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Four forces and spinor connection in general relativity

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This work is an extension of the recent spinor-connection theory of Szekeres, Cullinan, and Lynch. In that theory, a geometric model for the gravitational and electromagnetic fields was realized by use of both left- and right-connection groups acting on a 4×4 spinor tetrad. Here the right-connection group is enlarged in a natural way from a one-parameter to a three-parameter Lie group. This enlargement introduces two extra potential fields which may provide a simple model for the strong and weak fields in curved space-time. A solution to the new field equations is given for a neutral "pionlike" particle exhibiting the strong and gravitational fields.

I. INTRODUCTION

Szekeres, Cullinan, and Lynch[1] (henceforth SCL) have recently proposed a geometric model for the gravitational and electromagnetic fields in terms of left and right spinor-tetrad connection fields in general relativity. It has been demonstrated[2] that the field equations, derived from a single world Lagrangian function, may admit an essentially unique cosmological solution and a variety of particlelike solutions, with or without electric charge or spin.

However, the SCL theory exhibits no fields which can in any way account for either the weak or strong forces. There is, it is true, a short-range field in some way similar to the torsion field of U_4 theory, but such a field is interpreted as a mere spin contact interaction.[3]

We show here that the group of right connection in the SCL theory is by no means the most natural choice. With a natural and in a sense obligatory extension, from U(1) to $U(1)\otimes U(1)\otimes U(1)$, we obtain two extra four-potential fields which might feasibly provide a classical model for the weak and strong forces. This extended group is the largest group which leaves invariant the four left-ideal subspaces of the Clifford algebra used to represent the quartet of Dirac four-spinors (the matter field).

All other essential features of SCL are retained and, in particular, unlike in U_4 theory, the base manifold remains as the pseudo-Riemannian space of general relativity, with vector connection defined by means of the symmetric Riemann-Christoffel affinity. Physically, the group extension introduces two new "charges" as sources for the proposed strong and weak fields. The required short-range nature of these fields results from the imposition of appropriate boundary conditions on the solutions, so that the far-field Gaussian fluxes of the field intensities are zero. Thus the new "charges" are not observable in the same sense as is the usual electric charge. The relative strengths of the fields are accounted for by the relative magnitudes of the source "charges," which are conserved.

The field equations, though again derived from a single world Lagrangian function, should, in principle, admit a very great variety of classical particlelike solutions, depending on the choice of combination of the Clifford spinor-ideals and on the symmetries of the metric field. We give as an illustration of these potentialities a two-ideal, spherically symmetric, single-particle solution which manifests the strong and gravity fields only. The (dimensionless) strong coupling constant is assumed to be exactly unity, as it cannot it seems be determined from the field of this single particle. Except in the rather dramatic but well-behaved Schwarzschild region, where recourse to machine integration has been necessary, the solution can be given in explicit functional form and it can be carried right down to the origin. Identifying the solution as a classical "neutral pion," we calculate that near the "surface" of the particle the strong force is 1×10^{38} times as strong as that of gravity. Further, with this identification, the single fundamental mass required by the world Lagrangian, that is the mass whose Compton radius defines the scale of length for the geometry, is calculated to be within two percent of the μ-meson mass.

II. SPINOR CONNECTION AND CURVATURE

As in SCL, the base manifold is just the pseudo-Riemannian Einstein space E_4 with local coordinates x^μ, $\mu=1,2,3,4$. The manifold E_4 carries a variety of complex vector spaces in which representations of the proper, homogeneous Lorentz group L_4 are induced.

Firstly, there is the usual vierbein space V_4 which is equipped with orthonomal basis vectors and which has the group L_4 as a left operator domain. A vector $v\in V_4$ has components v_n, $n=1,2,3,4$ with $j_n v_n$ real, where $j_1=j_2=j_3=ij_4=i$, $i^2=-1$, and where the summation convention is suspended as always in expressions involving the j_n. The space V_4 induces the Minkowski representation of L_4, the action of $\sigma\in L_4$ on $v\in V_4$ being $(\sigma v)_m=M_{mn}v_n$, where M_{mn} has positive unit determinant and each component has the reality property that $j_m j_n M_{mn}$ is real. The local metric field

$$g=(g_{n\mu}),\quad ||g||=i\det(g_{n\mu})\neq0 \tag{2.1}$$

is a nonsingular linear mapping, $g_{n\mu}\lambda^\mu$, from the tangent space of E_4 to V_4. For arbitrary tangent vectors λ^μ,ϕ^ν, the inner product $g_{m\mu}g_{m\nu}\lambda^\mu\phi^\nu$ is a real scalar, invariant under both the action of L_4 in V_4 and the group of general coordinate transformations in E_4. The metric tensor of E_4 is taken as

$$g_{\mu\nu}=g_{m\mu}g_{m\nu}, \quad ||g||=(-\det g_{\mu\nu})^{1/2} \tag{2.2}$$

with $g^{\mu\nu}=g_m{}^{\mu}g_m{}^{\nu}$, where $g_m{}^{\mu}g_{m\nu}=\delta^{\mu}{}_{\nu}$ and $g_m{}^{\mu}g_{n\mu}=\delta_{mn}$.

As well as the vierbein space V_4 we need the usual skew-tensor space W_6 spanned by the wedge products $u \wedge v$ of V_4 vectors. A vector $w \in W_6$ has skew components w_{mn} with $j_m j_n w_{mn}$ real and the two-sided action of $\sigma \in L_4$ is expressed by $(\sigma w \sigma^{-1})_{mn}=M_{mp}M_{nq}w_{pq}$.

To represent the spinor-tetrad we employ a 16-dimensional complex Clifford algebra Ω generated by the Dirac matrices Γ_m, $m=1,2,3,4$, with $\Gamma_m\Gamma_n+\Gamma_n\Gamma_m=2\delta_{mn}I$. Following Eddington,[4] a convenient basis for Ω is obtained in terms of the 15 symbols $\Gamma_{ab}=-\Gamma_{ba}$, $0 \le a < b \le 5$, defined by

$$\Gamma_{0m}=\Gamma_m, \quad \Gamma_{05}=-\Gamma_1\Gamma_2\Gamma_3\Gamma_4 , \tag{2.3}$$

$$\Gamma_{ab}=-i\Gamma_{0a}\Gamma_{0b}, \quad 1 \le a < b \le 5 \tag{2.4}$$

with the general multiplication rule

$$\Gamma_{ab}\Gamma_{cd}=(\delta_{ac}\delta_{bd}-\delta_{ad}\delta_{bc})I+\tfrac{1}{2}E_{abcdef}\Gamma_{ef} +i(\delta_{ac}\Gamma_{bd}+\delta_{bd}\Gamma_{ac}-\delta_{ad}\Gamma_{bc}-\delta_{bc}\Gamma_{ad}) , \tag{2.5}$$

where E_{abcdef} is the alternating symbol on 0,1,2,3,4,5. The sixteen elements I,Γ_{ab} form a basis for Ω over the complex field, so that any element $\omega \in \Omega$ is written $\omega=\alpha I+\frac{1}{2}\gamma_{ab}\Gamma_{ab}$. The Γ_{ab} have a representation by 4×4 traceless Hermitian matrices (Appendix A). The Hermitian conjugate $\omega^{(*)}$ of ω is thus $\omega^{(*)}=\alpha^* I+\frac{1}{2}\gamma^*_{ab}\Gamma_{ab}$ where γ^*_{ab} is the complex conjugate of the number γ_{ab}.

