Test-particle motion in Einstein's unified field theory. II. Charged test particles

Coates R. Johnson
Department of Physics and Astronomy, The University of Georgia, Athens, Georgia 30602
(Received 8 September 1983)

In the preceding paper we developed a method for finding the exact equations of structure and motion of multipole test particles in Einstein's unified field theory—the theory of the nonsymmetric field. The method is also applicable to Einstein's gravitational theory. In the preceding paper we applied the method both in Einstein's unified field theory and in Einstein's gravitational theory and found the equations of structure and motion of neutral pole-dipole test particles possessing no electromagnetic multipole moments. In this paper we apply the method and find the equations of structure and motion of charged test particles in Einstein's unified field theory.

I. INTRODUCTION

In the preceding paper[1] (paper I) we developed a method for finding the exact equations of structure and motion of multipole test particles in Einstein's unified field theory—the theory of the nonsymmetric field—and in Einstein's gravitational theory. In the previous paper we applied the method in both Einstein's unified field theory and Einstein's gravitational theory and found the equations of structure and motion of neutral pole-dipole test particles possessing no electromagnetic multipole moments. In this paper we shall use the method to find the equations of structure and motion of charged test particles in Einstein's unified field theory. As discussed in the Introduction to paper I, finding the exact equations of structure and motion of a charged test particle in Einstein's unified field theory, in addition to being of interest in itself, can be regarded as a first step in an attempt to investigate the interaction of charged particles over microscopic distances in Einstein's theory.

In this paper we confine our investigation to charged test particles possessing no magnetic monopole moments. In a later paper (paper III) we shall investigate the interaction of charged test particles which possess magnetic monopole moments.

II. SIMPLE CHARGED TEST PARTICLES POSSESSING NO MAGNETIC MONOPOLE MOMENTS

As mentioned in the Introduction we shall confine our investigation in this paper to charged test particles possessing no magnetic monopole moments. By the condition that a charged particle possess no magnetic monopole moment we mean that the electromagnetic moment e^M associated with the particle vanishes.[2,3] In addition, we shall confine our investigation in this paper to charged particles which when isolated and possessing no spin can be represented through a time-independent spherically symmetric solution to Einstein's field equations in which the symmetric part of the fundamental field is flat at infinity.[4] By the above statement we do not mean that the particles cannot possess spin. What we mean is that we shall restrict our study to particles which when isolated possess only those multipole moments consistent with spherical symmetry with but one exception—and that exception is that the particles may possess spin. We shall also restrict our study in this paper to particles which under interaction develop the minimum number of higher multipole moments consistent with Einstein's field equations. We shall call the particles we shall be studying simple charged particles. Simple charged particles possessing no spins and no magnetic monopole moments have been studied in considerable detail in a previous paper by Johnson and Nance.[5] We shall refer to that paper as Johnson-Nance.

From the investigation presented in Johnson-Nance, from earlier work,[6] and from the work presented in paper I, it follows that in a harmonic coordinate system and keeping only terms linear in the multipole moments which characterize a particle, one finds the following for the fields $\gamma^*_{[\mu\nu]}$ and $\gamma_{(\mu\nu)}$ associated with an isolated simple charged particle possessing no magnetic monopole moment:[7]

$$\gamma^*_{[\mu\nu]}=\gamma_{\mu,\nu}-\gamma_{\nu,\mu}\ , \tag{2.1}$$

where

$$\gamma_\mu=\frac{1}{l}[\tfrac{1}{2}qr_\mu]_A+\frac{1}{l}[ql^2u_\mu(r_\rho u^\rho)^{-1}]_A\ , \tag{2.2}$$

and

$$\gamma_{(\mu\nu)}=4[(mu_\mu u_\nu+\tfrac{1}{2}\dot{s}_{\mu\rho}u^\rho u_\nu+\tfrac{1}{2}\dot{s}_{\nu\rho}u^\rho u_\mu)(r_\rho u^\rho)^{-1}]_A +4[(\tfrac{1}{2}s_{\mu\rho}u_\nu+\tfrac{1}{2}s_{\nu\rho}u_\mu)(r_\rho u^\rho)^{-1}]_A{}^{,\rho}\ . \tag{2.3}$$

We are using the notation

$$r^\mu=x^\mu-\xi^\mu,\ \ r_\mu=\eta_{\mu\rho}r^\rho,\ \ u^\mu=\frac{d\xi^\mu}{d\tau}\ ,$$
$$u_\mu=\eta_{\mu\rho}u^\rho,\ \ \dot{s}_{\mu\nu}=\frac{ds_{\mu\nu}}{d\tau}\ , \tag{2.4}$$

in (2.2) and (2.3). The points ξ^μ form the world line of the particle and are parametrized by a quantity τ defined through the equation

$$d\tau^2=\eta_{\mu\nu}d\xi^\mu d\xi^\nu \,. \tag{2.5}$$

We are also using the notation

$$[f]_A=a_{\rm ret}[f]_{\rm ret}+a_{\rm adv}[f]_{\rm adv}\,, \qquad a_{\rm ret}+a_{\rm adv}=1\,. \tag{2.6}$$

The quantities $a_{\rm ret}$ and $a_{\rm adv}$ in (2.6) are constants and can be regarded as characterizing the structure of the particle. The subscript ret indicates that in the expression in brackets those quantities which are associated with the particle are to be evaluated at the "retarded point"

$$(r_\rho r^\rho)=0,\quad r^4>0\,,$$

while the subscript adv indicates that the expression in brackets is to be multiplied by -1 and then in the expression in brackets those quantities associated with the particle are to be evaluated at the "advance point"

$$(r_\rho r^\rho)=0,\quad r^4<0\,.$$

The quantities l, q, m, and $s_{\mu\nu}$ in (2.2) and (2.3) characterize the particle. The quantity l is a constant and a universal length—the same for each particle.[8] The quantity q which is also a constant represents the charge of the particle,[9] the quantity m represents the mass of the particle,[10] and the quantity $s_{\mu\nu}$ represents the spin of the particle.[6]

From their definitions given in Sec. V of paper I, we find in a harmonic coordinate system for the quantities $i_\mu^{\rm kin}$ and $s_\mu^{\rm kin}$ associated with an isolated particle in Einstein's theory[1]

$$i_\mu^{\rm kin}=\tfrac{1}{6}\eta_{\mu\lambda}\epsilon^{\rho\sigma\kappa\lambda}\gamma^{*L}_{[\rho\sigma,\kappa]}\,, \tag{2.7}$$

$$s_\mu^{\rm kin}=\Box^2\gamma^{*L}_{[\nu\mu]}{}^{,\nu}\,, \tag{2.8}$$

and for the quantity $t_{\mu\nu}^{\rm kin}$ associated with the particle[1]

$$t_{\mu\nu}^{\rm kin}=\Box^2\gamma^L_{(\mu\nu)}\,. \tag{2.9}$$

The fields $\gamma^{*L}_{[\mu\nu]}$ and $\gamma^L_{(\mu\nu)}$ appearing in (2.7)–(2.9) are those parts of $\gamma^*_{[\mu\nu]}$ and $\gamma_{(\mu\nu)}$, respectively, which are linear in the multipole moments characterizing the particle.

Making use of Eqs. (2.1)–(2.3) in Eqs. (2.7)–(2.9), we find associated with a simple charged particle possessing no magnetic monopole moment[11]

$$i_\mu^{\rm kin}=0\,, \tag{2.10}$$

$$s_\mu^{\rm kin}=\frac{4\pi}{l}\left[\int qu_\mu\delta(x-\xi)d\tau - \int ql^2u_\mu\Box^2\delta(x-\xi)d\tau\right], \tag{2.11}$$

and

$$t_{\mu\nu}^{\rm kin}=16\pi\int[mu_\mu u_\nu+\tfrac{1}{2}\dot s_{\mu\rho}u^\rho u_\nu+\tfrac{1}{2}\dot s_{\nu\rho}u^\rho u_\mu]\delta(x-\xi)d\tau +16\pi\int[\tfrac{1}{2}s_{\mu\rho}u_\nu+\tfrac{1}{2}s_{\nu\rho}u_\mu]\delta^{,\rho}(x-\xi)d\tau\,. \tag{2.12}$$

Equation (2.11) can be written in the form

$$s_\mu^{\rm kin}=\frac{4\pi}{l}\left[\int qu_\mu\delta(x-\xi)d\tau + \int ql^2(\eta_{\mu\lambda}u_\kappa-\eta_{\kappa\lambda}u_\mu)\delta^{,\kappa\lambda}(x-\xi)d\tau\right]. \tag{2.13}$$

Comparing Eqs. (2.10), (2.13), and (2.12) with Eqs. (5.12)–(5.14) of paper I, we see that a simple charged particle possessing no magnetic monopole moment can be considered as characterized by a universal length l, an electromagnetic monopole moment e^E, an electromagnetic quadrupole moment $e^E_{[\mu\nu]\lambda}$, a mass monopole moment m^G, and a spin $S^G_{\mu\nu}$, where[12]

$$e^E=(c^2/l)q\,, \tag{2.14}$$

$$e^E_{[\mu\kappa]\lambda}=(c^2/l)ql^2(\eta_{\mu\lambda}u_\kappa-\eta_{\kappa\lambda}u_\mu)\,, \tag{2.15}$$

$$m^G=4mc^2\,, \tag{2.16}$$

$$S^G_{\mu\nu}=4s_{\mu\nu}c^2\,. \tag{2.17}$$

Making use of the relationship of $\mathbf{i}^\mu$ to $i_\mu^{\rm kin}$, of $\mathbf{s}^\mu$ to $s_\mu^{\rm kin}$, and of $\mathbf{T}^{\mu\nu}$ to $t_{\mu\nu}^{\rm kin}$ (these relationships are discussed in Sec. V of paper I) we see that in the test-particle limit a simple charged particle possessing no magnetic monopole moment will be associated with a vanishing electromagnetic current density $\mathbf{i}^\mu$, an electromagnetic current density $\mathbf{s}^\mu$ of the form[13]

$$\mathbf{s}^\mu=\int\tilde{s}^\mu(x)\delta(x-\xi)ds +\int[\tilde{s}^{\mu\kappa}(x)\delta(x-\xi)]_{;\kappa}ds +\int[\tilde{s}^{\mu\kappa\lambda}(x)\delta(x-\xi)]_{;\kappa\lambda}ds\,, \tag{2.18}$$

and an energy-momentum tensor density $\mathbf{T}^{\mu\nu}$ of the form

$$\mathbf{T}^{\mu\nu}=\int\tilde{T}^{(\mu\nu)}(x)\delta(x-\xi)ds +\int[\tilde{T}^{(\mu\nu)\kappa}(x)\delta(x-\xi)]_{;\kappa}ds\,. \tag{2.19}$$

We have retained dipoles terms in (2.18) although dipole terms are not present in (2.13). The reason we have retained such terms is that we wish at this stage of our analysis to leave open the possibility that through interaction with the background field the test particle might develop such terms. We do know, however, that no mulipole terms higher than those present in (2.18) and (2.19) will be generated through interaction with the background field. This follows from the general form of Eqs. (5.29)–(5.31) in paper I.

