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Matrix general relativity and Yang-Mills theory

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We have constructed a theory where the metric tensor $g_{\mu\nu}(x)$ and the connection $\Gamma^{\lambda}_{\mu\nu}(x)$ are now generalized to be matrix-valued instead of scalar functions. Such a theory can combine both Einstein's general relativity and Yang-Mills gauge theory. Similarly, the usual vierbein field $e^{a}_{\mu}(a)$ may be assumed to be also matrix-valued. This theory can possess two different kinds of local gauge transformations. In such a theory, there exists a possibility that gravity may be regarded as a bound state of a spin-$\frac{3}{2}$ fermion pair in some sense. Also, the torsion tensor does not behave covariantly under one of the local gauge symmetries.

I. MATRIX GENERAL RELATIVITY

One important problem in high-energy physics is how to unify the general relativity of Einstein with Yang-Mills gauge theory. The most popular and the most interesting approach for this problem is to consider the geometry of a high-dimensional manifold as in Kaluza-Klein theory.[1] However, there exists the second possibility which is less well known and is a generalization of the original idea by Einstein[2] where we restrict ourselves to consideration of the ordinary four-dimensional Minkowski manifold. Suppose that the connection $\Gamma^{\mu}_{\lambda\nu}(x)$ $(\lambda,\mu,\nu=0,1,2,3)$ is now an $N\times N$ matrix. The Yang-Mills gauge potential $A_{\lambda}(x)$ which is an $N\times N$ matrix may now be identified with the *Ansatz*[3,4]

$$\Gamma^{\mu}_{\lambda\nu}(x)={}^{(0)}\Gamma^{\mu}_{\lambda\nu}(x)E+\delta^{\mu}_{\nu}A_{\lambda}(x)\ , \tag{1.1}$$

where E is the $N\times N$ unit matrix and ${}^{(0)}\Gamma^{\mu}_{\lambda\nu}(x)$ is the standard scalar affine connection. Note that for $N=1$ Eq. (1.1) reproduces the original *Ansatz* of Einstein,[2] so that this is the straightforward generalization of the original idea of Einstein.

The purpose of this note is to generalize and explore the second alternative mentioned above, which we call matrix relativity. As we shall see shortly, Eq. (1.1) corresponds to a connection in a $4N$-dimensional vector bundle B over the Minkowski space-time base. The bundle group G_0 of B is a direct product $\mathrm{GL}(4,R)\otimes \mathrm{U}(N)$ or its subgroup, which implies no correlation between the space-time group $\mathrm{GL}(4,R)$ (or its subgroup) and the internal-symmetry group $\mathrm{U}(N)$ (or its subgroup). If we wish to introduce a possible correlation between the space-time and the internal symmetry, then we evidently have to extend the bundle group G_0 into a larger group G. Temporarily, we assume it to be the $\mathrm{GL}(4N,C)$ group. Then the corresponding general connection $\Gamma^{\mu}_{\lambda\nu}(x)$ is a general $N\times N$ matrix, which has in general a more complicated structure than that given by Eq. (1.1). We here consider only a purely affine geometry. A generalization of the matrix metric tensor based upon a generalized vierbein field will be discussed in the next section.

First, under the coordinate transformation

$$x^{\mu}\to x'^{\mu}\ , \tag{1.2}$$

the connection must transform as

$$\Gamma^{\mu}_{\lambda\nu}(x)\to\Gamma'^{\mu}_{\lambda\nu}(x')=\frac{\partial x'^{\mu}}{\partial x^{\alpha}}\frac{\partial x^{\beta}}{\partial x'^{\lambda}}\frac{\partial x^{\gamma}}{\partial x'^{\nu}}\Gamma^{\alpha}_{\beta\gamma}(x)+\frac{\partial x'^{\mu}}{\partial x^{\alpha}}\frac{\partial^2 x^{\alpha}}{\partial x'^{\lambda}\partial x'^{\nu}}E\ . \tag{1.3}$$

The local gauge transformation can be introduced in a familiar way as follows. Let $S^{\mu}{}_{\nu}(x)$ and $T^{\mu}{}_{\nu}(x)$ be arbitrary $N\times N$ matrices for each $\mu,\nu=0,1,2,3$, subject to the constraint

$$S^{\mu}{}_{\lambda}(x)T^{\lambda}{}_{\nu}(x)=T^{\mu}{}_{\lambda}(x)S^{\lambda}{}_{\nu}(x)=\delta^{\mu}{}_{\nu}E\ . \tag{1.4}$$

Then, the local gauge transformation for the connection $\Gamma^{\mu}_{\lambda\nu}(x)$ may be defined by

$$\Gamma^{\mu}_{\lambda\nu}(x)\to\widetilde{\Gamma}^{\mu}_{\lambda\nu}(x)=T^{\mu}{}_{\alpha}(x)\Gamma^{\alpha}_{\lambda\beta}(x)S^{\beta}{}_{\nu}(x)+T^{\mu}{}_{\alpha}(x)\partial_{\lambda}S^{\alpha}{}_{\nu}(x)\ . \tag{1.5}$$

When we define the $N\times N$ curvature matrix tensor $R^{\mu}{}_{\nu\alpha\beta}(x)$ by

$$R^{\mu}{}_{\nu\alpha\beta}(x)=\partial_{\alpha}\Gamma^{\mu}_{\beta\nu}(x)-\partial_{\beta}\Gamma^{\mu}_{\alpha\nu}(x)+\Gamma^{\mu}_{\alpha\lambda}(x)\Gamma^{\lambda}_{\beta\nu}(x)-\Gamma^{\mu}_{\beta\lambda}(x)\Gamma^{\lambda}_{\alpha\nu}(x)\ , \tag{1.6}$$

then it is straightforward to see the covariant transformation law

$$R^{\mu}{}_{\nu\alpha\beta}(x)\to\widetilde{R}^{\mu}{}_{\nu\alpha\beta}(x)=T^{\mu}{}_{\lambda}(x)R^{\lambda}{}_{\tau\alpha\beta}(x)S^{\tau}{}_{\nu}(x)\ , \tag{1.7}$$

under Eq. (1.5). We have to be careful of the order of the product of two connections in Eq. (1.6) since they are now matrices.

We may note that both coordinate transformation Eq. (1.3) and the gauge transformation Eq. (1.5) can be combined into a single form of

$$\Gamma^{\mu}_{\lambda\nu}(x)\to\widetilde{\Gamma}'^{\mu}_{\lambda\nu}(x')=\frac{\partial x^{\tau}}{\partial x'^{\lambda}}[T^{\mu}{}_{\alpha}(x)\Gamma^{\alpha}_{\tau\beta}(x)S^{\beta}{}_{\nu}(x)+T^{\mu}{}_{\alpha}(x)\partial_{\tau}S^{\alpha}{}_{\nu}(x)]\ . \tag{1.8}$$

For a pure local gauge transformation, we simply set $x'^{\mu}=x^{\mu}$, while we identify

$$T^{\mu}{}_{\nu}(x)=\frac{\partial x'^{\mu}}{\partial x^{\nu}}E,\quad S^{\mu}{}_{\nu}(x)=\frac{\partial x^{\mu}}{\partial x'^{\nu}}E \tag{1.9}$$

for the pure coordinate transformation Eq. (1.3). The Riemann curvature tensor $R^{\mu}{}_{\nu\alpha\beta}(x)$ transforms, of course, covariantly also under Eq. (1.8), since it behaves covariantly under pure coordinate transformations as in the usual theory.[5] We also remark that the existence of the local gauge transformation Eq. (1.5) for the scalar case of $N=1$ has been previously noted by some authors.[6]