The adjoint $\omega^{\dagger}$ of $\omega \in \Omega$ is defined by

$$\omega^{\dagger}=\Gamma_{04}\omega^{(*)}\Gamma_{04} . \tag{2.6}$$

The skew-adjoint ($\omega^{\dagger}=-\omega$) elements $\frac{1}{2}ij_m j_n\Gamma_{mn}$ span the real Lie algebra of the spin representation of the simply connected covering group S_4 of L_4. An element $\sigma \in S_4$ is represented in Ω by

$$\sigma=\exp(\tfrac{1}{4}iP_{mn}\Gamma_{mn}), \quad j_m j_n P_{mn} \text{ real} \tag{2.7}$$

so that $\sigma^{\dagger}=\sigma^{-1}$.

The algebra Ω admits S_4 as a left operator domain. This left representation space, denoted by Ω^l, can be decomposed into a direct sum of four minimal left ideals,[5] say $\Omega^{(\eta,\epsilon)}$ with $\eta=\pm1$, $\epsilon=\pm1$, so that

$$\Omega^l=\sum_{\eta,\epsilon}\Omega^{(\eta,\epsilon)}=\Omega^{(+,+)}\oplus\Omega^{(+,-)}\oplus\Omega^{(-,+)}\oplus\Omega^{(-,-)} . \tag{2.8}$$

For some fixed η,ϵ, an element $\psi^{(\eta,\epsilon)}$ of $\Omega^{(\eta,\epsilon)}$ is a Dirac four-spinor. A general element $\psi \in \Omega^l$ has a decomposition into a quartet of Dirac four-spinors

$$\psi=\sum_{\eta,\epsilon}\psi^{(\eta,\epsilon)} \tag{2.9}$$

and is called a spinor-tetrad.

The adjoint $\psi^{\dagger}$ of a spinor-tetrad $\psi \in \Omega^l$ is defined by

$$\psi^{\dagger}=\psi^{(*)}\Gamma_{04} . \tag{2.10}$$

If $\omega \in \Omega, \psi \in \Omega^l$, then $\omega\psi \in \Omega^l$, and it follows from (2.6) and (2.10) that $(\omega\psi)^{\dagger}=\psi^{\dagger}\omega^{\dagger}$ and from (2.7) that $(\sigma\psi)^{\dagger}=\psi^{\dagger}\sigma^{-1}$ if $\sigma \in S_4$. Thus, if ψ_1 and ψ_2 are spinor-tetrads then the inner product

$$\langle\psi_1\psi_2\rangle=\tfrac{1}{4}\mathrm{Tr}\psi_1^{\dagger}\psi_2 \tag{2.11}$$

is invariant under the action of S_4.

The decomposition (2.9) of Ω^l can be performed so that the constituent spinor ideals are mutually orthogonal under the inner product (2.11). Such a decomposition is given by

$$\psi^{(\eta,\epsilon)}=\sum_{n=1}^{4}u_n^{(\eta,\epsilon)}Y_n^{(\eta,\epsilon)} , \tag{2.12}$$

where the four $u_n^{(\eta,\epsilon)}$ are complex functions of the coordinates x^{μ} at the point of attachment to E_4 and the four $Y_m^{(\eta,\epsilon)}$ which span the ideal $\Omega^{(\eta,\epsilon)}$ are given by

$$Y_1^{(\eta,\epsilon)}=\tfrac{1}{4}(\Gamma_{01}+i\eta\Gamma_{15}+i\epsilon\Gamma_{02}-\eta\epsilon\Gamma_{25}) , \tag{2.13}$$

$$Y_2^{(\eta,\epsilon)}=\tfrac{1}{4}(\Gamma_{03}+i\eta\Gamma_{35}+i\eta\epsilon\Gamma_{04}-\epsilon\Gamma_{45}) , \tag{2.14}$$

$$Y_3^{(\eta,\epsilon)}=\tfrac{1}{4}(\Gamma_{23}+\eta\Gamma_{14}+i\epsilon\Gamma_{31}+i\eta\epsilon\Gamma_{24}) , \tag{2.15}$$

$$Y_4^{(\eta,\epsilon)}=\tfrac{1}{4}(\Gamma_{12}+\eta\Gamma_{34}-\eta\epsilon\Gamma_{05}-\epsilon I) . \tag{2.16}$$

The algebra Ω also induces the usual scalar, pseudoscalar, vector, pseudovector, and skew-tensor representations of S_4 with the irreducible representation spaces being spanned by the basis elements (I), (Γ_{05}), (Γ_{0n}), (Γ_{n5}), and (Γ_{mn}). In particular, if $v \in V_4$ and $w \in W_6$, then $v_n\Gamma_{0n}$ and $w_{mn}\Gamma_{mn}$ are self-adjoint ($\omega^{\dagger}=\omega$) and are here called vector and skew-tensor operators, respectively, with the transformation laws

$$\sigma(v_n\Gamma_{0n})\sigma^{-1}=M_{mp}v_p\Gamma_{0m} , \tag{2.17}$$

$$\sigma(w_{mn}\Gamma_{mn})\sigma^{-1}=M_{pq}M_{kl}w_{ql}\Gamma_{pk} \tag{2.18}$$

as can be verified from (2.7) with the Minkowski matrix $(M_{mn})=\exp(P_{mn})$, with (P_{mn}) infinitesimal.

The fact that the spinor-tetrad (2.12) is to play a more fundamental role in our theory than the usual Dirac four-spinor has crucial significance when we seek a natural choice for the group of spinor connection. Cullinan[6] noted that in undergoing an infinitesimal displacement along a curve in E_4, the spinor-tetrad should logically admit transformation from the right in addition to the conventional transformation from the left. If $\tau \in \Omega$ is such a right transformation then the inner product (2.11) is invariant provided $\tau\tau^{(*)}=\tau^{(*)}\tau=I$. Thus the right-acting group must be unitary or special unitary. This requirement is rather too general as a starting point and a more specific requirement[7] is that the minimal left ideals in the decomposition (2.8) of Ω^l should be invariant subspaces, that is $\Omega^{(\eta,\epsilon)}\tau \in \Omega^{(\eta,\epsilon)}$. In SCL the right group is taken to be the simplest that is admissible, namely, U(1). However, it is readily verified that the decomposition (2.12) allows by this criterion a larger, maximal group, namely, U(1)⊗U(1)⊗U(1), since the individual minimal left-ideals $\psi^{(\eta,\epsilon)}$ are eigenspaces under the right action of not one but precisely three (nontrivial) basis elements of Ω, with the multiplication rules

$$\psi^{(\eta,\epsilon)}\Gamma_{05}=\eta\psi^{(\eta,\epsilon)}\ , \tag{2.19}$$

$$\psi^{(\eta,\epsilon)}\Gamma_{12}=-\epsilon\psi^{(\eta,\epsilon)}\ , \tag{2.20}$$

$$\psi^{(\eta,\epsilon)}\Gamma_{34}=-\epsilon\eta\psi^{(\eta,\epsilon)}\ . \tag{2.21}$$