We have found that in the test-particle limit a simple charged particle possessing no magnetic monopole moment will be associated with a vanishing electromagnetic current density $\mathbf{i}^\mu$, an electromagnetic current density $\mathbf{s}^\mu$ of the form (2.18), and an energy-momentum tensor density $\mathbf{T}^{\mu\nu}$ of the form (2.19). From the analysis of Einstein's field equations contained in paper I, we also know that the electromagnetic current density $\mathbf{s}^\mu$ and the energy-momentum tensor density $\mathbf{T}^{\mu\nu}$ are subject to the equations

$$\mathbf{s}^\mu{}_{;\mu}=0\,, \tag{2.20}$$

$$\mathbf{T}^{\mu\nu}{}_{;\nu}=a^{\mu\rho}\gamma^*_{[\nu\rho]}\mathbf{s}^\nu\,. \tag{2.21}$$

We shall use Eqs. (2.20) and (2.21) along with Eqs. (2.18) and (2.19) and Eqs. (2.13) and (2.12) to find the equations of structure and motion of simple charged test particles possessing no magnetic monopole moments in Einstein's unified field theory.

III. EQUATIONS OF MOTION

A. Electromagnetic current density

We first investigate the constraint that Eqs. (2.20) place on a test particle characterized by an electromagnetic current density s^μ of the form (2.18). In doing this we obtain the quantities which characterize the electromagnetic structure of the test particle, and we also obtain the equations of structure satisfied by these quantities.

If we make use of the definition of covariant differentiation which is found in paper I, and also make use of the identities (6.8) and (6.9) in paper I, and the identity

$$f(x)\int g(s)\delta_{,\rho\sigma}(x-\xi)ds=\int[f(\xi)g(s)]\delta_{,\rho\sigma}(x-\xi)ds-\int[f_{,\rho}(\xi)g(s)]\delta_{,\sigma}(x-\xi)ds$$
$$-\int[f_{,\sigma}(\xi)g(s)]\delta_{,\rho}(x-\xi)ds+\int[f_{,\rho\sigma}(\xi)g(s)]\delta(x-\xi)ds\;, \tag{3.1}$$

Eqs. (2.18) can be put into the form

$$s^\mu=\int[\tilde{s}^{\mu\kappa\lambda}]\delta_{,\kappa\lambda}(x-\xi)ds+\int\left[\tilde{s}^{\mu\kappa}+\tilde{s}^{\mu\rho\sigma}\begin{Bmatrix}\kappa\\ \rho\sigma\end{Bmatrix}+\tilde{s}^{\rho\kappa\sigma}\begin{Bmatrix}\mu\\ \rho\sigma\end{Bmatrix}+\tilde{s}^{\rho\sigma\kappa}\begin{Bmatrix}\mu\\ \rho\sigma\end{Bmatrix}\right]\delta_{,\kappa}(x-\xi)ds$$
$$+\int\left[\tilde{s}^{\mu}+\tilde{s}^{\rho\sigma}\begin{Bmatrix}\mu\\ \rho\sigma\end{Bmatrix}-\tfrac{1}{2}\tilde{s}^{\rho\kappa\sigma}\begin{Bmatrix}\mu\\ \rho\kappa\end{Bmatrix}_{,\sigma}-\tfrac{1}{2}\tilde{s}^{\rho\kappa\sigma}\begin{Bmatrix}\mu\\ \rho\sigma\end{Bmatrix}_{,\kappa}+\tilde{s}^{\rho\kappa\sigma}\begin{Bmatrix}\lambda\\ \sigma\kappa\end{Bmatrix}\begin{Bmatrix}\mu\\ \rho\lambda\end{Bmatrix}\right.$$
$$\left.+\tfrac{1}{2}\tilde{s}^{\rho\kappa\sigma}\begin{Bmatrix}\lambda\\ \rho\kappa\end{Bmatrix}\begin{Bmatrix}\mu\\ \sigma\lambda\end{Bmatrix}+\tfrac{1}{2}\tilde{s}^{\rho\kappa\sigma}\begin{Bmatrix}\lambda\\ \rho\sigma\end{Bmatrix}\begin{Bmatrix}\mu\\ \kappa\lambda\end{Bmatrix}+\tfrac{1}{2}\tilde{s}^{\rho\kappa\sigma}R^{*\mu}{}_{\rho\kappa\sigma}\right]\delta(x-\xi)ds\;. \tag{3.2}$$

The quantities in the brackets in (3.2) are understood as evaluated along the world line ξ^μ of the test particle and are functions of s. In (3.2) we are using the notation

$$R^{*\mu}{}_{\rho\sigma\kappa}=\begin{Bmatrix}\mu\\ \rho\sigma\end{Bmatrix}_{,\kappa}-\begin{Bmatrix}\mu\\ \rho\kappa\end{Bmatrix}_{,\sigma}-\begin{Bmatrix}\mu\\ \lambda\sigma\end{Bmatrix}\begin{Bmatrix}\lambda\\ \rho\kappa\end{Bmatrix}+\begin{Bmatrix}\mu\\ \lambda\kappa\end{Bmatrix}\begin{Bmatrix}\lambda\\ \rho\sigma\end{Bmatrix}. \tag{3.3}$$

Making use of the definition of $s^\mu{}_{;\mu}$ given in paper I, one finds from (3.2) that

$$s^\mu{}_{;\mu}=\int[\tilde{s}^{\mu\kappa\lambda}]\delta_{,\mu\kappa\lambda}(x-\xi)ds+\int\left[\tilde{s}^{\mu\kappa}+\tilde{s}^{\mu\rho\sigma}\begin{Bmatrix}\kappa\\ \rho\sigma\end{Bmatrix}+\tilde{s}^{\rho\kappa\sigma}\begin{Bmatrix}\mu\\ \rho\sigma\end{Bmatrix}+\tilde{s}^{\rho\sigma\kappa}\begin{Bmatrix}\mu\\ \rho\sigma\end{Bmatrix}\right]\delta_{,\mu\kappa}(x-\xi)ds$$
$$+\int\left[\tilde{s}^{\mu}+\tilde{s}^{\rho\sigma}\begin{Bmatrix}\mu\\ \rho\sigma\end{Bmatrix}-\tfrac{1}{2}\tilde{s}^{\rho\kappa\sigma}\begin{Bmatrix}\mu\\ \rho\kappa\end{Bmatrix}_{,\sigma}-\tfrac{1}{2}\tilde{s}^{\rho\kappa\sigma}\begin{Bmatrix}\mu\\ \rho\sigma\end{Bmatrix}_{,\kappa}+\tilde{s}^{\rho\kappa\sigma}\begin{Bmatrix}\lambda\\ \sigma\kappa\end{Bmatrix}\begin{Bmatrix}\mu\\ \rho\lambda\end{Bmatrix}\right.$$
$$\left.+\tfrac{1}{2}\tilde{s}^{\rho\kappa\sigma}\begin{Bmatrix}\lambda\\ \rho\kappa\end{Bmatrix}\begin{Bmatrix}\mu\\ \sigma\lambda\end{Bmatrix}+\tfrac{1}{2}\tilde{s}^{\rho\kappa\sigma}\begin{Bmatrix}\lambda\\ \rho\sigma\end{Bmatrix}\begin{Bmatrix}\mu\\ \kappa\lambda\end{Bmatrix}+\tfrac{1}{2}\tilde{s}^{\rho\kappa\sigma}R^{*\mu}{}_{\rho\kappa\sigma}\right]\delta_{,\mu}(x-\xi)ds\;. \tag{3.4}$$

The quantities in brackets in (3.4) are functions of s. From (3.4) one can show, since (2.20) must be satisfied, that there is no loss in generality in choosing $\tilde{s}^{\mu\kappa\lambda}$ to be of the form[14]

$$\tilde{s}^{\mu\kappa\lambda}=\tilde{s}^{(\mu\kappa)\lambda}+\tilde{s}^{[\mu\kappa]\lambda}\;, \tag{3.5}$$

where $\tilde{s}^{(\mu\kappa)\lambda}$ and $\tilde{s}^{[\mu\kappa]\lambda}$ are oriented third-rank tensors characterizing the test particle, and

$$\tilde{s}^{(\mu\kappa)\lambda}+\tilde{s}^{(\kappa\lambda)\mu}+\tilde{s}^{(\lambda\mu)\kappa}=0\;. \tag{3.6}$$

We shall choose $\tilde{s}^{\mu\kappa\lambda}$ to be of this form.