In order to realize that the present theory is related to a vector bundle with the bundle group GL$(4N,C)$, it is more convenient to regard Minkowski indices μ,ν also as matrix indices as follows. Let A,B $(=1,2,\ldots,N)$ be the matrix indices of the N-dimensional internal space. We now define $4N\times 4N$ matrices $\Gamma_{\lambda}(x)$, and $R_{\alpha\beta}(x)$ as well as $S(x)$ and $T(x)$ by

$$\begin{aligned}
&\langle A,\mu\,|\,\Gamma_{\lambda}(x)\,|\,B,\nu\rangle=\langle A\,|\,\Gamma^{\mu}_{\lambda\nu}(x)\,|\,B\rangle\ ,\\
&\langle A,\mu\,|\,R_{\alpha\beta}(x)\,|\,B,\nu\rangle=\langle A\,|\,R^{\mu}{}_{\nu\alpha\beta}(x)\,|\,B\rangle\ ,\\
&\langle A,\mu\,|\,S(x)\,|\,B,\nu\rangle=\langle A\,|\,S^{\mu}{}_{\nu}(x)\,|\,B\rangle\ ,\\
&\langle A,\mu\,|\,T(x)\,|\,B,\nu\rangle=\langle A\,|\,T^{\mu}{}_{\nu}(x)\,|\,B\rangle\ ,
\end{aligned} \tag{1.10}$$

for $A,B=1,2,\ldots,N$ and $\mu,\nu=0,1,2,3$. Then, Eqs. (1.4)—(1.7) are rewritten as

$$S(x)T(x)=T(x)S(x)=I\ , \tag{1.4'}$$

$$\Gamma_{\lambda}(x)\to\tilde{\Gamma}_{\lambda}(x)=T(x)\Gamma_{\lambda}(x)S(x)+T(x)\partial_{\lambda}S(x)\ , \tag{1.5'}$$

$$R_{\alpha\beta}(x)=\partial_{\alpha}\Gamma_{\beta}(x)-\partial_{\beta}\Gamma_{\alpha}(x)+[\Gamma_{\alpha}(x),\Gamma_{\beta}(x)]\ , \tag{1.6'}$$

$$R_{\alpha\beta}(x)\to\tilde{R}_{\alpha\beta}(x)=T(x)R_{\alpha\beta}(x)S(x)\ , \tag{1.7'}$$

where I in Eq. (1.4') refers to the $4N\times 4N$ unit matrix. Since $T(x)=S^{-1}(x)$ by Eq. (1.4'), these equations are formally equivalent to the Yang-Mills gauge theory based upon the gauge group GL$(4N,C)$ with the connection $\Gamma_{\lambda}(x)$. However, the theory is not literally equivalent to the Yang-Mills theory, since the coordinate transformation Eq. (1.3) or (1.8) gives the mixing between the internal and Minkowski components in this $4N$-dimensional matrix notation. Indeed, $\Gamma^{\mu}_{\lambda\nu}(x)$ consists of Lorentz spin components with spins 3, 2, and 1, in contrast to the pure Lorentz spin-one Yang-Mills gauge potential.

Before going into further detail, we note that

$$J=\frac{1}{32\pi^2}\int d^4x\,\epsilon^{\mu\nu\alpha\beta}\mathrm{Tr}[R_{\mu\nu}(x)R_{\alpha\beta}(x)] \tag{1.11}$$

gives a topological invariant corresponding to the second Chern integral of the Chern class.[7] It is straightforward[7,8] to note

$$\tfrac{1}{8}\epsilon^{\mu\nu\alpha\beta}\mathrm{Tr}[R_{\mu\nu}(x)R_{\alpha\beta}(x)]=\partial_{\lambda}K^{\lambda}(x)\ , \tag{1.12a}$$

$$K^{\lambda}(x)=\epsilon^{\lambda\mu\alpha\beta}\mathrm{Tr}[\,\Gamma_{\mu}(x)R_{\alpha\beta}(x)-\tfrac{1}{3}\Gamma_{\mu}(x)\Gamma_{\alpha}(x)\Gamma_{\beta}(x)]\ , \tag{1.12b}$$

where the traces in these formulas refer to the $4N$-dimensional fiber space. Also, $\epsilon^{\mu\nu\alpha\beta}$ is the completely antisymmetric constant Levi-Civita symbol with $\epsilon^{0123}=1$.

Suppose for a moment that a Riemannian metric tensor $g_{\mu\nu}(x)$ is given, and we raise and/or lower indices by means of $g^{\mu\nu}(x)$ and $g_{\mu\nu}(x)$ as usual. We emphasize that $g_{\mu\nu}(x)$ is a c-number function but not a matrix. Then, the Yang-Mills Lagrangian will be given by

$$L(x)=\tfrac{1}{16}\mathrm{Tr}[R_{\mu\nu}(x)R^{\mu\nu}(x)] \tag{1.13}$$

which is invariant under $4N$-dimensional local gauge transformation Eqs. (1.5') and (1.7'). In the limit with the flat metric $g_{\mu\nu}(x)=\eta_{\mu\nu}$, Eq. (1.13) is a quadratic function of 64 $N\times N$ matrices $\Gamma^{\mu}_{\lambda\nu}(x)$ $(\lambda,\mu,\nu=0,1,2,3)$, whose spin contents are a mixture of spins 3, 2, and 1. Therefore, the theory will correspond to a new type of local gauge theory involving the highest spin 3 with the new local gauge transformation Eq. (1.5). Since the transformation law Eq. (1.5) mixes all spin components of 3, 2, and 1 in a nontrivial way, we cannot construct a pure spin-3 local gauge theory. Moreover, the Hamiltonian constructed from the Lagrangian Eq. (1.13) cannot be made to be positive definite for any gauge condition. Because of these facts, the theory described by Eq. (1.13) is rather pathological unless we somehow eliminate the spin-3 components of $\Gamma^{\lambda}_{\mu\nu}(x)$ as in the classical general relativity. One way to achieve this end is to assume Eq. (1.1). Another method is to use a generalization of metric condition to be explained in the next section. In passing, we note that it is generally believed[9] that consistent theories with gravity coupled to massless particles of spin greater than two do not exist.

Returning to the original discussion, we note that the group of our bundle space is GL$(4N,C)$. However, as we mentioned in the beginning, the physical group G_0 leading to Eq. (1.1) must be GL$(4,R)\otimes$U(N) or its subgroup. If we accept this fact from the beginning, then the connection $\Gamma_{\lambda}(x)$ must be an element of the Lie algebra of GL$(4,R)\otimes$U(N). In particular, this implies that $\Gamma^{\mu}_{\lambda\nu}(x)$ must be written in a special form of Eq. (1.1). Corresponding to this restriction, the local gauge transformation functions $S^{\mu}{}_{\nu}(x)$ and $T^{\mu}{}_{\nu}(x)$ will be restricted to forms of

$$\begin{aligned}
&S^{\mu}{}_{\nu}(x)=a^{\mu}_{\nu}(x)U(x)\ ,\\
&T^{\mu}{}_{\nu}(x)=b^{\mu}_{\nu}(x)U^{-1}(x)\ ,
\end{aligned} \tag{1.14a}$$

where $U(x)$ is an $N\times N$ unimodular matrix and where $a^{\mu}_{\nu}(x)$ and $b^{\mu}_{\nu}(x)$ are scalar (i.e., not matrix) functions of the coordinate, satisfying

$$a^{\mu}_{\lambda}(x)b^{\lambda}_{\nu}(x)=b^{\mu}_{\lambda}(x)a^{\lambda}_{\nu}(x)=\delta^{\mu}_{\nu}\ . \tag{1.14b}$$

The restricted local gauge transformation based upon Eq. (1.14) is now read as

$$\Gamma^{\mu}_{\lambda\nu}(x)\to\tilde{\Gamma}^{\mu}_{\lambda\nu}(x)=b^{\mu}_{\alpha}(x)U^{-1}(x)\Gamma^{\alpha}_{\lambda\beta}(x)U(x)a^{\beta}_{\nu}(x)+U^{-1}(x)\partial_\lambda U(x)\delta^{\mu}_{\nu}+b^{\mu}_{\alpha}(x)\partial_\lambda a^{\alpha}_{\nu}(x)E\ . \tag{1.15}$$

Now, we assume explicitly the Einstein *Ansatz* Eq. (1.1). Then, Eq. (1.6) becomes

$$R^{\mu}{}_{\nu\alpha\beta}(x)={}^{(0)}R^{\mu}{}_{\nu\alpha\beta}(x)E+\delta^{\mu}_{\nu}F_{\alpha\beta}(x)\ , \tag{1.16a}$$

$$F_{\alpha\beta}(x)=\partial_\alpha A_\beta(x)-\partial_\beta A_\alpha(x)+[A_\alpha(x),A_\beta(x)]\ , \tag{1.16b}$$

where ${}^{(0)}R^{\mu}{}_{\nu\alpha\beta}(x)$ is the classical affine curvature tensor constructed from the scalar affine connection ${}^{(0)}\Gamma^{\mu}_{\lambda\nu}(x)$. Since the nonzero-trace part of $A_\mu(x)$ can be always absorbed into ${}^{(0)}\Gamma^{\mu}_{\lambda\nu}(x)$, we may assume without loss of generality

$$\mathrm{Tr}A_\lambda(x)=0\ . \tag{1.17}$$

Then, Eq. (1.15) with Eq. (1.1) is equivalent to

$$A_\mu(x)\to\tilde{A}_\mu(x)=U^{-1}(x)A_\mu(x)U(x)+U^{-1}(x)\partial_\mu U(x)\ , \tag{1.18a}$$

$$F_{\mu\nu}(x)\to\tilde{F}_{\mu\nu}(x)=U^{-1}(x)F_{\mu\nu}(x)U(x)\ , \tag{1.18b}$$

$${}^{(0)}\Gamma^{\mu}_{\lambda\nu}(x)\to{}^{(0)}\tilde{\Gamma}^{\mu}_{\lambda\nu}(x)=b^{\mu}_{\alpha}(x){}^{(0)}\Gamma^{\alpha}_{\lambda\beta}(x)a^{\beta}_{\nu}(x)+b^{\mu}_{\alpha}(x)\partial_\lambda a^{\alpha}_{\nu}(x)\ , \tag{1.18c}$$