Thus we take here for the right group the three-parameter group U_3 of unitary transformations

$$\tau=\exp(i\lambda_{(x)}\Gamma_{(x)}),\ \ \lambda_{(x)}\text{ real},\ \ x=1,2,3 \tag{2.22}$$

$$\Gamma_{(x)}=(\Gamma_{(1)},\Gamma_{(2)},\Gamma_{(3)})=(\Gamma_{05},\Gamma_{12},\Gamma_{34})\ . \tag{2.23}$$

The effect of an element $\tau\in U_3$ is to provide each $\psi^{(\eta,\epsilon)}$ with a phase factor $\exp[i(\eta\lambda_{(1)}-\epsilon\lambda_{(2)}-\epsilon\eta\lambda_{(3)})]$. From (2.5), the $\Gamma_{(x)}$ have the commutation rules

$$[\Gamma_{(x)},\Gamma_{(y)}]_+=2\Gamma_{(z)},\ \ [\Gamma_{(x)},\Gamma_{(y)}]_-=0\ ,$$

where (x,y,z) is any permutation of (1,2,3). The decomposition (2.12) as specified by the $Y_n^{(\eta,\epsilon)}$ of (2.13) to (2.16) is not of course unique, but it can be shown that the corresponding $\Gamma_{(x)}$ exist for any decomposition. We require a fixed, global decomposition (i.e., the same decomposition at each point of attachment in E_4) and without loss of generality take it as given by (2.13) to (2.16).

Taking now $S_4\otimes U_3$ as the connection group, the covariant derivative of a spinor tetrad is

$$\psi_{|\mu}=\psi_{,\mu}-\tfrac{1}{4}iS_{mn\mu}\Gamma_{mn}\psi-\tfrac{1}{2}iK_{(x)\mu}\psi\Gamma_{(x)}\ , \tag{2.24}$$

where the comma derivative denotes the partial derivative ∂_μ and $\frac{1}{4}iS_{mn\mu}\Gamma_{mn}$ and $\frac{1}{2}iK_{(x)\mu}\Gamma_{(x)}$ are elements of the Lie algebras of S_4 and U_3, respectively, both covariant with respect to μ and with $j_mj_nS_{mn\mu}$ and $K_{(x)\mu}$ real. The condition $(\sigma\psi\tau)_{|\mu}=\sigma(\psi_{|\mu})\tau$ gives with (2.7) and (2.22) the connection field transformation laws

$$\widetilde{S}_{mn\mu}=M_{mp}M_{nq}S_{pq\mu}+M_{mp,\mu}M_{np}\ , \tag{2.25}$$

$$\widetilde{K}_{(x)\mu}=K_{(x)\mu}+\lambda_{(x),\mu}\ . \tag{2.26}$$

Thus $S_{mn\mu}$ is not a tensor of W_6 and $K_{(x)\mu}$ are Weyl gauge fields. One left- and three right-spin-curvature tensors follow from

$$\psi_{|\mu|\nu}-\psi_{|\nu|\mu}=-\tfrac{1}{4}iR_{mn\mu\nu}\Gamma_{mn}\psi-\tfrac{1}{2}iP_{(x)\mu\nu}\psi\Gamma_{(x)}\ , \tag{2.27}$$

$$R_{mn\mu\nu}=S_{mn\mu,\nu}-S_{mn\nu,\mu}+S_{mp\mu}S_{pn\nu}-S_{mp\nu}S_{pn\mu}\ , \tag{2.28}$$

$$P_{(x)\mu\nu}=K_{(x)\mu,\nu}-K_{(x)\nu,\mu}\ . \tag{2.29}$$

From (2.25) and (2.8) it follows that $R_{mn\mu\nu}\Gamma_{mn}$ is a tensor operator, of type (2.18), and hence

$$R_{\mu\nu\alpha\beta}=g_{m\mu}g_{n\nu}R_{mn\alpha\beta} \tag{2.30}$$

is a real left curvature tensor in E_4. The right curvature tensors (2.29), like the left, are invariant under all S_4 and U_3 transformations.

The left curvature tensor $R_{\mu\nu\alpha\beta}$ can be expressed in terms of the usual Riemann tensor $G_{\mu\nu\alpha\beta}$ and a spin-connection torsion tensor $T_{\lambda\mu\nu}$ as follows.[8] Defining

$$Q_{mn\nu}=g_m{}^{\rho}g_{n\mu}\Gamma^{\mu}_{\rho\nu}-g_m{}^{\lambda}g_{n\lambda,\nu}\ , \tag{2.31}$$

$$\Gamma^{\mu}_{\rho\nu}=\tfrac{1}{2}g^{\lambda\mu}(g_{\lambda\rho,\nu}+g_{\lambda\nu,\rho}-g_{\rho\nu,\lambda})\ , \tag{2.32}$$

$$T_{mn\mu}=S_{mn\mu}-Q_{mn\mu}\ , \tag{2.33}$$

it follows that $T_{mn\mu}$ is a skew tensor of W_6 for fixed μ, since $Q_{mn\mu}$ has the same left-connection transformation law (2.25) as $S_{mn\mu}$. Thus,

$$T_{\mu\nu\rho}=-T_{\nu\mu\rho}=g_{m\mu}g_{n\nu}T_{mn\rho} \tag{2.34}$$

is a real torsion tensor in E_4. From (2.28) and (2.33) it then follows that

$$R_{\mu\nu\rho\sigma}=G_{\mu\nu\rho\sigma}+T_{\mu\nu\rho;\sigma}-T_{\mu\nu\sigma;\rho}-T_{\mu\lambda\sigma}T^{\lambda}_{\ \nu\rho}+T_{\mu\lambda\rho}T^{\lambda}_{\ \nu\sigma}\ , \tag{2.35}$$

where the semicolon derivative is the usual covariant derivative with respect to the Riemann-Christoffel affinity $\Gamma^{\sigma}_{\beta\gamma}$ of (2.32) and $G^{\mu}_{\ \nu\rho\sigma}$ is the Riemann tensor constructed from this affinity, viz.,

$$G^{\mu}_{\ \nu\rho\sigma}=\Gamma^{\mu}_{\nu\sigma,\rho}-\Gamma^{\mu}_{\nu\rho,\sigma}+\Gamma^{\nu}_{\lambda\rho}\Gamma^{\lambda}_{\nu\sigma}-\Gamma^{\mu}_{\lambda\sigma}\Gamma^{\lambda}_{\nu\rho}\ . \tag{2.36}$$

The Ricci tensor $G_{\mu\nu}$ and the Ricci scalar $G=G^{\mu}_{\mu}$ are obtained from the Riemann tensor by the usual contraction of indices by use of $g_{\alpha\beta}$.