Making use of (3.5) and (3.6) in both (3.2) and (3.4), one finds

$$s^\mu=\int[\tilde{s}^{\mu\kappa\lambda}]\delta_{,\kappa\lambda}(x-\xi)ds+\int\left[\tilde{s}^{\mu\kappa}+\tilde{s}^{\mu\rho\sigma}\begin{Bmatrix}\kappa\\ \rho\sigma\end{Bmatrix}-\tilde{s}^{\kappa\rho\sigma}\begin{Bmatrix}\mu\\ \rho\sigma\end{Bmatrix}\right]\delta_{,\kappa}(x-\xi)ds$$
$$+\int\left[\tilde{s}^{\mu}+\tilde{s}^{\rho\sigma}\begin{Bmatrix}\mu\\ \rho\sigma\end{Bmatrix}+\tilde{s}^{\rho\kappa\sigma}R^{*\mu}{}_{[\rho\kappa]\sigma}+\tfrac{1}{3}\tilde{s}^{\rho\kappa\sigma}R^{*\mu}{}_{(\rho\kappa)\sigma}\right]\delta(x-\xi)ds\;, \tag{3.7}$$

$$s^\mu{}_{;\mu}=\int[\tilde{s}^{\mu\kappa}]\delta_{,\kappa\mu}(x-\xi)ds+\int\left[\tilde{s}^{\mu}+\tilde{s}^{\rho\sigma}\begin{Bmatrix}\mu\\ \rho\sigma\end{Bmatrix}+\tilde{s}^{\rho\kappa\sigma}R^{*\mu}{}_{[\rho\kappa]\sigma}+\tfrac{1}{3}\tilde{s}^{\rho\kappa\sigma}R^{*\mu}{}_{(\rho\kappa)\sigma}\right]\delta_{,\mu}(x-\xi)ds\;, \tag{3.8}$$

From (3.8), since (2.20) must be satisfied, one can show that there is no loss in generality in choosing $\tilde{s}^{\mu\kappa}$ to be of the form[15]

$$\tilde{s}^{\mu\kappa}=\tilde{s}^{[\mu\kappa]}\ , \tag{3.9}$$

where $\tilde{s}^{[\mu\kappa]}$ is an oriented antisymmetric second-rank tensor characterizing the test particle. Making this choice we find from (3.7)

$$\mathbf{s}^{\mu}=\int[\tilde{s}^{\mu\kappa\lambda}]\delta_{,\kappa\lambda}(x-\xi)ds+\int\left[\tilde{s}^{[\mu\kappa]}+\tilde{s}^{\mu\rho\sigma}\begin{Bmatrix}\kappa\\ \rho\sigma\end{Bmatrix}-\tilde{s}^{\kappa\rho\sigma}\begin{Bmatrix}\mu\\ \rho\sigma\end{Bmatrix}\right]\delta_{,\kappa}(x-\xi)ds$$
$$+\int[\tilde{s}^{\mu}+\tilde{s}^{\rho\kappa\sigma}R^{*\mu}{}_{[\rho\kappa]\sigma}+\tfrac{1}{3}\tilde{s}^{\rho\kappa\sigma}R^{*\mu}{}_{(\rho\kappa)\sigma}]\delta(x-\xi)ds\ , \tag{3.10}$$

and from (3.8)

$$\mathbf{s}^{\mu}{}_{;\mu}=\int[\tilde{s}^{\mu}+\tilde{s}^{\rho\kappa\sigma}R^{*\mu}{}_{[\rho\kappa]\sigma}+\tfrac{1}{3}\tilde{s}^{\rho\kappa\sigma}R^{*\mu}{}_{(\rho\kappa)\sigma}]\delta_{,\mu}(x-\xi)ds\ . \tag{3.11}$$

From (3.11) and making use of the fact that (2.20) must be satisfied, one can show that there is no loss in generality in choosing $\tilde{s}^{\mu}$ to be of the form[16]

$$\tilde{s}^{\mu}=eU^{\mu}-\tilde{s}^{\rho\kappa\sigma}R^{*\mu}{}_{[\rho\kappa]\sigma}-\tfrac{1}{3}\tilde{s}^{\rho\kappa\sigma}R^{*\mu}{}_{(\rho\kappa)\sigma}\ , \tag{3.12}$$

where e is an oriented scalar characterizing the test particle, and

$$U^{\mu}=\frac{d\xi^{\mu}}{ds}\ . \tag{3.13}$$

Placing (3.12) in (3.11) one finds

$$\mathbf{s}^{\mu}{}_{;\mu}=\int[eU^{\mu}]\delta_{,\mu}(x-\xi)ds=\int\left[\frac{de}{ds}\right]\delta(x-\xi)ds\ . \tag{3.14}$$

Since (2.20) must be satisfied, this means the quantity e will obey the equations of structure

$$\frac{de}{ds}=0\ . \tag{3.15}$$

We have thus found that with no loss in generality the electromagnetic structure of the test particle can be characterized by the quantities e, $\tilde{s}^{[\mu\kappa]}$, and $\tilde{s}^{\mu\kappa\lambda}$, where the quantity e is a constant, and $\tilde{s}^{(\mu\kappa)\lambda}$ satisfies Eqs. (3.6). If we place (3.12) in (3.10), we see that the electromagnetic current density $\mathbf{s}^{\mu}$ associated with the test particle is given by

$$\mathbf{s}^{\mu}=\int[\tilde{s}^{\mu\kappa\lambda}]\delta_{,\kappa\lambda}(x-\xi)ds+\int\left[\tilde{s}^{[\mu\kappa]}+\tilde{s}^{\mu\rho\sigma}\begin{Bmatrix}\kappa\\ \rho\sigma\end{Bmatrix}-\tilde{s}^{\kappa\rho\sigma}\begin{Bmatrix}\mu\\ \rho\sigma\end{Bmatrix}\right]\delta_{,\kappa}(x-\xi)ds+\int[eU^{\mu}]\delta(x-\xi)ds\ . \tag{3.16}$$

B. Energy-momentum tensor density

We shall next investigate the constraints that Eqs. (2.21) place on a test particle associated with a vanishing electromagnetic current density $\mathbf{i}^{\mu}$, an electromagnetic current density $\mathbf{s}^{\mu}$ of the form (2.18), and an energy-momentum tensor density of the form (2.19). In the process of doing this we shall find the quantities which characterize the gravitational structure of the test particle, the equations of structure satisfied by these quantities, and the equations of motion satisfied by the particle.

If we make use of the definition of covariant differentiation found in paper I and also make use of the identities (6.8) and (6.9) of that paper, Eqs. (2.19) can be put into the form

$$\mathbf{T}^{\mu\nu}=\int[\tilde{T}^{(\mu\nu)\kappa}]\delta_{,\kappa}(x-\xi)ds+\int\left[\tilde{T}^{(\mu\nu)}+\tilde{T}^{(\mu\rho)\sigma}\begin{Bmatrix}\nu\\ \rho\sigma\end{Bmatrix}+\tilde{T}^{(\nu\rho)\sigma}\begin{Bmatrix}\mu\\ \rho\sigma\end{Bmatrix}\right]\delta(x-\xi)ds\ , \tag{3.17}$$

where the quantities in the brackets in (3.17) are evaluated along the world line ξ^{μ} of the test particle and are functions of s. Making use of the definition of $\mathbf{T}^{\mu\nu}{}_{;\nu}$ given in paper I, one finds from (3.17) that

$$\mathbf{T}^{\mu\nu}{}_{;\nu}=\int[\tilde{T}^{(\mu\nu)\kappa}]\delta_{,\kappa\nu}(x-\xi)ds+\int\left[\tilde{T}^{(\mu\nu)}+\tilde{T}^{(\mu\rho)\sigma}\begin{Bmatrix}\nu\\ \rho\sigma\end{Bmatrix}+\tilde{T}^{(\nu\rho)\sigma}\begin{Bmatrix}\mu\\ \rho\sigma\end{Bmatrix}+\tilde{T}^{(\rho\sigma)\nu}\begin{Bmatrix}\mu\\ \rho\sigma\end{Bmatrix}\right]\delta_{,\nu}(x-\xi)ds$$
$$+\int\left[\tilde{T}^{(\rho\sigma)}\begin{Bmatrix}\mu\\ \rho\sigma\end{Bmatrix}-\tilde{T}^{(\rho\sigma)\kappa}\begin{Bmatrix}\mu\\ \rho\sigma\end{Bmatrix}_{,\kappa}+2\tilde{T}^{(\rho\sigma)\kappa}\begin{Bmatrix}\lambda\\ \rho\kappa\end{Bmatrix}\begin{Bmatrix}\mu\\ \sigma\lambda\end{Bmatrix}\right]\delta(x-\xi)ds\ , \tag{3.18}$$

where the quantities in brackets in (3.18) are functions of s.

From (3.16), again making use of the identities (6.8) and (6.9) of paper I and in addition the identity (3.1), one finds

$$\gamma^{*[\mu}{}_{\nu]}\mathrm{s}^{\nu}=\int[\gamma^{*[\mu}{}_{\rho]}\tilde{S}^{\rho\nu\kappa}]\delta_{,\kappa\nu}(x-\xi)ds$$
$$+\int\left[\gamma^{*[\mu}{}_{\rho]}\tilde{S}^{[\rho\nu]}-\gamma^{*[\mu}{}_{\rho],\sigma}\tilde{S}^{\rho\sigma\nu}-\gamma^{*[\mu}{}_{\rho],\sigma}\tilde{S}^{\rho\nu\sigma}+\gamma^{*[\mu}{}_{\rho]}\tilde{S}^{\rho\sigma\kappa}\begin{Bmatrix}\nu\\ \sigma\kappa\end{Bmatrix}-\gamma^{*[\mu}{}_{\rho]}\tilde{S}^{\nu\sigma\kappa}\begin{Bmatrix}\rho\\ \sigma\kappa\end{Bmatrix}\right]\delta_{,\nu}(x-\xi)ds$$
$$+\int\left[e\gamma^{*[\mu}{}_{\nu]}U^{\nu}-\gamma^{*[\mu}{}_{\nu],\kappa}\tilde{S}^{[\nu\kappa]}+\gamma^{*[\mu}{}_{\nu],\kappa\lambda}\tilde{S}^{\nu\kappa\lambda}+\gamma^{*[\mu}{}_{\nu],\kappa}\tilde{S}^{\kappa\sigma\lambda}\begin{Bmatrix}\nu\\ \sigma\lambda\end{Bmatrix}-\gamma^{*[\mu}{}_{\nu],\kappa}\tilde{S}^{\nu\sigma\lambda}\begin{Bmatrix}\kappa\\ \sigma\lambda\end{Bmatrix}\right]\delta(x-\xi)ds\ , \quad (3.19)$$