when we note $\mathrm{Tr}[U^{-1}(x)\partial_\lambda U(x)]=0$ because of the unimodularity of $U(x)$. Therefore, we may identify $A_\mu(x)$ as the Yang-Mills field, while the transformation law Eq. (1.18c) corresponds[6] to the usual affine connection. Inserting Eq. (1.16a) into Eq. (1.11), we find similarly

$$J=J_1+J_2\ , \tag{1.19a}$$

$$J_1=\frac{N}{32\pi^2}\int d^4x\ \epsilon^{\mu\nu\alpha\beta(0)}R^{\lambda}{}_{\tau\mu\nu}(x){}^{(0)}R^{\tau}{}_{\lambda\alpha\beta}(x)\ , \tag{1.19b}$$

$$J_2=\frac{1}{8\pi^2}\int d^4x\ \epsilon^{\mu\nu\alpha\beta}\mathrm{Tr}[F_{\mu\nu}(x)F_{\alpha\beta}(x)]\ , \tag{1.19c}$$

where the trace here in Eq. (1.19c) and hereafter refer to the N-dimensional internal space. Note that J_1 and J_2 are proportional to the second Pontrjagin integral[7] and the familiar winding number of the Yang-Mills gauge field, respectively. We remark that both J_1 and J_2 are intimately related[10] to triangle anomalies.

Next, if we insert Eq. (1.16a) into Eq. (1.13), then we find

$$L(x)=\tfrac{1}{4}NL_{\mathrm{WY}}(x)+\tfrac{1}{4}\mathrm{Tr}[F_{\mu\nu}(x)F^{\mu\nu}(x)]\ , \tag{1.20a}$$

$$L_{\mathrm{WY}}(x)=\tfrac{1}{4}{}^{(0)}R_{\mu\nu\alpha\beta}(x){}^{(0)}R^{\nu\mu\alpha\beta}(x)\ . \tag{1.20b}$$

Here, $L_{\mathrm{WY}}(x)$ is essentially the Weyl-Yang Lagrangian,[11] which gives the Yang-Mills gravity theory. It is now known[12,13] that the theory based upon $L_{\mathrm{WY}}(x)$ gives essentially the same result as the free Einstein relativity at least at the classical level. In passing, we simply mention that more complicated Lagrangians containing $L_{\mathrm{WY}}(x)$ have been studied by many authors both classically[13] and quantum mechanically.[14]

If we wish to work with the linear Hilbert-Palatine action rather than quadratic Weyl-Yang form, then we can do so by introducing the $N\times N$ matrix field $G_{\mu\nu}(x)$. The Reimannian metric $g_{\mu\nu}(x)$ may be identified as the trace part of $G_{\mu\nu}(x)$, i.e., by

$$g_{\mu\nu}(x)=\frac{1}{N}\mathrm{Tr}G_{\mu\nu}(x)\ . \tag{1.21}$$

We raise and/or lower indices as usual by means of $g^{\mu\nu}(x)$ and/or $g_{\mu\nu}(x)$. Consider the action integral

$$A=\frac{1}{N}\int d^4x[-g(x)]^{1/2}\mathrm{Tr}\{G^{\mu\nu}(x)R^{\lambda}{}_{\mu\lambda\nu}(x)+c_1[G^{\mu\nu}(x)G_{\mu\nu}(x)-4E]\}\ , \tag{1.22}$$

where c_1 is a constant. Note that Eq. (1.22) is still invariant under Eq. (1.15) restricted to $a^{\mu}_{\nu}(x)=b^{\mu}_{\nu}(x)=\delta^{\mu}_{\nu}$ provided that we assume

$$G_{\mu\nu}(x)\to\tilde{G}_{\mu\nu}(x)=U^{-1}(x)G_{\mu\nu}(x)U(x)\ . \tag{1.23}$$

Regarding $G_{\mu\nu}(x)$, ${}^{(0)}\Gamma^{\mu}_{\lambda\nu}(x)$, and $A_\lambda(x)$ to be independent variational variables, it is easy to see that the variational principle $\delta A=0$ for $\delta G_{\mu\nu}(x)$ and $\delta^{(0)}\Gamma^{\mu}_{\lambda\nu}(x)$ gives

$$G_{\mu\nu}(x)=g_{\mu\nu}(x)E-\frac{1}{2c_1}F_{\mu\nu}(x) \tag{1.24}$$

assuming $c_1\neq0$. Therefore, the Yang-Mills gauge field $F_{\mu\nu}(x)$ may be regarded as the antisymmetric component of the generalized matrix-metric tensor $G_{\mu\nu}(x)$. Inserting Eq. (1.24) into Eq. (1.22), we find

$$A=\int d^4x[-g(x)]^{1/2}\left\{{}^{(0)}R(x)-\frac{1}{4Nc_1}\times\mathrm{Tr}[F_{\mu\nu}(x)F^{\mu\nu}(x)]\right\}\ , \tag{1.25}$$

where ${}^{(0)}R(x)$ is the scalar curvature, i.e.,

$$ {}^{(0)}R(x)=g^{\mu\nu}(x){}^{(0)}R^{\lambda}{}_{\mu\lambda\nu}(x)={}^{(0)}R^{\lambda\nu}{}_{\lambda\nu}(x) \ . \tag{1.26}$$

Therefore, our action is simply an independent sum of the Hilbert-Palatine action and the Yang-Mills Lagrangian without any genuine interaction between them. We may also note that if we did not subtract a term proportional to $-4c_1\mathrm{Tr}E$ in Eq. (1.22), then the action Eq. (1.25) would contain a cosmological term.

In ending this section, consider a special Yang-Mills type transformation

$$\Gamma^{\mu}_{\lambda\nu}(x)\to\tilde{\Gamma}^{\mu}_{\lambda\nu}(x)=U^{-1}(x)\Gamma^{\mu}_{\lambda\nu}(x)U(x)+U^{-1}(x)\partial_\lambda U(x)\delta^{\mu}_{\nu} \tag{1.27}$$

in Eq. (1.15) with choice of $a^{\mu}_{\nu}(x)=b^{\mu}_{\nu}(x)=\delta^{\mu}_{\nu}$. Then, the curvature tensor transforms covariantly,

$$R^{\mu}{}_{\nu\alpha\beta}(x)\to\tilde{R}^{\mu}{}_{\nu\alpha\beta}(x)\to\tilde{R}^{\mu}{}_{\nu\alpha\beta}(x)=U^{-1}(x)R^{\mu}{}_{\nu\alpha\beta}(x)U(x) \ . \tag{1.28}$$

Also, setting

$$B_\lambda(x)=\tfrac{1}{4}\Gamma^{\mu}_{\lambda\mu}(x) \ , \tag{1.29}$$

it behaves as the Yang-Mills gauge field with the same transformation as Eq. (1.18a), even if we do not assume the Einstein's *Ansatz*. All nondiagonal elements $\Gamma^{\mu}_{\lambda\nu}(x)$ with $\mu\neq\nu$, as well as all differences of diagonal terms such as $\Gamma^{1}_{\lambda1}(x)-\Gamma^{2}_{\lambda2}(x)$, behave covariantly just like $F_{\mu\nu}(x)$. Nevertheless, for the pure coordinate transformation $B_\lambda(x)$ transforms as

$$B_\lambda(x)\to B'_\lambda(x')=\frac{\partial x^\nu}{\partial x'^\lambda}B_\nu(x)+\frac{\partial}{\partial x^\nu}\left[\frac{\partial x^\nu}{\partial x'^\lambda}\right]E \ , \tag{1.30}$$

so that it does not represent a genuine vector field. However, its traceless part defined by

$$A_\lambda(x)=B_\lambda(x)-\frac{1}{N}[\mathrm{Tr}B_\lambda(x)]E \tag{1.31}$$

behaves as a genuine vector, so that $A_\lambda(x)$ rather than $B_\lambda(x)$ represents the real Yang-Mills gauge field.

II. QUASIMETRIC THEORY

We now generalize the formalism of the previous section for matrix geometry without assuming the Einstein's *Ansatz* Eq. (1.1).

Hereafter, all lower case greek indices $\mu,\nu,\lambda,\ldots$ refer to $(n+1)$-dimensional space-time with $(n+1)$ values $0,1,2,\ldots,n$. Similarly, we introduce an auxiliary $(m+1)$-dimensional space, which is indexed by small latin indices $a,b,\ldots,$ with $(m+1)$ values $0,1,2,\ldots,m$. Although we will ultimately restrict ourselves to a special case of $n+1=m+1=4$, we will maintain this generality till the end.