III. FIELD EQUATIONS

The world Lagrangian real scalar density function is taken, with obvious classical analog, as

$$L=||g||\left[-g_m{}^{\mu}g_n{}^{\nu}R_{mn\mu\nu}-\frac{\Lambda^2}{T}b_{(x)}P_{(x)\mu\nu}P_{(x)}{}^{\mu\nu}+\frac{i}{2}\frac{\Lambda^2}{T}\left\langle\left[\Gamma^{\mu}\psi_{|\mu}+\frac{i}{\Lambda}\psi\right]\psi-\psi\left[\Gamma^{\mu}\psi_{|\mu}+\frac{i}{\Lambda}\psi\right]\right\rangle\right]\ , \tag{3.1}$$

where the repeated x are summed over 1,2,3 and Γ^{μ} is the vector operator, of type (2.17),

$$\Gamma^{\mu}=g_n{}^{\mu}\Gamma_{0n}\ . \tag{3.2}$$

The $b_{(x)},T$ are taken to be four dimensionless coupling constants[9] and the constant Λ is included to achieve dimensional consistency. It has the dimension of length and sets the scale of length for the geometry. We apply Hamilton's principle to the action $\int L\,d^4x$, where the independent variables are the left- and right-connection, metric and spinor-tetrad fields $S_{mn\mu}$, $K_{(x)\mu}$, $g_{m\mu}$, and ψ, respectively. The four sets of equations which result are

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$$T_{\lambda\mu\nu}=\frac{\Lambda^2}{8T}E_{\lambda\mu\nu\sigma}\langle\psi\Gamma_{05}\Gamma^{\sigma}\psi\rangle\ , \tag{3.3}$$

$$P_{(x)}{}^{\mu\nu}{}_{;\nu}=\frac{1}{8b_{(x)}}\langle\psi\Gamma^{\mu}\psi\Gamma_{(x)}\rangle\ \text{(no } x \text{ summation)}\ , \tag{3.4}$$

$$G_{\mu\nu}-\tfrac{1}{2}g_{\mu\nu}G=2b_{(x)}\frac{\Lambda^2}{T}(g^{\rho\sigma}P_{(x)\rho\mu}P_{(x)\sigma\nu}-\tfrac{1}{4}g_{\mu\nu}P_{(x)\alpha\beta}P_{(x)}{}^{\alpha\beta})+\tfrac{1}{2}g_{\mu\nu}T^{\alpha\beta\sigma}T_{\alpha\beta\sigma}$$
$$+\frac{\Lambda^2}{8T}(K_{(x)\mu}\langle\psi\Gamma_\nu\psi\Gamma_{(x)}\rangle+K_{(x)\nu}\langle\psi\Gamma_\mu\psi\Gamma_{(x)}\rangle)-\frac{i\Lambda^2}{8T}\langle\psi_{;\mu}\Gamma_\nu\psi+\psi_{;\nu}\Gamma_\mu\psi-\psi\Gamma_\nu\psi_{;\mu}-\psi\Gamma_\mu\psi_{;\nu}\rangle\ , \tag{3.5}$$

$$-i\Gamma^\mu\psi_{;\mu}-\tfrac{1}{4}T^{\alpha\beta\gamma}E_{\alpha\beta\gamma\mu}\Gamma_{05}\Gamma^\mu\psi-\tfrac{1}{2}K_{(x)\mu}\Gamma^\mu\psi\Gamma_{(x)}+\Lambda^{-1}\psi=0\ , \tag{3.6}$$

where summation over repeated x index is carried out everywhere except on the right of (3.4) and where

$$E_{\lambda\mu\nu\sigma}=||g||\mathrm{sig}(\lambda\mu\nu\sigma) \tag{3.7}$$

is the alternating tensor of the manifold and

$$\psi_{;\mu}=\psi_{,\mu}-\tfrac{1}{4}iQ_{mn\mu}\Gamma_{mn}\psi \tag{3.8}$$

with $Q_{mn\mu}$ being defined by (2.31).

The equations (3.4) are the differential equations for the three right-connection four-vector fields $K_{(x)\mu}$. In SCL there is just one of these right-connection fields and it is interpreted as the geometric image of the electromagnetic four-potential A_μ. Here we give this electromagnetic role to $K_{(2)\mu}$, associated with the Lie algebra element $iK_{(2)\mu}\Gamma_{(2)}$, although, in principle, any one of the generators $i\Gamma_{(x)}$ could be so labeled. We take $A_{(2)\mu}$ as the electromagnetic four-potential in Heaviside-Lorentz units and put

$$K_{(2)\mu}=2q_{(2)}(\hbar c)^{-1}A_{(2)\mu}\ , \tag{3.9}$$

$$P_{(2)\mu\nu}=2q_{(2)}(\hbar c)^{-1}F_{(2)\mu\nu}$$
$$=2q_{(2)}(\hbar c)^{-1}[A_{(2)\mu,\nu}-A_{(2)\nu,\mu}]\ , \tag{3.10}$$

$$\Lambda=\frac{\hbar}{M_0c}\ , \tag{3.11}$$

where $q_{(2)}$ and M_0 denote an electric charge and a rest mass, respectively, and where $F_{(2)\mu\nu}$ is Maxwell's field tensor. With these identifications, the flat-space version of (3.6) for a single minimal left ideal element $\psi^{(\eta,\epsilon)}$ and with neglect of torsion $T_{\lambda\mu\nu}$ and the potentials $K_{(1)\mu}$ and $K_{(3)\mu}$ becomes, on account of (2.20) and (2.23),

$$i\hbar c\Gamma^\mu\psi_{,\mu}-\epsilon q_{(2)}A_{(2)\mu}\Gamma^\mu\psi-M_0c^2\psi=0\ . \tag{3.12}$$

This equation is Dirac's equation for a charge $-\epsilon q_{(2)}$ and rest mass M_0. Equation (3.4) gives

$$F_{(2)}{}^{\mu\nu}{}_{,\nu}=\frac{-\epsilon\hbar c}{16q_{(2)}b_{(2)}}\langle\psi\Gamma^\mu\psi\rangle\ . \tag{3.13}$$

In the case of the spherically symmetric field of a charge $-\epsilon q_{(2)}$ with Coulomb's electrostatic field intensity (at large r)

$$E_{(2)}=\frac{-\epsilon q_{(2)}}{4\pi r^2} \tag{3.14}$$

we need the space volume integral of the divergence of the (global) field intensity $E_{(2)}$ to satisfy the Maxwell-Gauss condition

$$\int\vec{\nabla}\cdot\vec{\mathrm{E}}_{(2)}d^3x=\int F_{(2)}{}^{4i}{}_{,i}d^3x=-\epsilon q_{(2)}\ , \tag{3.15}$$

where i is summed over 1,2,3. Implicit here is the non-Coulomb requirement that $r^2E_{(2)}\to 0$ as $r\to 0$, since

$$\int\vec{\nabla}\cdot\vec{\mathrm{E}}_{(2)}d^3x=4\pi\int d(r^2E_{(2)})$$

for a radially symmetric field. To meet (3.15) we could take, for example, the conventional normalization,

$$\int||g||\langle\psi\Gamma^4\psi\rangle d^3x=1\ ,$$

in which case from (3.13) we would have $b_{(2)}=(64\pi\alpha)^{-1}$, where α is the electromagnetic fine-structure constant, $q_{(2)}{}^2(4\pi\hbar c)^{-1}$. If we now take

$$T=\hbar c(16\pi G_nM_0{}^2)^{-1}\ , \tag{3.16}$$

where G_n is Newton's gravitational constant, the field equations (3.5) are, in the appropriate weak-field limit, the Einstein-Maxwell equations for the free electromagnetic field.

In analogy with the above, the two fields $K_{(1)\mu},K_{(3)\mu}$ resulting from the enlarged connection group are now associated with "strong" and "weak" charges, respectively. We introduce as sources for these fields strong and weak "charges" $q_{(1)}$ and $q_{(3)}$, respectively. If the corresponding curvature tensors $P_{(1)\mu\nu}$ and $P_{(3)\mu\nu}$ are short range, decaying faster than inverse square at large r, the charges $q_{(1)}$ and $q_{(3)}$ will give no observable flux far from the sources, in contrast with the electric charge. Equation (3.15) will hold if the corresponding field intensities $E_{(1)}$ and $E_{(3)}$ are Coulomb-like as $r\to 0$ rather than as $r\to\infty$. This would be quite unacceptable in special relativity, causing for example infinite self-energy, since the energy density in that theory is given by $E_{(x)}{}^2$. We shall see however that the influence of the metric and torsion fields removes this infinite-energy problem in the present theory (at least in the particular case later demonstrated).