so that making use of (3.18) and (3.19) one has

$$\mathrm{T}^{\mu\nu}{}_{;\nu}-a^{\mu\rho}\gamma^{*}_{[\nu\rho]}\mathrm{s}^{\nu}=\int[\tilde{T}^{(\mu\nu)\kappa}+\gamma^{*[\mu}{}_{\rho]}\tilde{S}^{\rho\nu\kappa}]\delta_{,\kappa\nu}(x-\xi)ds$$
$$+\int\left[\tilde{T}^{(\mu\nu)}+\tilde{T}^{(\mu\rho)\sigma}\begin{Bmatrix}\nu\\ \rho\sigma\end{Bmatrix}+\tilde{T}^{(\nu\rho)\sigma}\begin{Bmatrix}\mu\\ \rho\sigma\end{Bmatrix}+\tilde{T}^{(\rho\sigma)\nu}\begin{Bmatrix}\mu\\ \rho\sigma\end{Bmatrix}\right.$$
$$+\gamma^{*[\mu}{}_{\rho]}\tilde{S}^{[\rho\nu]}-\gamma^{*[\mu}{}_{\rho],\sigma}\tilde{S}^{\rho\sigma\nu}-\gamma^{*[\mu}{}_{\rho],\sigma}\tilde{S}^{\rho\nu\sigma}$$
$$\left.+\gamma^{*[\mu}{}_{\rho]}\tilde{S}^{\rho\sigma\kappa}\begin{Bmatrix}\nu\\ \sigma\kappa\end{Bmatrix}-\gamma^{*[\mu}{}_{\rho]}\tilde{S}^{\nu\sigma\kappa}\begin{Bmatrix}\rho\\ \sigma\kappa\end{Bmatrix}\right]\delta_{,\nu}(x-\xi)ds$$
$$+\int\left[\tilde{T}^{(\rho\sigma)}\begin{Bmatrix}\mu\\ \rho\sigma\end{Bmatrix}-\tilde{T}^{(\rho\sigma)\kappa}\begin{Bmatrix}\mu\\ \rho\sigma\end{Bmatrix}_{,\kappa}+2\tilde{T}^{(\rho\sigma)\kappa}\begin{Bmatrix}\lambda\\ \rho\kappa\end{Bmatrix}\begin{Bmatrix}\mu\\ \sigma\lambda\end{Bmatrix}\right.$$
$$+e\gamma^{*[\mu}{}_{\nu]}U^{\nu}-\gamma^{*[\mu}{}_{\nu],\kappa}\tilde{S}^{[\nu\kappa]}+\gamma^{*[\mu}{}_{\nu],\kappa\lambda}\tilde{S}^{\nu\kappa\lambda}$$
$$\left.+\gamma^{*[\mu}{}_{\nu],\kappa}\tilde{S}^{\kappa\sigma\lambda}\begin{Bmatrix}\nu\\ \sigma\lambda\end{Bmatrix}-\gamma^{*[\mu}{}_{\nu],\kappa}\tilde{S}^{\nu\sigma\lambda}\begin{Bmatrix}\kappa\\ \sigma\lambda\end{Bmatrix}\right]\delta(x-\xi)ds\ . \quad (3.20)$$

We are using the definition

$$\gamma^{*[\mu}{}_{\nu]}=a^{\mu\rho}\gamma^{*}_{[\rho\nu]}\ . \quad (3.21)$$

Using the fact that (2.21) must be satisfied one can show from (3.20) that there is no loss in generality in choosing $\tilde{T}^{(\mu\nu)\kappa}$ to be of the form[17]

$$\tilde{T}^{(\mu\nu)\kappa}=\tfrac{1}{2}S^{\mu\kappa}U^{\nu}+\tfrac{1}{2}S^{\nu\kappa}U^{\mu}-\tfrac{1}{2}\gamma^{*[\mu}{}_{\rho]}\tilde{S}^{\rho\nu\kappa}-\tfrac{1}{2}\gamma^{*[\nu}{}_{\rho]}\tilde{S}^{\rho\mu\kappa}-\tfrac{1}{2}\gamma^{*[\mu}{}_{\rho]}\tilde{S}^{\rho\kappa\nu}-\tfrac{1}{2}\gamma^{*[\nu}{}_{\rho]}\tilde{S}^{\rho\kappa\mu}+\tfrac{1}{2}\gamma^{*[\kappa}{}_{\rho]}\tilde{S}^{\rho\nu\mu}+\tfrac{1}{2}\gamma^{*[\kappa}{}_{\rho]}\tilde{S}^{\rho\mu\nu}\ , \quad (3.22)$$

where $S^{\mu\nu}$ is an antisymmetric second-rank tensor characterizing the test particle. Placing (3.22) in (3.20) one finds

$$\mathrm{T}^{\mu\nu}{}_{;\nu}-a^{\mu\rho}\gamma^{*}_{[\nu\rho]}\mathrm{s}^{\nu}=\int[\tfrac{1}{2}S^{\mu\kappa}U^{\nu}]\delta_{,\kappa\nu}(x-\xi)ds+\int\left[\tilde{T}^{(\mu\nu)}+\tfrac{1}{2}S^{\mu\sigma}U^{\rho}\begin{Bmatrix}\nu\\ \rho\sigma\end{Bmatrix}-\tfrac{1}{2}S^{\nu\sigma}U^{\rho}\begin{Bmatrix}\mu\\ \rho\sigma\end{Bmatrix}\right.$$
$$\left.+\gamma^{*[\mu}{}_{\rho]}\tilde{S}^{[\rho\nu]}-\gamma^{*[\mu}{}_{\rho];\sigma}\tilde{S}^{\rho\sigma\nu}-\gamma^{*[\mu}{}_{\rho];\sigma}\tilde{S}^{\rho\nu\sigma}\right]\delta_{,\nu}(x-\xi)ds$$
$$+\int\left[\tilde{T}^{(\rho\sigma)}\begin{Bmatrix}\mu\\ \rho\sigma\end{Bmatrix}-\tfrac{1}{2}S^{\rho\sigma}U^{\kappa}R^{*\mu}{}_{\kappa\rho\sigma}+e\gamma^{*[\mu}{}_{\nu]}U^{\nu}-\gamma^{*[\mu}{}_{\nu];\kappa}\tilde{S}^{[\nu\kappa]}+\gamma^{*[\mu}{}_{\nu];\kappa\lambda}\tilde{S}^{\nu\kappa\lambda}-\gamma^{*[\rho}{}_{\nu]}\tilde{S}^{\nu\kappa\lambda}R^{*\mu}{}_{\lambda\kappa\rho}\right.$$
$$+\gamma^{*[\rho}{}_{\nu]}\tilde{S}^{[\nu\kappa]}\begin{Bmatrix}\mu\\ \rho\kappa\end{Bmatrix}-\gamma^{*[\rho}{}_{\nu];\kappa}\tilde{S}^{\nu\sigma\kappa}\begin{Bmatrix}\mu\\ \rho\sigma\end{Bmatrix}-\gamma^{*[\rho}{}_{\nu];\kappa}\tilde{S}^{\nu\kappa\sigma}\begin{Bmatrix}\mu\\ \rho\sigma\end{Bmatrix}$$
$$\left.+\tfrac{2}{3}\gamma^{*[\mu}{}_{\nu]}\tilde{S}^{\rho\kappa\lambda}R^{*\nu}{}_{(\rho\kappa)\lambda}\right]\delta(x-\xi)ds\ . \quad (3.23)$$

But since one also has

$$\int[S^{\mu\kappa}U^{\nu}]\delta_{,\kappa\nu}(x-\xi)ds=\int\left[\frac{dS^{\mu\nu}}{ds}\right]\delta_{,\nu}(x-\xi)ds\ , \quad (3.24)$$

we find, making use of (3.24) in (3.23),

$$\mathbf{T}^{\mu\nu}{}_{;\nu}-a^{\mu\rho}\gamma^{*}_{[\nu\rho]}\mathbf{s}^{\nu}=\int\left[\widetilde{T}^{(\mu\nu)}+\frac{1}{2}\frac{DS^{\mu\nu}}{Ds}+\gamma^{*[\mu}{}_{\rho]}\widetilde{S}^{[\rho\nu]}-\gamma^{*[\mu}{}_{\rho];\sigma}\widetilde{S}^{\rho\sigma\nu}-\gamma^{*[\mu}{}_{\rho];\sigma}\widetilde{S}^{\rho\nu\sigma}\right]\delta_{,\nu}(x-\xi)ds$$
$$+\int\Big[\widetilde{T}^{(\rho\sigma)}\begin{Bmatrix}\mu\\ \rho\sigma\end{Bmatrix}-\tfrac{1}{2}S^{\rho\sigma}U^{\kappa}R^{*\mu}{}_{\kappa\rho\sigma}+e\gamma^{*[\mu}{}_{\nu]}U^{\nu}-\gamma^{*[\mu}{}_{\nu];\kappa}\widetilde{S}^{[\nu\kappa]}+\gamma^{*[\mu}{}_{\nu];\kappa\lambda}\widetilde{S}^{\nu\kappa\lambda}-\gamma^{*[\rho}{}_{\nu]}\widetilde{S}^{\nu\kappa\lambda}R^{*\mu}{}_{\lambda\kappa\rho}$$
$$+\gamma^{*[\rho}{}_{\nu]}\widetilde{S}^{[\nu\kappa]}\begin{Bmatrix}\mu\\ \rho\kappa\end{Bmatrix}-\gamma^{*[\rho}{}_{\nu];\kappa}\widetilde{S}^{\nu\sigma\kappa}\begin{Bmatrix}\mu\\ \rho\sigma\end{Bmatrix}-\gamma^{*[\rho}{}_{\nu];\kappa}\widetilde{S}^{\nu\kappa\sigma}\begin{Bmatrix}\mu\\ \rho\sigma\end{Bmatrix}+\tfrac{2}{3}\gamma^{*[\mu}{}_{\nu]}\widetilde{S}^{\rho\kappa\lambda}R^{*\nu}{}_{(\rho\kappa)\lambda}\Big]\delta(x-\xi)ds, \tag{3.25}$$

where we are using the definition

$$\frac{DS^{\mu\nu}}{Ds}=\frac{dS^{\mu\nu}}{ds}+S^{\mu\rho}U^{\sigma}\begin{Bmatrix}\nu\\ \rho\sigma\end{Bmatrix}+S^{\rho\nu}U^{\sigma}\begin{Bmatrix}\mu\\ \rho\sigma\end{Bmatrix}, \tag{3.26}$$

in (3.26). The absolute derivative $DS^{\mu\nu}/Ds$ is a second-rank tensor.