Let $\Gamma^{a}_{\mu b}(x)$ for each $a,b=0,1,2,\ldots,m$ and $\mu=0,1,2,\ldots,n$ be an $M\times M$ matrix and define an $M\times M$ matrix-valued one-form $\omega^a{}_b$ by

$$\omega^a{}_b=\Gamma^{a}_{\mu b}(x)dx^\mu \ . \tag{2.1}$$

For the special case of $M=1$, this reduces, of course, to the standard Cartan one-form.[15,16] If we wish, we may define the $(m+1)M\times(m+1)M$ matrix-valued one-form ω by

$$\langle A,a\mid\omega\mid B,b\rangle=\langle A\mid\omega^a{}_b\mid B\rangle=\langle A\mid\Gamma^{a}_{\mu b}(x)\mid B\rangle dx^\mu \tag{2.2}$$

for $A,B=1,2,\ldots,M$ as in the previous section. Then, it is clear that we are considering a vector bundle with the bundle group G which is a subgroup of $\mathrm{GL}[(m+1)M,C]$.

The curvature two-form $R^a{}_b$ is defined then as usual by

$$R^a{}_b=d\omega^a{}_b+\omega^a{}_c\Lambda\omega^c{}_b \tag{2.3}$$

which is rewritten as

$$R^a{}_b=\tfrac{1}{2}R^a{}_{b\alpha\beta}(x)dx^\alpha\Lambda dx^\beta \ , \tag{2.4a}$$

$$R^a{}_{b\alpha\beta}(x)=\partial_\alpha\Gamma^{a}_{\beta b}(x)-\partial_\beta\Gamma^{a}_{\alpha b}(x)+\Gamma^{a}_{\alpha c}(x)\Gamma^{c}_{\beta b}(x)-\Gamma^{a}_{\beta c}(x)\Gamma^{c}_{\alpha b}(x) \ . \tag{2.4b}$$

In the matrix notation used in Eq. (2.2), we may also rewrite Eq. (2.3) as

$$R=d\omega+\omega\Lambda\omega \tag{2.3$'$}$$

by defining the $(m+1)M\times(m+1)M$ matrix-valued two-form R in the similar way. If we choose $m=n=3$ with identification $a=\mu$, and $b=\nu$ corresponding to the so-called coordinate basis,[15] then Eq. (2.4b) reproduces of course Eq. (1.6). However, for reasons which will be apparent, we will not consider such a special identification in this paper.

The local gauge transformation is defined by

$$\omega^a{}_b\to\tilde{\omega}^a{}_b=U^a{}_c(x)\omega^c{}_dW^d{}_b(x)+U^a{}_c(x)\partial_\lambda W^c{}_b(x)dx^\lambda \ , \tag{2.5}$$

where $M\times M$ matrices $U^a{}_b(x)$ and $W^b{}_a(x)$ are assumed to satisfy

$$U^a{}_c(x)W^c{}_b(x)=W^a{}_c(x)U^c{}_b(x)=\delta^a_bE_M \ . \tag{2.6}$$

Here E_M is the $M\times M$ identity matrix. The corresponding transformation property for $\Gamma^{a}_{\mu b}(x)$ is then given by

$$\Gamma^{a}_{\mu b}(x)\to\tilde{\Gamma}^{a}_{\mu b}(x)=U^a{}_c(x)\Gamma^{c}_{\mu d}(x)W^d{}_b(x)+U^a{}_c(x)\partial_\mu W^c{}_b(x) \ . \tag{2.7}$$

Under this transformation, $R^a{}_b$ transforms covariantly, i.e.,

$$R^a{}_b\to\tilde{R}^a{}_b=U^a{}_c(x)R^c{}_dW^d{}_b(x) \ . \tag{2.8}$$

Next, we introduce a generalization of the vierbein field. Let $L^{a}_{\mu}(x)$ be the $M\times N$ matrix for each $a=0,1,2,\ldots,m$ and $\mu=0,1,2,\ldots,n$, and define the $M\times N$ matrix-valued one-form ω^a by

$$\omega^a=L^{a}_{\mu}(x)dx^\mu \ . \tag{2.9}$$

For the special case of $N=M=1$ and $n=m=3$, then it reduces to the standard canonical one-form.[16] We define the $M\times N$ matrix-valued two-form T^a by

$$T^a = d\omega^a + \omega^a{}_b \Lambda \omega^b \tag{2.10}$$

which is clearly a generalization of the usual torsion two-form. Assuming the transformation law

$$\omega^a \to \widetilde{\omega}^a = U^a{}_b(x)\omega^b \tag{2.11a}$$

or

$$L^a_\mu(x) \to \widetilde{L}^a_\mu(x) = U^a{}_b(x) L^b_\mu(x) , \tag{2.11b}$$

we easily see that T^a behaves covariantly as

$$T^a \to \widetilde{T}^a = U^a{}_b(x) T^b . \tag{2.12}$$

In terms of the component, we have of course

$$T^a = \tfrac{1}{2} T^a_{\mu\nu}(x) dx^\mu \Lambda dx^\nu , \tag{2.13a}$$

$$T^a_{\mu\nu}(x) = \partial_\mu L^a_\nu(x) - \partial_\nu L^a_\mu(x) + \Gamma^a_{\mu b}(x) L^b_\nu(x) - \Gamma^a_{\nu b}(x) L^b_\mu(x) . \tag{2.13b}$$

Now, we will introduce a generalization of the metric tensor $g_{\mu\nu}(x)$. With this in mind, let the $M \times M$ matrix I_{ab} $(a,b,=0,1,2,\ldots,m)$ be a coordinate-independent Hermitian flat matrix, in a sense that it satisfies

$$(I_{ab})^\dagger = I_{ba} . \tag{2.14}$$

We introduce a quasimetric tensor $G_{\mu\nu}(x)$ by

$$G_{\mu\nu}(x) = [L^a_\mu(x)]^\dagger I_{ab} L^b_\nu(x) \tag{2.15}$$

which is clearly an $N \times N$ matrix satisfying

$$[G_{\mu\nu}(x)]^\dagger = G_{\nu\mu}(x) . \tag{2.16}$$

If $n+1 = m+1 = 4$, then the most natural choice for I_{ab} will be

$$I_{ab} = \eta_{ab} E_M , \tag{2.17}$$

where η_{ab} is the flat Minkowski metric.[15]

When we define $g_{\mu\nu}(x)$ by

$$g_{\mu\nu}(x) = \frac{1}{N} \mathrm{Tr} G_{\mu\nu}(x) , \tag{2.18a}$$

then Eq. (2.16) gives

$$g^*_{\mu\nu}(x) = g_{\nu\mu}(x) . \tag{2.18b}$$

We simply note that a theory satisfying the condition Eq. (2.18b) for the special case of $N = M = 1$ and $n = m = 3$ has been previously considered by Kunstatter *et al.*[17] and its possible physical consequences on somewhat related theory have been extensively studied by Moffat.[18] If we assume in addition the reality of $L^a_\mu(x)$, then $g_{\mu\nu}(x)$ is real and symmetric so that it may be identified with the usual Riemannian metric. This is the reason why we call $G_{\mu\nu}(x)$ a quasimetric tensor here. We may call our space also a quasi-Riemannian space.

Returning to the general case, it is necessary to impose an extra condition

$$[U^a{}_c(x)]^\dagger I_{ab} U^b{}_d(x) = I_{cd} \tag{2.19}$$

in order to maintain a covariant transformation law for $G_{\mu\nu}(x)$. Therefore, our local gauge group is a $\mathrm{U}(p,q)$ group with $p+q = (m+1)M$ for some non-negative integers p and q. However, we will often refer the $\mathrm{GL}[(m+1)M,C]$ group as the local gauge group for simplicity instead of the correct $\mathrm{U}(p,q)$ group. From Eqs. (2.6) and (2.19), we find

$$W^a{}_b(x) = (I^{-1})^{ac} [U^d{}_c(x)]^\dagger I_{db} , \tag{2.20}$$

where we assumed the existence of the inverse constant $M \times M$ matrix $(I^{-1})^{ab}$ satisfying

$$(I^{-1})^{ac} I_{cb} = I_{bc} (I^{-1})^{ca} = \delta^a_b E_M . \tag{2.21}$$

If we choose Eq. (2.17) for explicit form of I_{ab}, then $(I^{-1})^{ab}$ is clearly given by

$$(I^{-1})^{ab} = \eta^{ab} E_M . \tag{2.22}$$

Assuming the validity of Eq. (2.19), it is apparent that $G_{\mu\nu}(x)$ remains invariant under the local gauge transformation Eq. (2.11), i.e.,

$$G_{\mu\nu}(x) \to \widetilde{G}_{\mu\nu}(x) = G_{\mu\nu}(x) . \tag{2.23}$$

It is often more convenient to regard indices $a,b,\ldots$, etc., also as $(m+1)\times(m+1)$ matrix indices, and introduce $(m+1)M \times (m+1)M$ matrices U, $U^\dagger$, I, and R just like ω of Eq. (2.2) by

$$\begin{aligned} \langle a,A \mid U \mid b,B \rangle &= \langle A \mid U^a{}_b(x) \mid B \rangle , \\ \langle a,A \mid U^\dagger \mid b,B \rangle &= \langle A \mid [U^b{}_a(x)]^\dagger \mid B \rangle , \\ \langle a,A \mid I \mid b,B \rangle &= \langle A \mid I_{ab} \mid B \rangle , \\ \langle a,A \mid R \mid b,B \rangle &= \langle A \mid R^a{}_b \mid B \rangle , \end{aligned} \tag{2.24}$$

for $a,b = 0,1,2,\ldots,m$ and $A,B = 1,2,\ldots,M$. Then, Eqs. (2.8), (2.14), and (2.19) may be rewritten as

$$R \to \widetilde{R} = URW = UR(I^{-1}U^\dagger I) , \tag{2.8'}$$

$$I^\dagger = I , \tag{2.14'}$$

$$U^\dagger I U = I , \tag{2.19'}$$

in addition to the validity of Eq. (2.3′).