We note that the analogy made with Dirac's equation (3.12) is purely formal. The field potential A_μ in that equation refers to the so-called "external" field, whereas no such dichotomy is envisaged in Eq. (3.6), where the $K_{(x)\mu}$ field refers to the total field, including that of the source itself. Likewise, there is only superficial similarity between the torsion of (3.3) and the torsion of U_4 theory.[3] In U_4 theory, the torsion originates from Cartan's nonsymmetric connection for real vectors in the space-time manifold and is physically interpreted in terms of spinning matter. Our torsion has a totally different origin. Further, its presence need not indicate any spinning matter at all, as we shall shortly see.

In SCL the existence of positive and negative electric charge follows from the eigenproperties of the left ideals under the action of the right-connection group. Likewise, the strong and weak charges will appear with opposite signs in different ideals $\psi^{(\eta,\epsilon)}$. Using the right multiplica-

TABLE I. "Charge sign" table as constructed from the right multiplication rules (2.19) to (2.21).

Charge \ Ideal	$\psi^{(+,+)}$	$\psi^{(+,-)}$	$\psi^{(-,+)}$	$\psi^{(-,-)}$
Strong	+	+	−	−
Electric	−	+	−	+
Weak	−	+	+	−

tion rules (2.19) to (2.21) we can draw up a "charge sign" table as in Table I.

IV. NEUTRAL PIONLIKE PARTICLE

The π^0 has zero spin and takes part in the strong and gravitational interactions. We take as our model a field with exact spherical symmetry which carries the strong charge only. Numbering the coordinates x^μ by (r,θ,ϕ,t) and using Λ of (3.11) as the unit of length, the line element is

$$ds^2=-f_0{}^2dr^2-r^2(d\theta^2+\sin^2\theta\, d\phi^2)+g_0{}^2dt^2 , \quad (4.1)$$

where f_0 and g_0 are real functions of r only. Equations (2.2) are satisfied by the metric field

$$g_{m\mu}=\mathrm{diag}(if_0,ir,ir\sin\theta,g_0) . \quad (4.2)$$

To achieve zero electric and weak fields we see from (3.4) that our ansatz for the spinor field must be such that $\langle\psi\Gamma^\mu\psi\Gamma_{(x)}\rangle$ is zero for $x=2,3$. From Table I, we expect a spinor tetrad with nonvanishing components in just two of the four ideals $\psi^{(\eta,\epsilon)}$, of the form $\psi=\psi^{(+,+)}+\psi^{(+,-)}$ or $\psi=\psi^{(-,+)}+\psi^{(-,-)}$. Thus, for some fixed η, either $+1$ or -1, we set[2]

$$\psi=\frac{1}{\sqrt{2g_0}}e^{it}\sum_{\epsilon=\pm1}e^{i\epsilon\phi/2}\sum_{n=1}^{4}u_n^{(\eta,\epsilon)}Y_n^{(\eta,\epsilon)} \quad (4.3)$$

with the ansatz

$$u_1^{(\eta,\epsilon)}=i\epsilon Pe^{i\epsilon\theta/2}+\eta Qe^{-i\epsilon\theta/2} , \quad (4.4)$$

$$u_2^{(\eta,\epsilon)}=-i\epsilon Pe^{-i\epsilon\theta/2}-\eta Qe^{i\epsilon\theta/2} , \quad (4.5)$$

$$u_3^{(\eta,\epsilon)}=-iRe^{-i\epsilon\theta/2}-\eta\epsilon Se^{i\epsilon\theta/2} , \quad (4.6)$$

$$u_4^{(\eta,\epsilon)}=iRe^{i\epsilon\theta/2}+\eta\epsilon Se^{-i\epsilon\theta/2} , \quad (4.7)$$

where P, Q, R, and S are real functions of r only. All relevant inner products vanish except for

$$\langle\psi\Gamma_{45}\psi\rangle=\frac{i\eta A}{2g_0} , \quad (4.8)$$

$$\langle\psi\Gamma_{04}\psi\Gamma_{05}\rangle=\frac{\eta B}{2g_0} , \quad (4.9)$$

$$\langle\psi\psi\rangle=\frac{1}{g_0}(QR-PS) , \quad (4.10)$$

where we have put

$$A=P^2+Q^2-R^2-S^2 , \quad (4.11)$$

$$B=P^2+Q^2+R^2+S^2 . \quad (4.12)$$

The equations (3.4) are satisfied for $x=2,3$ by $K_{(2)\mu}=K_{(3)\mu}=0$. The left curvature scalar is given by

$$R^{\mu\nu}{}_{\mu\nu}=G+T^{\lambda\mu\nu}T_{\lambda\mu\nu}=\frac{-1}{2T}\langle\psi\psi\rangle , \quad (4.13)$$

where G is the Ricci scalar and the torsion scalar is

$$T^{\lambda\mu\nu}T_{\lambda\mu\nu}=\frac{-3A^2}{128T^2g_0{}^2} . \quad (4.14)$$

Writing $K_{(1)i}=0$, $i=1,2,3$, $K_{(1)4}=K$, and denoting differentiation with respect to r by a prime, the field equations give

$$K'=-f_0g_0J , \quad (4.15)$$

$$J'+\frac{2}{r}J=\frac{\eta}{16b_{(1)}}\frac{f_0}{g_0}B , \quad (4.16)$$

$$P'+\frac{1}{r}(1-f_0)P=\frac{f_0}{2g_0}(\eta K-2)Q-f_0R-\frac{3f_0}{32Tg_0}AQ , \quad (4.17)$$

$$Q'+\frac{1}{r}(1+f_0)Q=-\frac{f_0}{2g_0}(\eta K-2)P-f_0S+\frac{3f_0}{32Tg_0}AP , \quad (4.18)$$

$$R'+\frac{1}{r}(1+f_0)R=-\frac{f_0}{2g_0}(\eta K-2)S-f_0P-\frac{3f_0}{32Tg_0}AS , \quad (4.19)$$

$$S'+\frac{1}{r}(1-f_0)S=\frac{f_0}{2g_0}(\eta K-2)R-f_0Q+\frac{3f_0}{32Tg_0}AR , \quad (4.20)$$

$$\frac{2}{r}\frac{f_0'}{f_0}+\frac{1}{r^2}(f_0{}^2-1)=\frac{b_{(1)}}{T}f_0{}^2J^2+\frac{3f_0{}^2A^2}{256T^2g_0{}^2}-\frac{f_0{}^2}{8Tg_0{}^2}(\eta K-2)B , \quad (4.21)$$

$$-\frac{2}{r}\frac{g_0'}{g_0}+\frac{1}{r^2}(f_0^2-1)=\frac{b_{(1)}}{T}f_0{}^2J^2-\frac{3f_0{}^2A^2}{256T^2g_0{}^2}+\frac{f_0{}^2}{8Tg_0{}^2}(\eta K-2)B+\frac{f_0{}^2}{2Trg_0}(PQ+RS)+\frac{f_0{}^2}{2Tg_0}(PS-QR) . \quad (4.22)$$