The first integral on the right-hand side of (3.25) can be written in the form

$$\int[X^{(\mu\nu)}+Y^{[\mu\nu]}]\delta_{,\nu}(x-\xi)ds\ , \tag{3.27}$$

where

$$X^{(\mu\nu)}=\widetilde{T}^{(\mu\nu)}+\tfrac{1}{2}\gamma^{*[\mu}{}_{\rho]}\widetilde{S}^{[\rho\nu]}+\tfrac{1}{2}\gamma^{*[\nu}{}_{\rho]}\widetilde{S}^{[\rho\mu]}-\tfrac{1}{2}\gamma^{*[\mu}{}_{\rho];\sigma}\widetilde{S}^{\rho\sigma\nu}-\tfrac{1}{2}\gamma^{*[\nu}{}_{\rho];\sigma}\widetilde{S}^{\rho\sigma\mu}-\tfrac{1}{2}\gamma^{*[\mu}{}_{\rho];\sigma}\widetilde{S}^{\rho\nu\sigma}-\tfrac{1}{2}\gamma^{*[\nu}{}_{\rho];\sigma}\widetilde{S}^{\rho\mu\sigma}\ , \tag{3.28}$$

$$Y^{[\mu\nu]}=\frac{1}{2}\frac{DS^{\mu\nu}}{Ds}+\tfrac{1}{2}\gamma^{*[\mu}{}_{\rho]}\widetilde{S}^{[\rho\nu]}-\tfrac{1}{2}\gamma^{*[\nu}{}_{\rho]}\widetilde{S}^{[\rho\mu]}-\tfrac{1}{2}\gamma^{*[\mu}{}_{\rho];\sigma}\widetilde{S}^{\rho\sigma\nu}+\tfrac{1}{2}\gamma^{*[\nu}{}_{\rho];\sigma}\widetilde{S}^{\rho\sigma\mu}-\tfrac{1}{2}\gamma^{*[\mu}{}_{\rho];\sigma}\widetilde{S}^{\rho\nu\sigma}+\tfrac{1}{2}\gamma^{*[\nu}{}_{\rho];\sigma}\widetilde{S}^{\rho\mu\sigma}\ . \tag{3.29}$$

It can be shown that with no loss in generality one can always write[18]

$$X^{(\mu\nu)}={}^{*}X^{(\mu\nu)}+X^{\mu}U^{\nu}+X^{\nu}U^{\mu}+MU^{\mu}U^{\nu}\ , \tag{3.30}$$

$$Y^{[\mu\nu]}={}^{*}Y^{[\mu\nu]}+Y^{\mu}U^{\nu}-Y^{\nu}U^{\mu}\ , \tag{3.31}$$

where

$${}^{*}X^{(\mu\nu)}U_{\nu}=0,\quad X^{\mu}U_{\mu}=0\ , \tag{3.32}$$

$${}^{*}Y^{[\mu\nu]}U_{\nu}=0,\quad Y^{\mu}U_{\mu}=0\ , \tag{3.33}$$

and

$$U_{\mu}=a_{\mu\rho}U^{\rho}\ . \tag{3.34}$$

This means that the first integral on the right-hand side of (3.25) can always be written in the form

$$\int[{}^{*}X^{(\mu\nu)}+{}^{*}Y^{[\mu\nu]}+X^{\nu}U^{\mu}-Y^{\nu}U^{\mu}]\delta_{,\nu}(x-\xi)ds+\int\left[\frac{d}{ds}(MU^{\mu}+X^{\mu}+Y^{\mu})\right]\delta(x-\xi)ds\ , \tag{3.35}$$

so that if we make use of (3.35) in (3.25) we find

$$\mathbf{T}^{\mu\nu}{}_{;\nu}-a^{\mu\rho}\gamma^{*}_{[\nu\rho]}\mathbf{s}^{\nu}=\int[{}^{*}X^{(\mu\nu)}+{}^{*}X^{[\mu\nu]}+X^{\nu}U^{\mu}-Y^{\nu}U^{\mu}]\delta_{,\nu}(x-\xi)ds$$
$$+\int\Big[\frac{d}{ds}(MU^{\mu}+X^{\mu}+Y^{\mu})+\widetilde{T}^{(\rho\sigma)}\begin{Bmatrix}\mu\\ \rho\sigma\end{Bmatrix}-\tfrac{1}{2}S^{\rho\sigma}U^{\kappa}R^{*\mu}{}_{\kappa\rho\sigma}$$
$$+e\gamma^{*[\mu}{}_{\nu]}U^{\nu}-\gamma^{*[\mu}{}_{\nu];\kappa}\widetilde{S}^{[\nu\kappa]}+\gamma^{*[\mu}{}_{\nu];\kappa\lambda}\widetilde{S}^{\nu\kappa\lambda}-\gamma^{*[\rho}{}_{\nu]}\widetilde{S}^{\nu\kappa\lambda}R^{*\mu}{}_{\lambda\kappa\rho}$$
$$+\gamma^{*[\rho}{}_{\nu]}\widetilde{S}^{[\nu\kappa]}\begin{Bmatrix}\mu\\ \rho\kappa\end{Bmatrix}-\gamma^{*[\rho}{}_{\nu];\kappa}\widetilde{S}^{\nu\sigma\kappa}\begin{Bmatrix}\mu\\ \rho\sigma\end{Bmatrix}-\gamma^{*[\rho}{}_{\nu];\kappa}\widetilde{S}^{\nu\kappa\sigma}\begin{Bmatrix}\mu\\ \rho\sigma\end{Bmatrix}$$
$$+\tfrac{2}{3}\gamma^{*[\mu}{}_{\nu]}\widetilde{S}^{\rho\kappa\lambda}R^{*\nu}{}_{(\rho\kappa)\lambda}\Big]\delta(x-\xi)ds\ . \tag{3.36}$$

Since (2.21) must be satisfied, we see from (3.36) that one must have[19]

$$ {}^{*}X^{(\mu\nu)}+{}^{*}Y^{[\mu\nu]}+X^{\nu}U^{\mu}-Y^{\nu}U^{\mu}=0\ , \tag{3.37}$$

which is equivalent to the requirement

$$ {}^{*}X^{(\mu\nu)}=0,\quad {}^{*}Y^{[\mu\nu]}=0\ ,\quad Y^{\mu}=X^{\mu}\ . \tag{3.38}$$

Making use of (3.38) in (3.36) we find

$$\begin{aligned} \mathbf{T}^{\mu\nu}{}_{;\nu}-a^{\mu\rho}\gamma^{*}_{[\nu\rho]}\mathbf{s}^{\nu}=\int\Bigg[&\frac{D}{Ds}(MU^{\mu}+2X^{\mu})-\frac{1}{2}S^{\rho\sigma}U^{\kappa}R^{*\mu}{}_{\kappa\rho\sigma}+e\gamma^{*[\mu}{}_{\nu]}U^{\nu}-\gamma^{*[\mu}{}_{\nu];\kappa}\tilde{S}^{[\nu\kappa]}+\gamma^{*[\mu}{}_{\nu];\kappa\lambda}\tilde{S}^{\nu\kappa\lambda}-\gamma^{*[\rho}{}_{\nu]}\tilde{S}^{\nu\kappa\lambda}R^{*\mu}{}_{\lambda\kappa\rho}\\ &+\gamma^{*[\rho}{}_{\nu]}\tilde{S}^{[\nu\kappa]}\begin{Bmatrix}\mu\\ \rho\kappa\end{Bmatrix}-\gamma^{*[\rho}{}_{\nu];\kappa}\tilde{S}^{\nu\sigma\kappa}\begin{Bmatrix}\mu\\ \rho\sigma\end{Bmatrix}-\gamma^{*[\rho}{}_{\nu];\kappa}\tilde{S}^{\nu\kappa\sigma}\begin{Bmatrix}\mu\\ \rho\sigma\end{Bmatrix}+\tfrac{2}{3}\gamma^{*[\mu}{}_{\nu]}\tilde{S}^{\rho\kappa\lambda}R^{*\nu}{}_{(\rho\kappa)\lambda}\\ &+\tilde{T}^{(\rho\sigma)}\begin{Bmatrix}\mu\\ \rho\sigma\end{Bmatrix}-(MU^{\rho}+2X^{\rho})U^{\sigma}\begin{Bmatrix}\mu\\ \rho\sigma\end{Bmatrix}\Bigg]\delta(x-\xi)ds\ , \end{aligned}\tag{3.39}$$

where we have used the definition of the absolute derivative DC^{μ}/Ds of a vector C^{μ} defined along the world line ξ^{μ},

$$\frac{DC^{\mu}}{Ds}=\frac{dC^{\mu}}{ds}+C^{\rho}U^{\sigma}\begin{Bmatrix}\mu\\ \rho\sigma\end{Bmatrix}\ . \tag{3.40}$$