Therefore, Q defined by

$$Q = -I^{-1} R^\dagger I \tag{2.25}$$

transforms also covariantly as

$$Q \to \widetilde{Q} = UQW . \tag{2.26}$$

For a reason which will be explained shortly, it is tempting to assume

$$I\omega + \omega^\dagger I = 0 \tag{2.27a}$$

or equivalently

$$I_{ac}\omega^c{}_b + (\omega^c{}_a)^\dagger I_{cb} = 0 , \tag{2.27b}$$

or

$$I_{ac}\Gamma^c_{\mu b}(x) + [\Gamma^c_{\mu a}(x)]^\dagger I_{cb} = 0 . \tag{2.27c}$$

Then, it is easy to see that we have

$$Q = R \tag{2.28a}$$

or

$$(R^b{}_a)^\dagger = -I_{ac}R^c{}_d(I^{-1})^{db} . \tag{2.28b}$$

It is convenient to lower and raise the small latin indices $a,b,\ldots$ by means of I_{ab} and $(I^{-1})^{ab}$, respectively. Then, setting

$$\omega_{ab} = I_{ac}\omega^c{}_b , \tag{2.29}$$

Eq. (2.27b) gives

$$\omega_{ab} + (\omega_{ba})^\dagger = 0 . \tag{2.30}$$

Therefore, for the special case of $N=M=1$ and $n=m=3$ with real ω_{ab}, Eq. (2.30) is recognized as the condition that the bundle under consideration is now the orthonormal frame bundle,[16] leading to existence of a Riemannian geometry. Because of this, we call the condition Eq. (2.27) the quasimetric (or quasi-Riemannian) condition. Similarly, we may define R^{ab} and R_{ab} by

$$R_{ab} = I_{ac}R^c{}_b , \tag{2.31a}$$

$$R^{ab} = R^a{}_c(I^{-1})^{cb} . \tag{2.31b}$$

Then, Eq. (2.28b) gives

$$(R_{ab})^\dagger = -R_{ba} , \tag{2.32a}$$

$$(R^{ab})^\dagger = -R^{ba} , \tag{2.32b}$$

which is clearly a generalization of the standard formula for the special case of $M=1$ with real R_{ab}. Also, the local gauge transformation Eq. (2.5) leads to

$$R_{ab} \to \widetilde{R}_{ab} = [W^c{}_a(x)]^\dagger R_{cd} W^d{}_b(x) , \tag{2.33a}$$

$$R^{ab} \to \widetilde{R}^{ab} = U^a{}_c(x)R^{cd}[U^b{}_d(x)]^\dagger . \tag{2.33b}$$

We are now in a position to derive a generalization of the Cartan equation. Let us define an $N\times N$ matrix-valued one-form $\omega_{\mu\nu}$ by

$$\omega_{\mu\nu} = [L^a_\mu(x)]^\dagger I_{ab}[\omega^b{}_c L^c_\nu(x) + \partial_\lambda L^b_\nu(x)dx^\lambda] . \tag{2.34}$$

Then, it is straightforward to verify the identity

$$\omega_{\mu\nu} + (\omega_{\nu\mu})^\dagger = [L^a_\mu(x)]^\dagger[I_{ac}\omega^c{}_b + (\omega^c{}_a)^\dagger I_{cb}]L^b_\nu(x) + \partial_\lambda G_{\mu\nu}(x)dx^\lambda .$$

Therefore, if the quasimetric condition Eq. (2.27b) is valid, then this gives

$$\omega_{\mu\nu} + (\omega_{\nu\mu})^\dagger = d\, G_{\mu\nu}(x) \tag{2.35}$$

which is recognized as a generalization of the Cartan equation[15,16] for the Riemannian metric. In order to simplify Eq. (2.34), we assume hereafter that the quasimetric tensor $G_{\mu\nu}(x)$ has its inverse matrix $H^{\mu\nu}(x)$ in a sense that it satisfies

$$H^{\mu\lambda}(x)G_{\lambda\nu}(x) = G_{\nu\lambda}(x)H^{\lambda\mu}(x) = \delta^\mu_\nu E_N , \tag{2.36}$$

where E_N is the $N\times N$ unit matrix. Then, defining an $N\times M$ matrix $M^\mu_a(x)$ by

$$M^\mu_a(x) = H^{\mu\nu}(x)[L^b_\nu(x)]^\dagger I_{ba} , \tag{2.37}$$

we see readily a relation

$$M^\mu_a(x)L^a_\nu(x) = \delta^\mu_\nu E_N , \tag{2.38}$$

so that $M^\mu_a(x)$ is the left inverse of $L^a_\nu(x)$. We note that we *need not*, in general, have the right inverse relation

$$L^a_\mu(x)M^\mu_b(x) = \delta^a_b E_M , \tag{2.39}$$

unless we have a special condition

$$(n+1)N = (m+1)M . \tag{2.40a}$$

Note that the validity of Eq. (2.38) requires only a weaker condition

$$(n+1)N \le (m+1)M . \tag{2.40b}$$

To maintain a generality, we will *not* assume hereafter the condition Eq. (2.39) unless it is stated otherwise. Note that if Eq. (2.40a) is assumed, then Eq. (2.39) follows from Eq. (2.38).

We may now define $\omega^\mu{}_\nu$ and $R^\mu{}_\nu$ by

$$\begin{aligned} \omega^\mu{}_\nu &= M^\mu{}_a(x)\omega^a{}_b L^b_\nu(x) + M^\mu_a(x)\partial_\lambda L^a_\nu(x)dx^\lambda , \\ R^\mu{}_\nu &= M^\mu_a(x)R^a{}_b L^b_\nu(x) , \end{aligned} \tag{2.41}$$

which are $N\times N$ matrix-valued differential forms. We readily derive the familiar Cartan structure equation

$$R^\mu{}_\nu = d\omega^\mu{}_\nu + \omega^\mu{}_\lambda \Lambda \omega^\lambda{}_\nu . \tag{2.42}$$

Defining the $N\times N$ matrix $\Gamma^\mu_{\lambda\nu}(x)$ by

$$\omega^\mu{}_\nu = \Gamma^\mu_{\lambda\nu}(x)dx^\lambda \tag{2.43a}$$

or

$$\Gamma^\mu_{\lambda\nu}(x) = M^\mu_a(x)\Gamma^a_{\lambda b}(x)L^b_\nu(x) + M^\mu_a(x)\partial_\lambda L^a_\nu(x) , \tag{2.43b}$$

Eq. (2.42) reproduces Eq. (1.6) when we set

$$R^\mu{}_\nu = \tfrac{1}{2}R^\mu{}_{\nu\alpha\beta}(x)dx^\alpha \Lambda dx^\beta . \tag{2.44}$$

Now, the relation between $\omega_{\mu\nu}$ introduced by Eq. (2.34) and $\omega^\mu{}_\nu$ is easily found to be

$$\omega_{\mu\nu} = G_{\mu\lambda}(x)\omega^\lambda{}_\nu = G_{\mu\lambda}(x)\Gamma^\lambda_{\alpha\nu}(x)dx^\alpha \tag{2.45}$$

as in the usual Riemannian space. The Cartan equation Eq. (2.35) is rewritten as

$$\partial_\lambda G_{\mu\nu}(x) = G_{\mu\alpha}(x)\Gamma^\alpha_{\lambda\nu}(x) + [\Gamma^\alpha_{\lambda\mu}(x)]^\dagger G_{\alpha\nu}(x) . \tag{2.46}$$

The general solution of Eq. (2.46) for $\Gamma^\mu_{\lambda\nu}(x)$ when $G_{\mu\nu}(x)$ is given is found to be

$$\Gamma^\lambda_{\mu\nu}(x) = \tfrac{1}{2}H^{\lambda\alpha}(x)[\partial_\mu G_{\alpha\nu}(x) + \partial_\nu G_{\alpha\mu}(x) - \partial_\alpha G_{\mu\nu}(x)] + \Sigma^\lambda_{\mu\nu}(x) \tag{2.47}$$

where $\Sigma^\lambda_{\mu\nu}(x)$ is an arbitrary $N\times N$ matrix, subject to constraint

$$G_{\mu\alpha}(x)\Sigma^\alpha_{\lambda\nu}(x) + [\Sigma^\alpha_{\lambda\mu}(x)]^\dagger G_{\alpha\nu}(x) = 0 . \tag{2.48}$$