From the known short-range character of the strong field, we want K and J to drop away with increasing r about as quickly as the spinor field functions P,Q,R,S. At large r, where all these quantities are negligible, the equations admit Schwarzschild's solution

$$g_0{}^2=f_0{}^{-2}=1-\frac{\mu}{8\pi Tr} , \quad (4.23)$$

where the mass of our field generating object is taken as

$$m=\mu M_0 \tag{4.24}$$

with μ some constant and M_0 the fundamental mass of (3.11). We now rescale the radial coordinate and the field functions as follows:

$$y=rT , \tag{4.25}$$

$$K_0=\frac{\eta K}{T},\quad J_0=\frac{\eta\gamma}{T^2}y^2J , \tag{4.26}$$

$$(P_0,Q_0,R_0,S_0)=\frac{y}{T}(P,Q,R,S) , \tag{4.27}$$

where γ is a constant defined by

$$\gamma^2=b_{(1)}T . \tag{4.28}$$

If we write out now the field equations for the zero-indexed functions we find that η (which determines the sign of the strong charge) does not appear. Thus the equations describe a classical particle-antiparticle pair having the same mass, the same zero spin, but opposite charge. The equations have been found too complicated for a global solution in terms of explicit functions. However, we can find a solution with the help of machine integration if we neglect quantities with coefficient T^{-1}. For this purpose it is convenient to split all field quantities into a "principal" part and a "small" part with coefficient T^{-1} by the scheme

$$(f_0,g_0,J_0,K_0)=(f_1,g_1,J_1,K_1)+\frac{1}{T}(f_2,g_2,J_2,K_2) , \tag{4.29}$$

$$(P_0,Q_0,R_0,S_0)=(P_1,Q_1,R_1,S_1)+\frac{1}{T}(P_2,Q_2,R_2,S_2) . \tag{4.30}$$

The principal part of the field equations allows two equivalent, well-behaved solutions, one with $R_1=S_1=0$, the other with $P_1=Q_1=0$. In each case the principal part of the left curvature scalar (4.13) vanishes identically by (4.10), but torsion is present and the solution can be taken to $r=0$. In contrast the torsion-free case arising from $P_0=S_0\neq 0, Q_0=R_0\neq 0$ gives rise to a highly singular left curvature scalar and no way has been found to continue the solution to the origin to give a particle with finite mass. With the presence of torsion and the allowed setting $R_1=S_1=0$, the principal equations are (with the prime now denoting differentiation with respect to y)

$$K_1'=-\frac{f_1g_1}{\gamma}\frac{J_1}{y^2} , \tag{4.31}$$

$$J_1'=\frac{1}{16\gamma}\frac{f_1}{g_1}B_1 , \tag{4.32}$$

$$P_1'=\frac{f_1P_1}{y}+\frac{1}{2}\frac{f_1}{g_1}\left[K_1-\frac{3}{16}\frac{B_1}{y^2}\right]Q_1 , \tag{4.33}$$

$$Q_1'=-\frac{f_1Q_1}{y}-\frac{1}{2}\frac{f_1}{g_1}\left[K_1-\frac{3}{16}\frac{B_1}{y^2}\right]P_1 , \tag{4.34}$$

$$f_1'=\frac{f_1(1-f_1^2)}{2y}+\frac{f_1^3J_1^2}{2y^3}+\frac{f_1^3}{g_1^2}\frac{y}{16}\left[\frac{3A_1^2}{32y^4}-\frac{K_1B_1}{y^2}\right] , \tag{4.35}$$

$$g_1'=\frac{-g_1(1-f_1^2)}{2y}-\frac{g_1f_1^2J_1^2}{2y^3}+\frac{f_1^2}{g_1}\frac{y}{16}\left[\frac{3A_1^2}{32y^4}-\frac{K_1B_1}{y^2}\right]-\frac{f_1^2P_1Q_1}{4y^2} , \tag{4.36}$$

where now from (4.11) and (4.12)

$$A_1=B_1=P_1^2+Q_1^2 . \tag{4.37}$$

For large y, the system (4.31) to (4.36) has a far-field solution

$$g_1^2=f_1^{-2}=1-\frac{\mu}{8\pi y} , \tag{4.38}$$

$$K_1=\frac{-a^2}{32\gamma^2y^2}\left[1+\frac{4}{3}\frac{\mu}{16\pi y}\right] , \tag{4.39}$$

$$J_1=\frac{-a^2}{16\gamma y}\left[1+2\frac{\mu}{16\pi y}\right] , \tag{4.40}$$

$$P_1=\frac{a^3}{192\gamma^2y^2}\left[1+3\frac{\mu}{16\pi y}\right] , \tag{4.41}$$

$$Q_1=\frac{a}{y}\left[1+\frac{\mu}{16\pi y}\right] , \tag{4.42}$$

where a is a constant. This far-field solution is chosen so that the strong field quantities K_1 and J_1 drop off in unison with the spinor field quantities P_1 and Q_1.

To specify the solution we must fix the three constants μ, γ, and a. To fix γ we first normalize the spinor field by the condition

$$\int ||g||\langle\psi_1\Gamma^4\psi_1\rangle d^3y=1 ,$$

which gives

$$2\pi\int_0^\infty\frac{f_1}{g_1}B_1dy=1 . \tag{4.43}$$

Integrating (4.32) we have

$$\int_0^\infty dJ_1=\frac{1}{16\gamma}\int_0^\infty\frac{f_1}{g_1}B_1dy . \tag{4.44}$$

Since by (4.40) $J_1\to 0$ as $y\to\infty$, it follows from (4.43) and (4.44) that we must require

$$\lim_{y\to 0}J_1(y)=J_1(0)=\frac{-1}{32\pi\gamma} \tag{4.45}$$

so that from (4.26) the original, unscaled J field is Coulomb-like as $r\to 0$ with

$$\lim_{r\to 0}J(r)=\frac{-\eta}{32\pi\gamma^2r^2} . \tag{4.46}$$

By analogy with (3.10) and using (4.46), the physical strong field intensity $E_{(1)}$ as $r\to 0$ is

$$E_{(1)}(r)=\frac{\hbar c}{2q_{(1)}}J(r)=\frac{-\eta\hbar c}{64\pi\gamma^2 q_{(1)}r^2} \tag{4.47}$$

while by analogy with (3.14) and (3.15) we have

$$E_{(1)}(r)=\frac{-\eta q_{(1)}}{4\pi r^2} \tag{4.48}$$

so that by comparison of (4.47) and (4.48) we get

$$\frac{q_{(1)}{}^2}{4\pi\hbar c}=\frac{1}{64\pi\gamma^2} . \tag{4.49}$$

The equations (4.31) to (4.36) can be numerically integrated if $q_{(1)}{}^2(4\pi\hbar c)^{-1}$ is roughly of order one. We shall now assume that this "strong fine-structure constant" is exactly one and take from (4.49)