Making use of (3.30) and (3.31) and (3.38) in (3.28) and (3.29) we find

$$\begin{aligned} \tilde{T}^{(\mu\nu)}=MU^{\mu}U^{\nu}+X^{\mu}U^{\nu}+X^{\nu}U^{\mu}&-\tfrac{1}{2}\gamma^{*[\mu}{}_{\rho]}\tilde{S}^{[\rho\nu]}-\tfrac{1}{2}\gamma^{*[\nu}{}_{\rho]}\tilde{S}^{[\rho\mu]}+\tfrac{1}{2}\gamma^{*[\mu}{}_{\rho];\sigma}\tilde{S}^{\rho\sigma\nu}+\tfrac{1}{2}\gamma^{*[\nu}{}_{\rho];\sigma}\tilde{S}^{\rho\sigma\mu}\\ &+\tfrac{1}{2}\gamma^{*[\mu}{}_{\rho];\sigma}\tilde{S}^{\rho\nu\sigma}+\tfrac{1}{2}\gamma^{*[\nu}{}_{\rho];\sigma}\tilde{S}^{\rho\mu\sigma}\ , \end{aligned}\tag{3.41}$$

$$\frac{DS^{\mu\nu}}{Ds}=2X^{\mu}U^{\nu}-2X^{\nu}U^{\mu}-\gamma^{*[\mu}{}_{\rho]}\tilde{S}^{[\rho\nu]}+\gamma^{*[\nu}{}_{\rho]}\tilde{S}^{[\rho\mu]}+\gamma^{*[\mu}{}_{\rho];\sigma}\tilde{S}^{\rho\sigma\nu}-\gamma^{*[\nu}{}_{\rho];\sigma}\tilde{S}^{\rho\sigma\mu}+\gamma^{*[\mu}{}_{\rho];\sigma}\tilde{S}^{\rho\nu\sigma}-\gamma^{*[\nu}{}_{\rho];\sigma}\tilde{S}^{\rho\mu\sigma}\ , \tag{3.42}$$

and from (3.42) one has

$$\begin{aligned} X^{\mu}=\frac{1}{2}\frac{DS^{\mu\rho}}{Ds}U_{\rho}&+\tfrac{1}{2}\gamma^{*[\mu}{}_{\rho]}\tilde{S}^{[\rho\nu]}U_{\nu}-\tfrac{1}{2}\gamma^{*[\nu}{}_{\rho]}\tilde{S}^{[\rho\mu]}U_{\nu}-\tfrac{1}{2}\gamma^{*[\mu}{}_{\rho];\sigma}\tilde{S}^{\rho\sigma\nu}U_{\nu}\\ &+\tfrac{1}{2}\gamma^{*[\nu}{}_{\rho];\sigma}\tilde{S}^{\rho\sigma\mu}U_{\nu}-\tfrac{1}{2}\gamma^{*[\mu}{}_{\rho];\sigma}\tilde{S}^{\rho\nu\sigma}U_{\nu}+\tfrac{1}{2}\gamma^{*[\nu}{}_{\rho];\sigma}\tilde{S}^{\rho\mu\sigma}U_{\nu}\ . \end{aligned}\tag{3.43}$$

Making use of (3.43) in (3.41) we find

$$\begin{aligned} \tilde{T}^{(\mu\nu)}=MU^{\mu}U^{\nu}+\frac{1}{2}\frac{DS^{\mu\rho}}{Ds}U_{\rho}U^{\nu}+\frac{1}{2}\frac{DS^{\nu\rho}}{Ds}U_{\rho}U^{\mu}&-\tfrac{1}{2}\gamma^{*[\mu}{}_{\rho]}\tilde{S}^{[\rho\nu]}-\tfrac{1}{2}\gamma^{*[\nu}{}_{\rho]}\tilde{S}^{[\rho\mu]}+\tfrac{1}{2}\gamma^{*[\mu}{}_{\rho]}\tilde{S}^{[\rho\sigma]}U_{\sigma}U^{\nu}\\ &+\tfrac{1}{2}\gamma^{*[\nu}{}_{\rho]}\tilde{S}^{[\rho\sigma]}U_{\sigma}U^{\mu}-\tfrac{1}{2}\gamma^{*[\rho}{}_{\sigma]}\tilde{S}^{[\sigma\mu]}U_{\rho}U^{\nu}-\tfrac{1}{2}\gamma^{*[\rho}{}_{\sigma]}\tilde{S}^{[\sigma\nu]}U_{\rho}U^{\mu}+\tfrac{1}{2}\gamma^{*[\mu}{}_{\rho];\sigma}\tilde{S}^{\rho\sigma\nu}\\ &+\tfrac{1}{2}\gamma^{*[\nu}{}_{\rho];\sigma}\tilde{S}^{\rho\sigma\mu}+\tfrac{1}{2}\gamma^{*[\mu}{}_{\rho];\sigma}\tilde{S}^{\rho\nu\sigma}+\tfrac{1}{2}\gamma^{*[\nu}{}_{\rho];\sigma}\tilde{S}^{\rho\mu\sigma}-\tfrac{1}{2}\gamma^{*[\mu}{}_{\rho];\sigma}\tilde{S}^{\rho\sigma\kappa}U_{\kappa}U^{\nu}-\tfrac{1}{2}\gamma^{*[\nu}{}_{\rho];\sigma}\tilde{S}^{\rho\sigma\kappa}U_{\kappa}U^{\mu}\\ &-\tfrac{1}{2}\gamma^{*[\mu}{}_{\rho];\sigma}\tilde{S}^{\rho\kappa\sigma}U_{\kappa}U^{\nu}-\tfrac{1}{2}\gamma^{*[\nu}{}_{\rho];\sigma}\tilde{S}^{\rho\kappa\sigma}U_{\kappa}U^{\mu}+\tfrac{1}{2}\gamma^{*[\rho}{}_{\sigma];\kappa}\tilde{S}^{\sigma\mu\kappa}U_{\rho}U^{\nu}\\ &+\tfrac{1}{2}\gamma^{*[\rho}{}_{\sigma];\kappa}\tilde{S}^{\sigma\nu\kappa}U_{\rho}U^{\mu}+\tfrac{1}{2}\gamma^{*[\rho}{}_{\sigma];\kappa}\tilde{S}^{\sigma\kappa\mu}U_{\rho}U^{\nu}+\tfrac{1}{2}\gamma^{*[\rho}{}_{\sigma];\kappa}\tilde{S}^{\sigma\kappa\nu}U_{\rho}U^{\mu}\ , \end{aligned}\tag{3.44}$$

while making use of (3.43) in (3.42) gives as the equations of structure satisfied by $S^{\mu\nu}$,

$$\frac{DS^{\mu\nu}}{Ds}-\frac{DS^{\mu\rho}}{Ds}U_{\rho}U^{\nu}+\frac{DS^{\nu\rho}}{Ds}U_{\rho}U^{\mu}=N^{\mu\nu}\ , \tag{3.45}$$

where

$$\begin{aligned}N^{\mu\nu}= & -\gamma^{*[\mu}{}_{\rho]}\tilde{S}^{[\rho\nu]}+\gamma^{*[\nu}{}_{\rho]}\tilde{S}^{[\rho\mu]}+\gamma^{*[\mu}{}_{\rho]}\tilde{S}^{[\rho\sigma]}U_\sigma U^\nu-\gamma^{*[\nu}{}_{\rho]}\tilde{S}^{[\rho\sigma]}U_\sigma U^\mu-\gamma^{*[\rho}{}_{\sigma]}\tilde{S}^{[\sigma\mu]}U_\rho U^\nu \\ & +\gamma^{*[\rho}{}_{\sigma]}\tilde{S}^{[\sigma\nu]}U_\rho U^\mu+\gamma^{*[\mu}{}_{\rho];\sigma}\tilde{S}^{\rho\sigma\nu}-\gamma^{*[\nu}{}_{\rho];\sigma}\tilde{S}^{\rho\sigma\mu}+\gamma^{*[\mu}{}_{\rho];\sigma}\tilde{S}^{\rho\nu\sigma}-\gamma^{*[\nu}{}_{\rho];\sigma}\tilde{S}^{\rho\mu\sigma} \\ & -\gamma^{*[\mu}{}_{\rho];\sigma}\tilde{S}^{\rho\sigma\kappa}U_\kappa U^\nu+\gamma^{*[\nu}{}_{\rho];\sigma}\tilde{S}^{\rho\sigma\kappa}U_\kappa U^\mu-\gamma^{*[\mu}{}_{\rho];\sigma}\tilde{S}^{\rho\kappa\sigma}U_\kappa U^\nu+\gamma^{*[\nu}{}_{\rho];\sigma}\tilde{S}^{\rho\kappa\sigma}U_\kappa U^\mu \\ & +\gamma^{*[\rho}{}_{\sigma];\kappa}\tilde{S}^{\sigma\mu\kappa}U_\rho U^\nu-\gamma^{*[\rho}{}_{\sigma];\kappa}\tilde{S}^{\sigma\nu\kappa}U_\rho U^\mu+\gamma^{*[\rho}{}_{\sigma];\kappa}\tilde{S}^{\sigma\kappa\mu}U_\rho U^\nu-\gamma^{*[\rho}{}_{\sigma];\kappa}\tilde{S}^{\sigma\kappa\nu}U_\rho U^\mu\ .\end{aligned} \tag{3.46}$$