Similarly, when we define a two-form $R_{\mu\nu}$ by

$$R_{\mu\nu} = G_{\mu\lambda}(x)R^\lambda{}_\nu , \tag{2.49}$$

then the condition Eq. (2.32a) leads to

$$R_{\mu\nu} + (R_{\nu\mu})^\dagger = 0 \tag{2.50}$$

which is again a generalization of the familiar antisymmetric property of $R_{\mu\nu}$ for the exchange of μ and ν in the standard theory. In passing, we note that Eq. (2.50) is the Frobenius integrability condition for Eq. (2.35), i.e., the relation $dd=0$ gives Eq. (2.50), when we operate d to Eq. (2.35). Also, our $R_{\mu\nu}$ is a two-form and has nothing to do with $R_{\alpha\beta}$ introduced in the previous section in spite of the same notation. Similarly, we can define an $N\times N$ matrix-valued torsion tensor $T^{\lambda}_{\mu\nu}(x)$ by

$$T^{\lambda}_{\mu\nu}(x)=M^{\lambda}_{a}(x)T^{a}_{\mu\nu}(x) \tag{2.51}$$

which is rewritten in the familiar form of

$$T^{\lambda}_{\mu\nu}(x)=\Gamma^{\lambda}_{\mu\nu}(x)-\Gamma^{\lambda}_{\nu\mu}(x) . \tag{2.52}$$

Under the local gauge transformation Eq. (2.5) with Eq. (2.11), all $R_{\mu\nu}$ and $T^{\lambda}_{\mu\nu}(x)$ are invariant, i.e.,

$$\begin{aligned} &R_{\mu\nu}\to\widetilde{R}_{\mu\nu}=R_{\mu\nu} ,\\ &T^{\lambda}_{\mu\nu}(x)\to\widetilde{T}^{\lambda}_{\mu\nu}(x)=T^{\lambda}_{\mu\nu}(x) , \end{aligned} \tag{2.53}$$

just as $G_{\mu\nu}(x)$. These facts appear to be at variance with the result of the previous section. However, this is not actually the case, since there exists another local gauge transformation involving only greek indices. Consider the transformation

$$L^{a}_{\mu}(x)\to\widetilde{L}^{a}_{\mu}(x)=L^{a}_{\lambda}(x)S^{\lambda}{}_{\mu}(x) , \tag{2.54a}$$

$$\omega^{a}{}_{b}\to\widetilde{\omega}^{a}{}_{b}=\omega^{a}{}_{b} , \tag{2.54b}$$

where $S^{\lambda}{}_{\mu}(x)$ and its inverse $[S^{-1}(x)]^{\lambda}{}_{\mu}$ are $N\times N$ matrix functions satisfying

$$[S^{-1}(x)]^{\nu}{}_{\lambda}S^{\lambda}{}_{\mu}(x)=S^{\nu}{}_{\lambda}(x)[S^{-1}(x)]^{\lambda}{}_{\mu}=\delta^{\nu}_{\mu}E_{N} . \tag{2.55}$$

Then, it is easy to see that $\omega^{\mu}{}_{\nu}$ and $R^{\mu}{}_{\nu}$ now transform as

$$\omega^{\mu}{}_{\nu}\to\widetilde{\omega}^{\mu}{}_{\nu}=[S^{-1}(x)]^{\mu}{}_{\alpha}\omega^{\alpha}{}_{\beta}S^{\beta}{}_{\nu}(x)+[S^{-1}(x)]^{\mu}{}_{\alpha}d\,S^{\alpha}{}_{\nu}(x) , \tag{2.56a}$$

$$R^{\mu}{}_{\nu}\to\widetilde{R}^{\mu}{}_{\nu}=[S^{-1}(x)]^{\mu}{}_{\alpha}R^{\alpha}{}_{\beta}S^{\beta}{}_{\nu}(x) . \tag{2.56b}$$

Note that Eq. (2.56a) is equivalent to Eq. (1.5) of the previous section with $T^{\mu}{}_{\nu}(x)=[S^{-1}(x)]^{\mu}{}_{\nu}$ while Eq. (2.56b) gives Eq. (1.7). Similarly, we find

$$G_{\mu\nu}(x)\to\widetilde{G}_{\mu\nu}(x)=[S^{\alpha}{}_{\mu}(x)]^{\dagger}G_{\alpha\beta}(x)S^{\beta}{}_{\nu}(x) , \tag{2.57a}$$

$$R_{\mu\nu}(x)\to\widetilde{R}_{\mu\nu}(x)=[S^{\alpha}{}_{\mu}(x)]^{\dagger}R_{\alpha\beta}S^{\beta}{}_{\nu}(x) , \tag{2.57b}$$

which generalize Eqs. (1.23) and (1.28). Therefore, the new local gauge transformation defined by Eqs. (2.54) and (2.55) is a subgroup of GL[$(n+1)N,C$] while the old transformation based upon Eq. (2.11) is a subgroup of GL[$(m+1)M,C$]. We call these two local gauge transformations as external and internal ones, respectively. We have seen that both $R^{\mu}{}_{\nu}$ and $G_{\mu\nu}(x)$ transform covariantly under the new external gauge transformation. However, the torsion tensor $T^{\lambda}_{\mu\nu}(x)$ does *not* transform covariantly under Eq. (2.54) even for a special restriction of $S^{\mu}{}_{\nu}(x)=\delta^{\mu}_{\nu}U(x)$ as in Eq. (1.27). This is due to the fact that the canonical one-form ω^{a} does not possess a consistent transformation law into itself under the external transformation. Therefore, we should not consider the torsion tensor $T^{\lambda}_{\mu\nu}(x)$, if we insist invariance of theory under some form of the external local gauge symmetry.

We briefly mention that the Bianchi identities can be readily derived as usual by operating the differential operator d, for example, to Eqs. (2.42) and (2.10) to find

$$dR^{\mu}{}_{\nu}=R^{\mu}{}_{\lambda}\Lambda\omega^{\lambda}{}_{\nu}-\omega^{\mu}{}_{\lambda}\Lambda R^{\lambda}{}_{\nu} , \tag{2.58a}$$

$$d\,T^{a}=R^{a}{}_{b}\Lambda\omega^{b}-\omega^{a}{}_{b}\Lambda T^{b} , \tag{2.58b}$$

etc., when we note $dd=0$.

So far, we did not discuss the notion of the covariant derivatives. It is well known[7,16] that the differential geometry can be equally well discussed in terms of covariant derivatives. To this end, we note that internal indices $a,b,\dots$ are related to the internal group GL[$(m+1)M,C$] while the external greek indices $\mu,\nu,\dots$ are concerned with the external gauge group GL[$(n+1)N,C$]. First, let $\xi_{\mu}(x)$ be either an $N\times N$ matrix or an N-row vector, or more generally, any $N'\times N$ matrix for some positive integer N'. Then, under the local external group GL[$(n+1)N,C$], it may be assumed to transform as

$$\xi_{\mu}(x)\to\widetilde{\xi}_{\mu}(x)=\xi_{\nu}(x)S^{\nu}{}_{\mu}(x) . \tag{2.59}$$

We may define a external covariant derivative ∇_{μ} by

$$\nabla_{\mu}\xi_{\nu}(x)=\partial_{\mu}\xi_{\nu}(x)-\xi_{\lambda}(x)\Gamma^{\lambda}_{\mu\nu}(x) . \tag{2.60}$$

Then, it is readily computed to behave covariantly as

$$\nabla_{\mu}\xi_{\nu}(x)\to\widetilde{\nabla}_{\mu}\widetilde{\xi}_{\nu}(x)=[\nabla_{\mu}\xi_{\lambda}(x)]S^{\lambda}_{\nu}(x) , \tag{2.61}$$

under Eqs. (2.56a) and (2.59).