$$\gamma=\frac{-1}{8\sqrt{\pi}} . \tag{4.50}$$

The requirement (4.45) becomes

$$\lim_{y\to 0} J_1(y)=J_1(0)=\frac{1}{4\sqrt{\pi}} . \tag{4.51}$$

For later reference we introduce a "linear charge density" C, defined by

$$C=\frac{\sqrt{\pi}}{2}\frac{f_1}{g_1}B_1 , \tag{4.52}$$

which from (4.44) must satisfy the integral condition

$$\int_0^\infty C\,dy=\frac{1}{4\sqrt{\pi}} . \tag{4.53}$$

We can also introduce a "linear mass-energy density" D, with the use of a standard procedure of general relativity. In particular,[10] if the field is static and the metric tensor is Lorentzian at spatial infinity, $g_{\mu\nu}=\mathrm{diag}(-1,-1,-1,1)$, then the rest mass of the field-generating object is given in (cgs units) by

$$m=\frac{c^2}{8\pi G_n}\int\left[\frac{\partial U}{\partial g^{4\mu}{}_{,i}}g^{4\mu}\right]_{,i} d^3x , \tag{4.54}$$

where i is summed over 1,2,3 and U is the pseudoscalar density which can be specified by

$$U=||g||g^{\rho\sigma}(\Gamma^\alpha_{\sigma\rho}\Gamma^\beta_{\alpha\beta}-\Gamma^\alpha_{\beta\rho}\Gamma^\beta_{\alpha\sigma}) . \tag{4.55}$$

After some calculation[11] we find that D can be defined in terms of the mass parameter μ of (4.24) by

$$\mu=\int_0^\infty D\,dy , \tag{4.56}$$

where D is given in terms of the present field functions by

$$D=8\pi f_1 g_1\frac{J_1{}^2}{y^2}+\frac{\pi f_1{}^3}{g_1}\left[\frac{3A_1{}^2}{32y^2}-K_1B_1\right]$$
$$+2\pi f_1(1-f_1{}^2)\frac{P_1Q_1}{y} . \tag{4.57}$$

We must now integrate the system (4.31) to (4.36) with the initial ($y\to\infty$) conditions (4.38) to (4.42), where the constants a and μ must be chosen to satisfy the boundary ($y\to 0$) conditions (4.51), (4.53), and (4.56). Numerical investigation[12] has shown that all requirements can be met by a (numerically) unique pair of values (a,μ), namely,

$$a=2.4862\times 10^{-3},\quad \mu=1.3026 . \tag{4.58}$$

The solution so obtained shows no significant deviation from the far-field solution of (4.38) to (4.42) until we approach quite close to the Schwarzschild region, with the Schwarzschild radius itself being located at

$$y=y_S=\mu(8\pi)^{-1}=0.0518 .$$

As can be seen in Table II, the metric functions f_1 and g_1 follow the corresponding Schwarzschild functions (4.38) very closely until $y\sim 1.02y_S$, when a dramatic but well-behaved deviation occurs. The function f_1, rather than tending to the Schwarzschild infinity, reaches a sharp but finite maximum of magnitude 10.3 in the immediate vicinity of y_S. Likewise, the function g_1, rather than falling unphysically through zero to imaginary values, retains a real positive behavior. Here too the density D reaches a sharp maximum of magnitude 1.7×10^3, with some 98% of the total mass being concentrated between $1.02y_S$ and $0.96y_S$, according to (4.56).

Moving inwards from the Schwarzschild vicinity, the functions f_1 and D join g_1 in plunging smoothly, settling to their final analytic form at about $0.86y_S$, where the spinor intensity B_1 becomes constant. The presence of torsion ($A_1{}^2$ term) dominates in the differential equations for f_1 and g_1, and the whole system (4.31) to (4.36) can now be solved in terms of explicit functions which are well behaved right down to the center at $y=0$. For $0\le y\lesssim 0.86y_S$, the solution is given by

$$g_1=G_0 y^{(\delta_0\Gamma_0{}^2-1/2)}=\frac{f_1}{\Gamma_0 y} , \tag{4.59}$$

$$J_1=\frac{1}{4\sqrt{\pi}}(1-\Gamma_0 N_0{}^2y^2) , \tag{4.60}$$

$$K_1=k_0+O(y^{\delta_0\Gamma_0{}^2}) , \tag{4.61}$$

$$P_1=N_0\cos(\beta_0\ln y+\phi_0) , \tag{4.62}$$

$$Q_1=N_0\sin(\beta_0\ln y+\phi_0) , \tag{4.63}$$

$$D=\frac{2\pi f_1P_1Q_1}{y},\quad C=\frac{\sqrt{\pi}}{2}\Gamma_0N_0{}^2y . \tag{4.64}$$

Here the zero-indexed quantities are all constants. The following relationships hold between these constants, analytically derived and numerically verified:

$$\delta_0=\frac{3N_0{}^4}{512},\quad \beta_0=\frac{3\Gamma_0N_0{}^2}{32} . \tag{4.65}$$

The actual values of the constants as determined by numerical integration are

$$\Gamma_0=3.1899\times 10^3,\quad \log_{10}G_0=96.96 ,$$

$$\delta_0=7.6395\times 10^{-6},\quad N_0{}^2=3.61082\times 10^{-2} ,$$

$$\beta_0=10.798,\quad k_0=-7.80291\times 10^{-2} ,$$

and

$$\phi_0 = -1.257 \ .$$

We note from (4.64), (4.62), and (4.63) that in some regions deep inside the Schwarzschild radius, the mass density D takes negative values due to its sinusoidal oscillation with respect to the radial y coordinate. However, the absolute value of D is here extremely small and the total integrated mass within this deep interior zone is zero, with all the observed mass ($\mu = 1.3026$) being positively distributed outside and beyond $y \sim 0.86 y_S$. In contrast, the charge density C of (4.64) is everywhere positive and it is not until reaching the center at $y = 0$ that all of the charge $q_{(1)}$ is finally accounted for.

From (4.10), (4.13), and (4.14) we see that although the left curvature scalar $R^{\mu\nu}{}_{\mu\nu}$ is everywhere finite (namely, zero), both the Ricci scalar $G^{\mu}{}_{\mu}$ and the torsion scalar $T^{\lambda\mu\nu}T_{\lambda\mu\nu}$ have extremely strong but off-setting singularities at $y = 0$, at least to the level of accuracy (T^{-1}) here considered. In contrast, the right curvature scalar $P_{(1)}{}^{\mu\nu}P_{(1)\mu\nu}$ is singular, behaving as $J(r)^2 \propto r^{-4}$. In special relativity this singularity would lead to an infinite self-energy. That embarrassment does not arise in the present case, since although this J^2 term appears in the energy density (4.57), it has a multiplicative metric factor of $f_1 g_1$ which drags the term very strongly to zero at the center.

V. THE STRENGTH OF THE STRONG FIELD

At a distance r (in units of Λ) from a source of mass $m = \mu M_0$, the strength of the gravitational field in the far-field region can be measured by Newton's

$$F_G = G_n \mu^2 M_0{}^2 r^{-2} \ .$$

In terms of the y coordinate and the metric this is

$$F_G = \mu \hbar c T g_1 \frac{dg_1}{dy} \ . \tag{5.1}$$

In like manner, the strength of the strong field at a distance r from a source of charge $q_{(1)}$ can be measured by $F_s = q_{(1)} E_{(1)}(r)$, where $E_{(1)}(r)$ is the field intensity. Using (4.47) we can write this as $F_s = \frac{1}{2}\hbar c J(r)$, and using (4.26) we obtain in the y coordinate

$$F_s = \frac{\hbar c T^2}{2\gamma} \frac{J_1}{y^2} \ . \tag{5.2}$$

Thus a measure of the strength of the strong force as compared to gravity in the far-field region is

$$S_g = \frac{F_s}{F_G} = \frac{T J_1}{2\mu\gamma y g_1{}^2 g_1'} \sim \frac{6 T J_1}{y^2 g_1{}^2 g_1'} \ , \tag{5.3}$$

where we have used the values of μ and γ obtained previously.

