_0}(x,t)=\sum_{n=0}^{\infty}a_n(t)\phi_n(x)\ , \qquad (12)$$

where

$$a_n=\left[\frac{\sigma^n}{n!}\right]^{1/2}\exp\left[-\frac{\sigma}{2}+\frac{i}{h}\left[\frac{p_cx_c}{2}+\hbar n\varphi-g\right]\right]\ , \qquad (12a)$$

$$\tan\varphi=\frac{p_c+(e/c)A_{\rm ext}}{m\omega_0x_c}\ , \qquad (12b)$$

and

$$\hbar\omega_0\sigma=\tfrac{1}{2}m\dot{x}_c^2+\tfrac{1}{2}m\omega_0^2x_c^2\ . \qquad (12c)$$

If $\tau=\beta=0$, that is, when the radiation-reaction force is neglected, one can show that the set of states $\psi_{x_0,p_0}(x,t)$ is complete.[8] The completeness relation in this case is written as

$$\frac{1}{2\pi\hbar}\int_{-\infty}^{\infty}dx_0\int_{-\infty}^{\infty}dp_0\psi_{x_0,p_0}(y,t)\psi^*_{x_0,p_0}(x,t)=\sum_{n=0}^{\infty}\phi_n(x)\phi_n^*(y)=\delta(x-y). \qquad (13)$$

When $\tau\neq0$ the set $\psi_{x_0,p_0}(x,t)$ is complete only at $t=0$. The reason for this is that for $t>0$ the damping factors $\exp(-\beta t/2)$ in (11) cause the terms which depend on the arbitrary initial conditions to disappear from $x_c(t)$.

The propagator of an arbitrary solution $\psi_{x_0,p_0}(x,t)$, denoted by $K(x,x'\,|\,t)$, and defined as

$$\psi_{x_0,p_0}(x,t)=\int_{-\infty}^{\infty}dx'K(x,x'\,|\,t)\psi_{x_0,p_0}(x',0)\ , \qquad (14)$$

can be easily obtained since we know the analytical expressions of the infinite set of solutions $\psi_{x_0,p_0}(x,t)$ of the Schrödinger equation. We can express $K(x,x'\,|\,t)$ in a closed form, namely

$$K(x,x'\,|\,t)\left[\frac{m\omega_0y_0^2}{2\pi\hbar(y_c^2-y_0^2)}\right]^{-1/2}=\exp\left[\frac{m\omega_0}{2\hbar}(y_c^2-y_0^2+x_0^2-x_c^2)-\frac{ig(t)}{\hbar}+\frac{m\omega_0y_c^2}{y_0^2-y_c^2}\left[x'-x\frac{y_0}{y_c}\right]^2\right]\ , \qquad (15)$$

where

$$y_c(t)=x_c(t)+i\left[\frac{p_c(t)+(e/c)A_{\rm ext}(t)}{m\omega_0}\right]$$

with analogous expression for y_0 [x_0 and p_0 replaces x_c and p_c, respectively, and $A_{\rm ext}(0)=0$].

IV. PHASE-SPACE DISTRIBUTION AT FINITE TEMPERATURES

The Wigner[9,10] distribution associated with the coherent state $\psi_{x_0,p_0}(x,t)$ has a simple form:

$$W_0(x,p,t)\equiv\frac{1}{\pi\hbar}\int_{-\infty}^{\infty}dy\,\psi^*_{x_0,p_0}(x+y,t)\psi_{x_0,p_0}(x-y,t)\exp\left[\frac{2ipy}{\hbar}\right]=\frac{1}{\pi\hbar}\exp\left[-\frac{m\omega_0}{\hbar}(x-x_c)^2-\frac{(p-p_c)^2}{\hbar m\omega_0}\right]. \tag{16}$$

This distribution coincides exactly with the phase-space probability distribution of SED (Refs. 1 and 3) at zero temperature, because in this case the variances of the fluctuating coordinate x_f and fluctuating kinetic momentum $p_f\equiv m\dot{x}_f$ are $\langle x_f^2\rangle=\hbar/2m\omega_0$ and $\langle p_f^2\rangle=\hbar m\omega_0/2$, respectively, as can be seen from (3).