Making use of (3.43) in (3.39) one finds

$$\begin{aligned}\mathbf{T}^{\mu\nu}{}_{;\nu}-a^{\mu\rho}\gamma^*_{[\nu\rho]}\mathbf{s}^\nu=\int\Bigg[& \frac{D}{Ds}\Bigg\{MU^\mu+\frac{DS^{\mu\rho}}{Ds}U_\rho+\gamma^{*[\mu}{}_{\rho]}\tilde{S}^{[\rho\sigma]}U_\sigma-\gamma^{*[\rho}{}_{\sigma]}\tilde{S}^{[\sigma\mu]}U_\rho-\gamma^{*[\mu}{}_{\rho];\sigma}\tilde{S}^{\rho\sigma\kappa}U_\kappa \\ & +\gamma^{*[\rho}{}_{\sigma];\kappa}\tilde{S}^{\sigma\kappa\mu}U_\rho-\gamma^{*[\mu}{}_{\rho];\sigma}\tilde{S}^{\rho\kappa\sigma}U_\kappa+\gamma^{*[\rho}{}_{\sigma];\kappa}\tilde{S}^{\sigma\mu\kappa}U_\rho\Bigg\}-\tfrac{1}{2}S^{\rho\sigma}U^\kappa R^{*\mu}{}_{\kappa\rho\sigma} \\ & +e\gamma^{*[\mu}{}_{\nu]}U^\nu-\gamma^{*[\mu}{}_{\nu];\kappa}\tilde{S}^{[\nu\kappa]}+\gamma^{*[\mu}{}_{\nu];\kappa\lambda}\tilde{S}^{\nu\kappa\lambda}-\gamma^{*[\rho}{}_{\nu]}\tilde{S}^{\nu\kappa\lambda}R^{*\mu}{}_{\lambda\kappa\rho} \\ & +\frac{2}{3}\gamma^{*[\mu}{}_{\nu]}\tilde{S}^{\rho\kappa\lambda}R^{*\nu}{}_{(\rho\kappa)\lambda}\Bigg]\delta(x-\xi)ds\ .\end{aligned} \tag{3.47}$$

Since (2.21) must be satisfied we see from (3.47) that the test particle must obey the equations of motion

$$\frac{DP^\mu}{Ds}+\frac{1}{2}S^{\rho\sigma}U^\kappa R^{*\mu}{}_{\kappa\sigma\rho}=F^\mu\ , \tag{3.48}$$

where

$$\begin{aligned}P^\mu= & MU^\mu+\frac{DS^{\mu\rho}}{Ds}U_\rho+\gamma^{*[\mu}{}_{\rho]}\tilde{S}^{[\rho\sigma]}U_\sigma-\gamma^{*[\rho}{}_{\sigma]}\tilde{S}^{[\sigma\mu]}U_\rho-\gamma^{*[\mu}{}_{\rho];\sigma}\tilde{S}^{\rho\sigma\kappa}U_\kappa \\ & +\gamma^{*[\rho}{}_{\sigma];\kappa}\tilde{S}^{\sigma\kappa\mu}U_\rho-\gamma^{*[\mu}{}_{\rho];\sigma}\tilde{S}^{\rho\kappa\sigma}U_\kappa+\gamma^{*[\rho}{}_{\sigma];\kappa}\tilde{S}^{\sigma\mu\kappa}U_\rho\ ,\end{aligned} \tag{3.49}$$

$$F^\mu=-e\gamma^{*[\mu}{}_{\nu]}U^\nu+\gamma^{*[\mu}{}_{\nu];\kappa}\tilde{S}^{[\nu\kappa]}-\gamma^{*[\mu}{}_{\nu];\kappa\lambda}\tilde{S}^{\nu\kappa\lambda}+\gamma^{*[\rho}{}_{\nu]}\tilde{S}^{\nu\kappa\lambda}R^{*\mu}{}_{\lambda\kappa\rho}-\tfrac{2}{3}\gamma^{*[\mu}{}_{\nu]}\tilde{S}^{\rho\kappa\lambda}R^{*\mu}{}_{(\rho\kappa)\lambda}\ . \tag{3.50}$$

If we place (3.22) and (3.44) in (3.17) we find that the energy-momentum tensor density $\mathbf{T}^{\mu\nu}$ associated with the test particle is given by

$$\begin{aligned}\mathbf{T}^{\mu\nu}= & \int[\tfrac{1}{2}S^{\mu\kappa}U^\nu+\tfrac{1}{2}S^{\nu\kappa}U^\mu-\tfrac{1}{2}\gamma^{*[\mu}{}_{\rho]}\tilde{S}^{\rho\nu\kappa}-\tfrac{1}{2}\gamma^{*[\nu}{}_{\rho]}\tilde{S}^{\rho\mu\kappa}-\tfrac{1}{2}\gamma^{*[\mu}{}_{\rho]}\tilde{S}^{\rho\kappa\nu}-\tfrac{1}{2}\gamma^{*[\nu}{}_{\rho]}\tilde{S}^{\rho\kappa\mu} \\ & +\tfrac{1}{2}\gamma^{*[\kappa}{}_{\rho]}\tilde{S}^{\rho\nu\mu}+\tfrac{1}{2}\gamma^{*[\kappa}{}_{\rho]}\tilde{S}^{\rho\mu\nu}]\delta_{,\kappa}(x-\xi)ds \\ & +\int\Bigg[MU^\mu U^\nu+\frac{1}{2}\frac{DS^{\mu\rho}}{Ds}U_\rho U^\nu+\frac{1}{2}\frac{DS^{\nu\rho}}{Ds}U_\rho U^\mu-\tfrac{1}{2}\gamma^{*[\mu}{}_{\rho]}\tilde{S}^{[\rho\nu]}-\tfrac{1}{2}\gamma^{*[\nu}{}_{\rho]}\tilde{S}^{[\rho\mu]}+\tfrac{1}{2}\gamma^{*[\mu}{}_{\rho]}\tilde{S}^{[\rho\sigma]}U_\sigma U^\nu \\ & +\tfrac{1}{2}\gamma^{*[\nu}{}_{\rho]}\tilde{S}^{[\rho\sigma]}U_\sigma U^\mu-\tfrac{1}{2}\gamma^{*[\rho}{}_{\sigma]}\tilde{S}^{[\sigma\mu]}U_\rho U^\nu-\tfrac{1}{2}\gamma^{*[\rho}{}_{\sigma]}\tilde{S}^{[\sigma\nu]}U_\rho U^\mu+\tfrac{1}{2}\gamma^{*[\mu}{}_{\rho];\sigma}\tilde{S}^{\rho\sigma\nu} \\ & +\tfrac{1}{2}\gamma^{*[\nu}{}_{\rho];\sigma}\tilde{S}^{\rho\sigma\mu}+\tfrac{1}{2}\gamma^{*[\mu}{}_{\rho];\sigma}\tilde{S}^{\rho\nu\sigma}+\tfrac{1}{2}\gamma^{*[\nu}{}_{\rho];\sigma}\tilde{S}^{\rho\mu\sigma}-\tfrac{1}{2}\gamma^{*[\mu}{}_{\rho];\sigma}\tilde{S}^{\rho\sigma\kappa}U_\kappa U^\nu \\ & -\tfrac{1}{2}\gamma^{*[\nu}{}_{\rho];\sigma}\tilde{S}^{\rho\sigma\kappa}U_\kappa U^\mu-\tfrac{1}{2}\gamma^{*[\mu}{}_{\rho];\sigma}\tilde{S}^{\rho\kappa\sigma}U_\kappa U^\nu-\tfrac{1}{2}\gamma^{*[\nu}{}_{\rho];\sigma}\tilde{S}^{\rho\kappa\sigma}U_\kappa U^\mu+\tfrac{1}{2}\gamma^{*[\rho}{}_{\sigma];\kappa}\tilde{S}^{\sigma\mu\kappa}U_\rho U^\nu \\ & +\tfrac{1}{2}\gamma^{*[\rho}{}_{\sigma];\kappa}\tilde{S}^{\sigma\nu\kappa}U_\rho U^\mu+\tfrac{1}{2}\gamma^{*[\rho}{}_{\sigma];\kappa}\tilde{S}^{\sigma\kappa\mu}U_\rho U^\nu+\tfrac{1}{2}\gamma^{*[\rho}{}_{\sigma];\kappa}\tilde{S}^{\sigma\kappa\nu}U_\rho U^\mu+\tfrac{1}{2}S^{\mu\rho}U^\sigma\left\{{\nu \atop \rho\sigma}\right\} \\ & +\tfrac{1}{2}S^{\nu\rho}U^\sigma\left\{{\mu \atop \rho\sigma}\right\}-\gamma^{*[\mu}{}_{\kappa]}\tilde{S}^{\kappa\rho\sigma}\left\{{\nu \atop \rho\sigma}\right\}-\gamma^{*[\nu}{}_{\kappa]}\tilde{S}^{\kappa\rho\sigma}\left\{{\mu \atop \rho\sigma}\right\}\Bigg]\delta(x-\xi)ds\ .\end{aligned} \tag{3.51}$$

We see that the gravitational structure of the test particle is characterized by the quantities M and $S^{\mu\nu}$. The quantity M is constrained by the equations of motion (3.48)—(3.50), and the quantity $S^{\mu\nu}$ obeys the equations of structure (3.45) and (3.46). The form of the equations of motion and the form of the equations of structure allow us to identify M with the mass of the particle and $S^{\mu\nu}$ with its spin.