Similarly, let $\eta^{\mu}(x)$ be an $N\times N'$ matrix, which transforms as

$$\eta^{\mu}(x)\to\widetilde{\eta}^{\mu}(x)=[S^{-1}(x)]^{\mu}{}_{\lambda}\eta^{\lambda}(x) . \tag{2.62}$$

Then, defining its covariant derivative by

$$\nabla_{\nu}\eta^{\mu}(x)=\partial_{\nu}\eta^{\mu}(x)+\Gamma^{\mu}_{\nu\lambda}(x)\eta^{\lambda}(x) , \tag{2.63}$$

it transforms covariantly as

$$\nabla_{\nu}\eta^{\mu}(x)\to[S^{-1}(x)]^{\mu}{}_{\lambda}\nabla_{\nu}\eta^{\lambda}(x) . \tag{2.64}$$

We can define the covariant derivative of $G_{\mu\nu}(x)$ by

$$\nabla_{\lambda}G_{\mu\nu}(x)=\partial_{\lambda}G_{\mu\nu}(x)-[\Gamma^{\alpha}_{\lambda\mu}(x)]^{\dagger}G_{\alpha\nu}(x)-G_{\mu\alpha}(x)\Gamma^{\alpha}_{\lambda\nu}(x) \tag{2.65}$$

which transforms covariantly as $G_{\mu\nu}(x)$ as in Eq. (2.57a). Then, the metric condition Eq. (2.46) can be reinterpreted in the familiar form of

$$\nabla_{\lambda}G_{\mu\nu}(x)=0 . \tag{2.66}$$

We may define the covariant derivatives for quantities involving internal space indices $a,b,\dots$. Let $\xi_{a}(x)$ and $\eta^{a}(x)$ be any $M'\times M$ and $M\times M'$ matrices, respectively, which transform as

$$\xi_{a}(x)\to\widetilde{\xi}_{a}(x)=\xi_{b}(x)W^{b}{}_{a}(x) , \tag{2.67a}$$

$$\eta^{a}(x)\to\widetilde{\eta}^{a}(x)=U^{a}{}_{b}(x)\eta^{b}(x) , \tag{2.67b}$$

then their covariant derivatives defined by

$$\nabla_\mu \xi_a(x) = \partial_\mu \xi_a(x) - \xi_b(x)\Gamma^b_{\mu a}(x)\ , \tag{2.68a}$$

$$\nabla_\mu \eta^a(x) = \partial_\mu \eta^a(x) + \Gamma^a_{\mu b}(x)\eta^b(x)\ , \tag{2.68b}$$

transform exactly in the same way as $\xi_a(x)$ and $\eta^a(x)$, respectively, under the internal local gauge transformation Eqs. (2.67). Since $L^a_\mu(x)$ transforms as

$$L^a_\mu(x) \to \widetilde{L}^a_\mu(x) = U^a{}_b(x) L^b_\nu(x) S^\nu{}_\mu(x) \tag{2.69}$$

under the combined external and internal local gauge symmetries, its covariant derivative should be defined by

$$\nabla_\lambda L^a_\mu(x) = \partial_\lambda L^a_\mu(x) + \Gamma^a_{\lambda b}(x) L^b_\mu(x) - L^a_\nu(x)\Gamma^\nu_{\lambda\mu}(x)\ . \tag{2.70}$$

We can verify easily that it transforms covariantly exactly the same as $L^a_\mu(x)$. Moreover, we calculate

$$\nabla_\lambda L^a_\mu(x) = [\delta^a_b E_M - L^a_\lambda(x) M^\lambda_b(x)] \times [\partial_\mu L^b_\nu(x) + \Gamma^b_{\mu c}(x) L^c_\nu(x)] \tag{2.71}$$

so that we find

$$\nabla_\lambda L^a_\mu(x) = 0 \tag{2.72}$$

if we have the validity of the extra condition Eq. (2.39).

So far, everything goes smoothly in our theory. However, we encounter the following complication in contrast to the usual theory. Consider the covariant derivative Eq. (2.60). It will be tempting to define the higher-order covariant derivative $\nabla_\mu \nabla_\nu \xi_\lambda(x)$ as usual by

$$\nabla_\mu \nabla_\nu \xi_\lambda(x) = \partial_\mu[\nabla_\nu \xi_\lambda(x)] - [\nabla_\alpha \xi_\lambda(x)]\Gamma^\alpha_{\mu\nu}(x) - [\nabla_\nu \xi_\alpha(x)]\Gamma^\alpha_{\mu\lambda}(x) \tag{2.73}$$

which transforms as a third-rank tensor under the pure coordinate transformation. However, it can be readily shown that $\nabla_\mu \nabla_\nu \xi_\lambda(x)$ does not transform covariantly under the external local gauge group. Nevertheless, we find the familiar identity

$$[\nabla_\mu, \nabla_\nu]\xi_\lambda(x) = -\xi_\alpha(x) R^\alpha{}_{\lambda\mu\nu}(x) - [\nabla_\alpha \xi_\lambda(x)] T^\alpha_{\mu\nu}(x)\ . \tag{2.74}$$

The reason why $\nabla_\mu \nabla_\nu \xi_\lambda(x)$ is no longer covariant is due to noncommutativity of matrices, $\Gamma^\mu_{\lambda\nu}(x)$, $S^\mu_\nu(x)$, and $S^\alpha_\beta(x)$. This fact is also connected with the noncovariance of the torsion tensor $T^\lambda_{\mu\nu}(x)$ as we may see from Eq. (2.74). If we redefine $\nabla_\mu \nabla_\nu \xi_\lambda(x)$ by Eq. (2.73) but without the second term on its right-hand side, it transforms covariantly under Eqs. (2.56a) and (2.59), but *not* then under the pure coordinate transformation.

Our covariant derivative is related to the algebra-valued tangent-space formulation of Mann.[19] He introduces a notion of quaternion-valued or more generally any algebra-valued tangent space, and of the related covariant derivative of the tangent vectors. However, if we note the well-known isomorphism of any associative algebra with the unit element to a subalgebra of a (square) matrix algebra, then the relation between the present theory and the one given in Ref. 19 will become clear. In particular, the quaternion case corresponds to $N=2$. Let V be a vector space over a noncommutative skew field F such as quaternion division algebra. We may define the covariant derivative $\nabla_Y X$ for any F-valued tangent vectors X and Y as usual.[16] However, since a construction of the tensor product $V \otimes V$ is no longer straightforward for any noncommutative skew field F, we will have in general a difficulty of extending[16] the definition of the covariant derivatives for higher-order tensors. Note that we have in general

$$(\lambda X) \otimes Y \neq X \otimes (\lambda Y)$$

for $\lambda \in F$ because of the noncommutativity of the field F. This is another reason why we have a problem of consistently defining covariant derivatives for higher-order tensors such as $\nabla_\mu \nabla_\nu \xi_\lambda(x)$.

Up to now, we did not explicitly discuss the pure coordinate transformation. Both one-forms ω_a and $\omega^a{}_b$ must be invariant under it, so that both $L^a_\mu(x)$ and $\Gamma^a_{\mu b}(x)$ behave as vectors with respect to the greek index μ. Then, $G_{\mu\nu}(x)$ and $\nabla_\mu \xi_\nu(x)$ are clearly second-order tensors. It is not difficult to see that $\Gamma^\mu_{\lambda\nu}(x)$ defined by Eq. (2.43b) transforms as in Eq. (1.3) under the coordinate transformation. In this paper, we have made an implicit distinction between the coordinate transformation and local gauge transformation. This distinction is logical, if the local gauge group has nothing to do with the underlying space-time just as the usual Yang-Mills case. However, for discussions of general relativity, we have to consider either the Lorentz or Poincaré group as a part of the local gauge symmetry. For such a case, we have to make a distinction of two types of coordinate transformations which we may call passive and active. In the differential geometry, we make a coordinate transformation between two coordinate patches[16] covering a given open domain of the space-time. However by this transformation, the real physical points remain unmoved, but only their coordinates are differently labeled in each coordinate patch. We call this kind of transformation as the passive one. On the other hand, either local Lorentz or Poincaré group will map the physical space-time point into another point. We may call this as the active transformation. Therefore for gauge theories involving Lorentz or Poincaré group, we have to consider two types of co-ordinate transformations, strictly speaking. It so turns out that both active and passive transformations are not necessarily independent of each other, although the former must be considered in association with the local Lorentz or Poincaré transformation. However, such a complication will be discussed elsewhere, since the present formalism in this paper is essentially unaffected by this subtlety of the interpretation on the two kinds of coordinate transformations, if proper care is taken.

III. DISCUSSION

In the previous section, we succeeded in constructing a matrix geometry having a quasimetric tensor $G_{\mu\nu}(x)$. As we have emphasized in Sec. I, the spin-3 component of $\Gamma^\lambda_{\mu\nu}(x)$ must be eliminated in order to make a consistent theory. We could do this by assuming either the Einstein *Ansatz* Eq. (1.1) or the quasimetric condition Eq. (2.46).

Moreover, the identification of the Yang-Mills gauge field through Eq. (1.24) is quite tempting. Unfortunately, however, the validity of all these *Ansätze* Eqs. (1.1), (1.24), and (2.46) leads to unacceptable identity $\nabla_\lambda F_{\mu\nu}(x)=0$. Hence, we must abandon or modify at least one of these conditions. In this section, we will consider the possibility that the quasimetric condition Eq. (2.46) as well as possibly Eq. (1.24) is valid but not Eq. (1.1).