The continuity equation[9] for $W_0(x,p,t)$ can be written as

$$\frac{\partial W_0}{\partial t}+\hat{L}(t)W_0=0\,, \tag{17}$$

where the operator

$$\hat{L}(t)\equiv\dot{x}_c(t)\frac{\partial}{\partial x}+\dot{p}_c(t)\frac{\partial}{\partial p}$$

can be used in order to compute the time evolution of the probability distribution in phase space. This can be done by means of the formula

$$W_0(x,p,t)=\exp\left[-\int_0^t dt'\hat{L}(t')\right]W_0(x,p,0)\,. \tag{18}$$

This result is general since it follows from the local conservation of matter.

This general law of local conservation of the probability distribution will help us to extend formula (16) for nonzero temperatures. First, we recall previous results obtained by some authors[1,7] for the Wigner distribution of the free harmonic oscillator at temperature T, namely

$$W_T(,x,p,0)=\frac{\exp\left[-\dfrac{x^2}{2\langle x_f^2\rangle}-\dfrac{p^2}{2\langle p_f^2\rangle}\right]}{2\pi(\langle x_f^2\rangle\langle p_f^2\rangle)^{1/2}}\,, \tag{19}$$

which is valid when $A_{\text{ext}}(t)=0$ $(t\le 0)$. Second, we propagate $W_T(x,p,0)$ according to the general law (18). The result is

$$W_T(x,p,t)=\frac{\exp\left[-\dfrac{(x-x_c)^2}{2\langle x_f^2\rangle}-\dfrac{(p-p_c)^2}{2\langle p_f^2\rangle}\right]}{2\pi(\langle x_f^2\rangle\langle p_f^2\rangle)^{1/2}} \tag{20}$$

which is, as it should be, exactly the phase-space distribution which is obtained in SED directly from (1), (2), (3), and the central-limit theorem.

V. ANOTHER EXAMPLE

Before passing to our final comments let us briefly discuss another example which is the motion of the charged particle in a constant magnetic field **B**, but also subjected to an external force $-(e/c)\partial\mathbf{A}_{\text{ext}}/\partial t$ with arbitrary time dependence. If **B** is parallel to the z direction we have free motion along this axis when $t<0$. By studying the random motion of this system in the (x,y) plane by means of SED we conclude that[1,3]

$$Q_0(x,y,t)=\frac{\exp\left[-\dfrac{[x-x_c(t)]^2+[y-y_c(t)]^2}{2\hbar/m\omega_B}\right]}{2\pi\hbar/m\omega_B} \tag{21}$$

is the probability distribution at zero temperature because now $\langle x_f^2\rangle=\langle y_f^2\rangle=\hbar/m\omega_B$. Here $\omega_B=eB/mc$. As before $x_c(t)$ and $y_c(t)$ are the projections of the classical deterministic trajectories on the x and y axes.

In QM the corresponding Schrödinger equation is

$$i\hbar\frac{\partial\psi}{\partial t}=\frac{1}{2m}\left[-i\hbar\nabla-\frac{e}{c}\frac{\mathbf{B}\times\mathbf{r}}{2}-\frac{e}{c}\mathbf{A}\right]^2\psi\,, \tag{22}$$

where

$$\mathbf{A}(t)=\mathbf{A}_{\text{ext}}(t)+\mathbf{A}_{\text{rad}}(t)$$

includes the dissipation through the action of the radiation-reaction potential $\mathbf{A}_{\text{rad}}(t)$.

Comparison with the SED result (21) suggests that we look for an exact solution in the form

$$\psi(\mathbf{r},t)=u_0(x-x_c(t),y-y_c(t))\times\exp\left[-\frac{i\tilde{g}}{\hbar}+\frac{i}{\hbar}\mathbf{r}\cdot\mathbf{p}_c\right], \tag{23}$$

where $u_0(x,y)$ is the ground-state wave function of a charged particle in a constant magnetic field **B**, namely,[11]

$$u_0(x,y)=\frac{\exp\left[-\dfrac{(x^2+y^2)m\omega_B}{4\hbar}\right]}{(2\pi\hbar/m\omega_B)^{1/2}}\,. \tag{24}$$