Now the far-field of our particle begins just outside the Schwarzschild radius. At

$$y = 0.054 = 1.04 y_S \ ,$$

where less than 1% of the total mass has been penetrated, the metric field is already very close to Schwarzschild's vacuum solution and we shall identify this coordinate value as giving the "surface" of the particle. As can be deduced from Table II, the value of S_g at this y is $0.3T$. For values of $y \gg 1$, or $y \gg 20 y_S$, where the approximation (4.40) for J_1 is accurate, (5.3) gives $S_g \sim 10^{-3} T y^{-1}$. Thus, at the Compton radius, where the spinor field slowly begins to emerge from the vacuum (or more accurately[2] from the cosmological background field, $T^{-1}e^{it}$), S_g is very small, of order 10^{-3}, since $r = yT^{-1} \sim 1$ at the Compton radius $\hbar(\mu M_0 c)^{-1}$.

To specify S_g near the "surface" of our particle we need an estimate for T. To what extent the "radius" of the neutral pion has been objectively measured we do not know. However, to the extent that our particle has zero spin and exhibits the strong and gravitational force fields only, we tentatively identify it as a π^0, with rest mass some 264 electron masses.

Assuming forthwith that $m = \mu M_0 = 264 m_e$ (with m_e = electronic rest mass), we get by (4.58) that $M_0 \sim 203 m_e$. This value for M_0 is within 2% of $207 m_e$, which is the muon mass. From (3.16) we calculate $T \sim 2.8 \times 10^{38}$, so that at the "surface" of the particle the strong force has relative strength 1×10^{38}.

TABLE II. The field behavior in the vicinity of the Schwarzschild length at $y = 0.0518$. The function g_S is Schwarzschild's $(1 - \mu/8\pi y)^{1/2}$.

y	g_S	g_1	$\frac{dg_1}{dy}$	$f_1 g_1$	D	J_1	B_1
1	0.974	0.974	0.03	1.00	1×10^{-9}	5.8×10^{-6}	6.5×10^{-6}
0.1	0.694	0.696	3.7	1.00	3×10^{-4}	1.1×10^{-4}	1.2×10^{-3}
0.058	0.326	0.334	23	1.00	0.47	6.4×10^{-4}	9.3×10^{-3}
0.055	0.240	0.252	34	1.00	3.22	9.8×10^{-4}	1.3×10^{-2}
0.054	0.201	0.215	41	0.99	8.49	1.2×10^{-3}	1.6×10^{-2}
0.053	0.149	0.169	53	0.98	33.1	1.6×10^{-3}	1.9×10^{-2}
0.052	0.057	0.103	82	0.89	328	2.6×10^{-3}	2.5×10^{-2}
0.051	not real	0.020	43	0.08	515	7.2×10^{-3}	3.4×10^{-2}
0.050		0.003	5.0	1×10^{-3}	7.30	1.3×10^{-2}	3.6×10^{-2}
0.046		3×10^{-6}	5×10^{-3}	1×10^{-9}	4×10^{-4}	3.3×10^{-2}	3.6×10^{-2}

VI. CONCLUSION

The theory presented here is viewed as a logically necessary enlargement of the SCL model. It would appear to be the simplest classical generally relativistic theory which might hope to embrace the four forces of nature in a unified manner.

The present enlargement of the spinor-tetrad connection group has a very simple but most satisfying consequence from the cosmological point of view. The cosmological solution of SCL does not *per se* demand the use of all four ideals of the fundamental spinor-tetrad.[2] The requirement that the universe should be in overall "neutral" can be met in SCL by utilizing only two ideals of the tetrad (each ideal representing opposite electric charge). In the present theory, neutrality of the universe in the now extended weak and strong sense requires that all four members of the fundamental tetrad must certainly contribute to the cosmological field, as is obvious from Table I. This is clearly a most desired result in a theory aspiring to describe matter via the spinor-tetrad field of a Mie world Lagrangian function. Apart from this enforced change in the intrinsic structure of the cosmological spinor-tetrad, it is easy to find that the SCL cosmological metric solution (namely, the zero-space-curvature Friedmann solution) still holds.

Going now from the cosmological to the particle level, computational complexity makes it very difficult to see how our theory might ever deal with particle collision phenomena. However, at the more tractable single-particle level, it appears from Table I that, in principle, a large number of different classical particles should be possible, although some of these are unrecognized by experiment. For example, the four-ideal electrically neutral spinless particle dealt with in SCL (Ref. 2) would appear in the present theory as an undetected massive gravitational boson with zero weak and strong as well as electric charge. We note too that some accepted decay phenomena cannot ever occur by the present theory. For example, the simple decay $\pi^0 \to \gamma + \gamma$ violates the charge conservation law (3.4).

One pointed question raised by our theory concerns the predicted nature of the neutrino, in particular the value of its rest mass. If the neutrino is to interact via the weak (and gravity) force only, we would naturally assume it to carry the weak charge only and so to be built from two parallel spin-$\frac{1}{2}$ ideals $\psi^{(\eta,\epsilon)}$ with the product $\eta\epsilon$ being either $+1$ or -1 in both ideals according to Table I. It seems that the field equations cannot admit a solution interpretable as such a charge with rest mass zero and speed c. Indeed it appears that the only true rest-mass-zero radiation fields observable in our theory are the long-range gravitational and electromagnetic fields.

APPENDIX A

A convenient representation of the Γ_{ab} is obtained from the following Γ_{0m} and (2.5):

$$\Gamma_{01}=\begin{bmatrix} 0 & 0 & i & 0 \\ 0 & 0 & 0 & -i \\ -i & 0 & 0 & 0 \\ 0 & i & 0 & 0 \end{bmatrix},$$

$$\Gamma_{02}=\begin{bmatrix} 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{bmatrix},$$

$$\Gamma_{03}=\begin{bmatrix} 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \\ 0 & 0 & 0 & i \\ 0 & 0 & -i & 0 \end{bmatrix},$$

$$\Gamma_{04}=\begin{bmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}.$$

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[9]The $b_{(x)}$ and T might also be regarded as Lagrange multipliers which are possibly functions of the cosmic epoch as in Ref. 1 above, but this is of no relevance to the present particle application.

[10]R. Adler, M. Bazin, and M. Schiffer, *Introduction to General Relativity* (McGraw-Hill, New York, 1965), Chap. 10.

[11]The formulas (4.54) and (4.55) are not applicable in spherical polar coordinates, so we use a rectangular Cartesian frame with line element $ds^2=-dx^2-dy^2-dz^2-(f_0{}^2-1)dr^2+g_0{}^2dt^2$, where $r^2=x^2+y^2+z^2$ in obtaining the formulas (4.56) and (4.57).

[12]The computations were done using a fourth-order Runge-Kutta algorithm and Simpson's rule on a 14-decimal digit machine, with the computed results meeting the stated requirements to better than three parts per million. It need not necessarily follow that the values stated in (4.58) have this same accuracy.