To be definite, we assume hereafter $n=3$ so that we have the usual four-dimensional space-time. Also, the reality condition Eq. (2.16) requires $A_\mu(x)$ to be purely anti-Hermitian with c_1 being real in Eq. (1.24). First consider the action given by Eq. (1.22), but not Eq. (1.1). It is invariant under the full internal local gauge group $GL[(m+1)M,C]$ as well as a suitable physical subgroup of the external symmetry $GL(4N,C)$. Assuming Eq. (1.24) for $G_{\mu\nu}(x)$ and Eq. (2.47) for $\Gamma^\mu_{\lambda\nu}(x)$, the action Eq. (1.22) may be regarded now as a quadratic function of $\partial_\lambda G_{\mu\nu}(x)$. Choosing $G_{\mu\nu}(x)$ and possibly $\Sigma^\mu_{\lambda\nu}(x)$ in Eq. (2.47) as independent variational variables, we can find again a generalization of the Einstein-Yang-Mills equation. In this formalism, the quasimetric condition Eq. (2.46) is automatically satisfied but not the Einstein *Ansatz* Eq. (1.1). However, the resulting Lagrangian is a very complicated nonpolynomial function of $F_{\mu\nu}(x)$, which will be perhaps not of any great physical interest. A better method is either the Palatine formulation in which both $G_{\mu\nu}(x)$ and $\Gamma^\mu_{\lambda\nu}(x)$ are independent variational variables or to introduce a Lagrangian-multiplier field $H^{\lambda\mu\nu}(x)$ and add an additional Lagrangian of form

$$L'=H^{\lambda\mu\nu}(x)\{\partial_\lambda G_{\mu\nu}(x)-G_{\mu\alpha}(x)\Gamma^\alpha_{\lambda\nu}(x)-[\Gamma^\alpha_{\lambda\mu}(x)]^\dagger G_{\alpha\nu}(x)\}+\text{H.c.} \tag{3.1}$$

Then, regarding all $G_{\mu\nu}(x)$, $\Gamma^\mu_{\lambda\nu}(x)$, and $H^{\lambda\mu\nu}(x)$ to be now independent variational variables, the quasimetric condition Eq. (2.46) will result automatically from the variation of $H^{\lambda\mu\nu}(x)$.

Next, let us consider a generalization of Trautman's action[20] with $m=3$ of form

$$A=\epsilon_{abcd}\int \mathrm{Tr}[(\omega^a)^\dagger\Lambda R^{bc}\Lambda\omega^d] \tag{3.2}$$

which is real in view of Eq. (2.32b). However, this is not invariant under the general internal $GL(4M,C)$ local gauge transformation. For this reason, we may consider the following modified Trautman action:

$$A=\int\sqrt{-g}\ \mathrm{Tr}[(\omega^a)^\dagger\Lambda^* R_{ab}\Lambda\omega^b]\ , \tag{3.3}$$

where $^*R_{ab}$ is the dual of R_{ab} defined by

$$^*R_{ab}=\tfrac{1}{2}{}^*R_{ab\mu\nu}(x)dx^\mu\Lambda dx^\nu\ , \tag{3.4a}$$

$$^*R_{ab\mu\nu}(x)=\tfrac{1}{2}\epsilon_{\mu\nu\lambda\tau}g^{\lambda\alpha}(x)g^{\beta\tau}(x)R_{ab\alpha\beta}(x)\ . \tag{3.4b}$$

Note that we need not assume $m=3$ and that A given by Eq. (3.3) is real in view of Eq. (2.32a). Moreover, it is clearly invariant under the full internal local gauge transformation $GL[(m+1)M,C]$. We can rewrite Eq. (3.3) as

$$A=\int d^4x[-g(x)]^{1/2}\,\mathrm{Tr}[R_{\alpha\beta}{}^{\alpha\beta}(x)]\ , \tag{3.5a}$$

$$R_{\alpha\beta}{}^{\alpha\beta}(x)=g^{\mu\alpha}(x)g^{\beta\nu}(x)R_{\alpha\beta\mu\nu}(x)\ . \tag{3.5b}$$

For $N=1$, this reduces of course to the standard Hilbert-Palatine action. We can express $\Gamma^\lambda_{\mu\nu}(x)$ in terms of $G_{\mu\nu}(x)$ as in Eq. (2.47), and choose $G_{\mu\nu}(x)$ to be independent variational variable. Or we may add the Lagrange-multiplier term Eq. (3.1). Another method is to regard $L^a_\mu(x)$ and $\Gamma^a_{\mu b}(x)$ as variational variables with constraint Eq. (2.27c). However, the details of calculations will be given elsewhere.

We would like to make the following speculation. First, there appears to be no compelling necessity why we must assume that $L^a_\mu(x)$ corresponds to a generalization of the basis for the fiber of the orthonormal frame bundle.[16] Especially, the internal latin indices $a,b,\ldots,$ etc., may be chosen to represent Dirac spinor indices rather than the Lorentz vector ones for the case of $n+1=m+1=4$ even with $N=M=1$. In that case, $L^a_\mu(x)$ may be regarded as a spin-$\frac{3}{2}$ spinor of Rarita-Schwinger type. Then, the gravity field $g_{\mu\nu}(x)$ as well as possibly Yang-Mills field $F_{\mu\nu}(x)$ associated with $G_{\mu\nu}(x)$ may be viewed, in a sense, to be bound-states of spin-$\frac{3}{2}$ fermion pair. Note that we have to choose $I_{ab}=(\gamma_0)_{ab}E_M$ for this case in contrast to Eq. (2.17), where γ_μ $(\mu=0,1,2,3)$ are usual 4×4 Dirac matrices. Also, $\xi_a(x)$ and $\eta^a(x)$ introduced in the previous sections are regarded as spinors, whose covariant derivatives are simply defined by Eq. (2.67) with spin connection coefficient $\Gamma^b_{\mu a}(x)$. In contrast, the covariant derivative of a spinor field will be somewhat more involved, if the internal latin indices $a,b,\ldots,$ etc., are now considered to be vector indices of, say, the $SO(m,1)$ group as in the orthonormal frame. For such a case, let $\psi(x)$ be an $M'\times M''$ matrix-valued spinor of the $SO(m,1)$ group. Then, the covariant derivative $\nabla_\mu\psi(x)$ may be defined by

$$\nabla_\mu\psi(x)=\partial_\mu\psi(x)-\Gamma_\mu(x)\psi(x) \tag{3.6}$$

for some matrix spin connection $\Gamma_\mu(x)$. The explicit form of $\Gamma_\mu(x)$ must be determined by requiring the covariant transformation law

$$\psi(x)\to\tilde\psi(x)=S(x)\psi(x)\ ,\qquad \nabla_\mu\psi(x)\to\tilde\nabla_\mu\tilde\psi(x)=S(x)\nabla_\mu\psi(x)\ , \tag{3.7}$$

for some matrix $S(x)$ belonging to a suitable but yet unspecified class. This requires the validity of

$$\Gamma_\mu(x)\to\tilde\Gamma_\mu(x)=S^{-1}(x)\Gamma_\mu(x)S(x)+S^{-1}(x)\partial_\mu S(x)\ . \tag{3.8}$$

However, the problem is that a group generated by all $S(x)$ will not in general coincide with the internal local gauge group discussed in the previous section. Moreover, the existence of $\Gamma_\mu(x)$ compatible with the $SO(m,1)$ spinor character of $\psi(x)$ is by no means self-evident except for the usual case of $N=M=1$ and $M'=M''=1$. One way of circumventing these problems is to assume that $\psi(x)$ is a spinor of a noncompact form of the larger $SO[(m+1)M]$ group, since then the construction of $\Gamma_\mu(x)$ is straightforward.[5,15]

Concluding this paper, we may generalize the choice of the internal indices $a,b,\ldots,$ etc., to include the Grassmann degree of freedom. Then, we may be able to construct a generalization of supergravity theory. However, these possibilities and speculations will be left for future investigations. Also, we noted that in our theory the torsion tensor $T^{\lambda}_{\mu\nu}(x)$ as well as higher-order covariant derivatives such as $\nabla_\mu \nabla_\nu \xi_\lambda(x)$ do not behave covariantly even under the restricted transformation Eq. (1.27). This may imply that these quantities should not enter in theory, especially in construction of the Lagrangian. This will impose some restrictions in contrast to those considered in Refs. 13 and 14. Hence, the noncovariance may be viewed as a welcome addition to the theory rather than a defect.

Note added. After this paper was written, it came to our attention that the subject matter of Ref. 19 has also been discussed by J. W. Moffat, J. Math. Phys. **25**, 347 (1984) and lectures given at the Sir Arthur Eddington Centenary Symposium, Nagpur, India (to be published by World Scientific, Singapore). Also, gravity as a gauge theory is discussed in a paper by S. J. Gates and J. A. Shapiro, Phys. Rev. D **18**, 2768 (1978). The authors would like to express their gratitude to Professor Moffat and Professor Shapiro for calling our attention to these references.

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