We have checked that (23) is solution of (22) provided that $\mathbf{p}_c$ is the deterministic vector function defined by

$$\mathbf{p}_c\equiv m\dot{\mathbf{r}}_c+\frac{e}{c}\frac{\mathbf{B}\times\mathbf{r}_c}{2}+\frac{e}{c}\mathbf{A}(t) \tag{25}$$

which must be constructed by integrating the Abraham-Lorentz equation of motion

$$m\ddot{\mathbf{r}}_c=\frac{e}{c}\dot{\mathbf{r}}_c\times\mathbf{B}-\frac{e}{c}\frac{\partial}{\partial t}(\mathbf{A}_{\text{ext}}+\mathbf{A}_{\text{rad}})\,. \tag{26}$$

The function $\tilde{g}$ must be such that

$$\dot{\tilde{g}}(t)=\frac{\hbar\omega_B}{2}+\frac{1}{2m}\left[\mathbf{p}_c(t)-\frac{e}{c}\mathbf{A}(t)\right]^2+\frac{e^2}{8mc^2}(\mathbf{B}\times\mathbf{r}_c)^2\,. \tag{27}$$

The above results are valid at zero temperature. The extension to $T>0$ can be easily done within the realm of SED (Refs. 1 and 3) by replacing $\langle x_f{}^2\rangle=\hbar/m\omega_B$ at $T=0$ by

$$\langle x_f{}^2\rangle=\frac{\hbar}{m\omega_B}\coth\left[\frac{\hbar\omega_B}{2kT}\right]=\langle y_f{}^2\rangle \ . \tag{28}$$

We do not intend to discuss the details of such extension in QM for this particular example because of the great analogy with the preceding case of a one-dimensional harmonic oscillator.

VI. CONCLUSIONS

We want to finish our discussion with a few remarks. First, we note the strong similarity between SED and QM for those two simple examples. This has been known since 1963 from the work by Marshall[1] on the free harmonic oscillator. Another important point concerning the similarity between SED and QM is the transitions, to the excited states of the harmonic oscillator, induced by the external field. We have found that it is not possible to excite the particle to a pure state $\phi_n(x)$ $(n>0)$, if we start from the ground state and disturb the system with a controllable deterministic external force, despite its arbitrary time dependence. What we have found is that a coherent state is generated and all the excited states are instantaneously populated according to the Poisson distribution $P_n=\sigma^n\exp(-\sigma)/n!$ as can be seen from (12a). This observation raises again an interesting question concerning a fundamental difference[3] between SED and QM. In SED there are no excited states, with discrete and sharp energy levels, as there are in time-independent QM. The energy is continuously distributed.[3] Despite this fundamental difference both theories are up to now indistinguishable[3,12] from the experimental point of view as far as the harmonic oscillator is concerned. In our QM theoretical analysis of the forced harmonic oscillator we have not been able to decide affirmatively about the real existence of pure excited states. We have concluded that any time-dependent deterministic external disturbance excites quantum coherent states out of the ground state. This is entirely consistent with SED as far as probability distributions are concerned.

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[1]T. W. Marshall, Proc. R. Soc. London **A276**, 475 (1963).
[2]T. H. Boyer, Phys. Rev. D **11**, 790 (1975).
[3]T. H. Boyer, Phys. Rev. D **11**, 809 (1975).
[4]P. W. Milonni, Phys. Rep. **25C**, 1 (1976).
[5]L. de la Peña, *Stochastic Processes Applied to Physics and Other Related Fields* (World Scientific, Singapore, 1983), p. 428.
[6]H. M. França and G. C. Santos, Ref. 5, p. 593.
[7]R. W. Davies and K. T. R. Davies, Ann. Phys. (N.Y.) **89**, 261 (1975).
[8]P. Carruthers and M. M. Nieto, Am. J. Phys. **33**, 537 (1965).
[9]E. P. Wigner, Phys. Rev. **40**, 749 (1932).
[10]R. F. O'Connell and E. P. Wigner, Phys. Lett. **83A**, 145 (1981).
[11]L. Landau and E. Lifshitz, *Mécanique Quantique* (Editions Mir, Moscow, 1966), p. 496.
[12]G. H. Goedecke, Found. Phys. **14**, 41 (1984).