C. Simple charged test particles possessing no magnetic monopole moments

In Sec. II we found that if one neglects its interaction with other particles, a simple charged particle possessing no magnetic monopole moment is characterized by a mass monopole moment m^G, a spin $S^G_{\mu\nu}$, an electromagnetic monopole moment e^E, and an electromagnetic quadrupole moment $e^E_{[\mu\kappa]\lambda}$, where

$$m^G=4mc^2 , \tag{3.52}$$

$$S^G_{\mu\nu}=4s_{\mu\nu}c^2 , \tag{3.53}$$

$$e^E=(c^2/l)q , \tag{3.54}$$

$$e^E_{[\mu\kappa]\lambda}=(c^2/l)ql^2(\eta_{\mu\lambda}u_\kappa-\eta_{\kappa\lambda}u_\mu) . \tag{3.55}$$

There are no additional multipole moments associated with the particle. We thus have associated with the particle

$$i^{\rm kin}_\mu=0 , \tag{3.56}$$

and

$$s^{\rm kin}_\mu=\frac{4\pi}{c^2}\int e^E u_\mu \delta(x-\xi)d\tau + \frac{4\pi}{c^2}\int e^E_{[\mu\kappa]\lambda}\delta^{,\kappa\lambda}(x-\xi)d\tau , \tag{3.57}$$

$$t^{\rm kin}_{\mu\nu}=\frac{4\pi}{c^2}\int [m^G u_\mu u_\nu+\tfrac{1}{2}\dot S^G_{\mu\rho}u^\rho u_\nu+\tfrac{1}{2}\dot S^G_{\nu\rho}u^\rho u_\mu]\delta(x-\xi)d\tau + \frac{4\pi}{c^2}\int [\tfrac{1}{2}S^G_{\mu\rho}u_\nu+\tfrac{1}{2}S^G_{\nu\rho}u_\mu]\delta^{,\rho}(x-\xi)d\tau , \tag{3.58}$$

where the quantities m^G, $S^G_{\mu\nu}$, e^E, and $e^E_{[\mu\kappa]\lambda}$ are given in (3.52)–(3.55). Making use of Eqs. (3.16) and (3.51) and the relationship of i^μ to $i^{\rm kin}_\mu$, of s^μ to $s^{\rm kin}_\mu$, and of $\mathrm{T}^{\mu\nu}$ to $t^{\rm kin}_\mu$ (those relationships are discussed in Sec. V of paper I) we see that in an external field a simple charged test particle possessing no magnetic monopole moment will be characterized by a mass M, a spin $S^{\mu\nu}$, an electromagnetic monopole moment e, and an electromagnetic quadrupole moment $\tilde{s}^{\mu\kappa\lambda}$, where

$$M=8\pi(m+\Delta m) , \tag{3.59}$$

$$S^{\mu\nu}=8\pi s^{\mu\nu} , \tag{3.60}$$

$$e=(2\pi/l)q , \tag{3.61}$$

$$\tilde{s}^{\mu\kappa\lambda}=(2\pi/l)ql^2(a^{\mu\lambda}u^\kappa-a^{\kappa\lambda}u^\mu) . \tag{3.62}$$

The particle will possess no additional multipole moments in the external field. In (3.61) and (3.62), the length l is a universal constant, and the quantity q represents the charge of the particle. The quantities m and $s^{\mu\nu}$ represent, respectively, the mass and the spin of the particle. The quantity Δm is at this point of our study arbitrary and represents a certain freedom one always has in defining the mass of a particle in the presence of an external field.

In arriving at (3.62) as the exact expression for $\tilde{s}^{\mu\kappa\lambda}$ we have assumed that in a locally inertial coordinate system one finds

$$\tilde{s}^{\mu\kappa\lambda}=(2\pi/l)ql^2(\eta^{\mu\lambda}u^\kappa-\eta^{\kappa\lambda}u^\mu) , \tag{3.63}$$

that is, $\tilde{s}^{\mu\kappa\lambda}$ takes its flat space value. We are here defining a locally inertial coordinate system at a point x^μ as a coordinate system such that at x^μ

$$a_{\mu\nu}=\eta_{\mu\nu},\quad a_{\mu\nu,\lambda}=0 . \tag{3.64}$$

It is always possible to introduce a locally inertial coordinate system at any given point x^μ. At that point and in such a coordinate system the acceleration of a neutral pole test particle possessing no electromagnetic multipole moments will vanish[20]—this is, of course, the reason for the name locally inertial coördinate system. That in a locally inertial coordinate system $\tilde{s}^{\mu\kappa\lambda}$ takes its flat space value is a natural condition to place on the structure of a charged test particle, and we shall restrict our study from now on to charged test particles satisfying this condition.

At this point, we therefore place an additional restriction on what we mean by a simple charged test particle in Einstein's unified field theory. In addition to such a particle being the test particle limit of what we have called a simple charged particle, we also require that the electromagnetic multipole moments e, $\tilde{s}^{[\mu\nu]}$, and $\tilde{s}^{\mu\nu\lambda}$ associated with the particle take their flat space values in a locally inertial coordinate system.

We first investigate the equations of spin satisfied by such a test particle. Making use of (3.15), we see from (3.61) that q is constant even in the presence of an external field. Placing (3.62) in (3.46) and making use of the fact that $\tilde{s}^{[\mu\nu]}=0$, we find that $N^{\mu\nu}=0$. This means that the equations of spin (3.45) satisfied by a simple charged test particle possessing no magnetic monopole moment take the form

$$\frac{DS^{\mu\nu}}{Ds}-\frac{DS^{\mu\rho}}{Ds}U_\rho U^\nu+\frac{DS^{\nu\rho}}{Ds}U_\rho U^\mu=0 . \tag{3.65}$$

We next investigate the equations of mass and motion satisfied by the particle. Placing (3.59)–(3.62) in (3.48)–(3.50) and making use of the fact that $\tilde{s}^{[\mu\nu]}=0$, we find as the equations of mass and motion of a simple charged test particle possessing no magnetic monopole moment

$$\frac{DP^\mu}{Ds}+\tfrac{1}{2}S^{\rho\sigma}U^\kappa R^{*\mu}{}_{\kappa\sigma\rho}=F^\mu , \tag{3.66}$$

where

$$P^\mu=8\pi\left[(m+\Delta m)U^\mu+\frac{Ds^{\mu\rho}}{Ds}U_\rho + \left[\frac{1}{4l}\right]ql^2(\gamma^{*[\rho\sigma]}{}_{;\sigma}U_\rho U^\mu-\gamma^{*[\mu\sigma]}{}_{;\rho})\right] , \tag{3.67}$$

$$F^\mu=(2\pi/l)[q\gamma^{*[\nu\mu]}U_\nu+ql^2(a^{\kappa\lambda}\gamma^{*[\mu\nu]}{}_{;\kappa\lambda}U_\nu-\gamma^{*[\mu\lambda]}{}_{;\kappa\lambda}U^\kappa -\gamma^{*[\rho\sigma]}R^{*\mu}{}_{\rho\kappa\sigma}U^\kappa -a^{\mu\kappa}\gamma^{*[\rho\sigma]}R^*_{\kappa\rho}U_\sigma)] . \tag{3.68}$$

We have made use of the definitions

$$R^*_{\mu\nu}=R^{*\kappa}{}_{\mu\nu\kappa},\quad \gamma^{*[\mu\nu]}=a^{\nu\rho}\gamma^{*[\mu}{}_{\rho]}\ . \tag{3.69}$$

Equations (3.66) take a simpler form if we choose

$$\Delta m=-\left[\frac{1}{4l}\right]ql^2\gamma^{*[\rho\sigma]}{}_{;\sigma}U_\rho\ , \tag{3.70}$$

and introduce the notation

$$p^\mu=\frac{1}{8\pi}P^\mu+\left[\frac{1}{4l}\right]ql^2\gamma^{*[\mu\rho]}{}_{;\rho}\ , \tag{3.71}$$

$$f^\mu=\frac{1}{8\pi}F^\mu+\left[\frac{1}{4l}\right]ql^2\gamma^{*[\mu\rho]}{}_{;\rho\sigma}U^\sigma\ . \tag{3.72}$$

Using (3.70) and (3.71), we find the equations of mass and motion (3.66) can be written in the form

$$\frac{Dp^\mu}{Ds}+\tfrac{1}{2}s^{\rho\sigma}U^\kappa R^{*\mu}{}_{\kappa\sigma\rho}=f^\mu\ , \tag{3.73}$$

where

$$p^\mu=mU^\mu+\frac{Ds^{\mu\rho}}{Ds}U_\rho\ , \tag{3.74}$$

$$f^\kappa=\left[\frac{1}{4l}\right][q\gamma^{*[\nu\mu]}U_\nu+ql^2\gamma^{*[\mu\nu];\kappa}{}_{;\kappa}U_\nu -ql^2(\gamma^{*[\rho\sigma]}R^{*\mu}{}_{\kappa\rho\sigma}U^\kappa+\gamma^{*[\mu\rho]}R^*_{\rho\lambda}U^\lambda +\gamma^*_{[\rho\sigma]}R^{*\mu\rho}U^\sigma)]\ . \tag{3.75}$$

In (3.75) we are using the definition

$$R^{*\mu\nu}=a^{\mu\rho}a^{\nu\sigma}R^*_{\rho\sigma}\ . \tag{3.76}$$

One finds from (3.75) that

$$f^\mu U_\mu=0\ . \tag{3.77}$$

Although both the universal length l and the charge q associated with a simple test particle possessing no magnetic monopole moment are constant, we see from (3.73)–(3.75), making use of (3.77), and from (3.65) that the mass m and spin $s^{\mu\nu}$ associated with the particle are in general not constant. They obey the equations of structure

$$\frac{dm}{ds}=\frac{DU_\rho}{Ds}\frac{D}{Ds}(s^{\rho\mu}U_\mu)\ , \tag{3.78}$$

$$\frac{Ds^{\mu\nu}}{Ds}-\frac{Ds^{\mu\rho}}{Ds}U_\rho U^\nu+\frac{Ds^{\nu\rho}}{Ds}U_\rho U^\mu=0\ . \tag{3.79}$$

IV. ELECTROMAGNETIC CURRENT DENSITIES AND ENERGY-MOMENTUM TENSOR DENSITY

We have found that in Einstein's unified field theory a simple charged test particle possessing no magnetic monopole moment is characterized by a mass m, a spin $s^{\mu\nu}$, a charge q, and a universal length l. In this section we wish to give the current densities $\mathbf{i}^\mu$ and $\mathbf{s}^\mu$ and the energy-momentum tensor density $\mathbf{T}^{\mu\nu}$ associated with these particles.

We have already seen that $\mathbf{i}^\mu$ vanishes. Making use of (3.61), (3.62), and the fact that $\tilde{s}^{[\mu\nu]}$ vanishes, we find from (3.16) that

$$\mathbf{s}^\mu=(2\pi/l)\int[qU^\mu]\delta(x-\xi)ds+(2\pi/l)ql^2\int\left[a^{\rho\sigma}U^\kappa\begin{Bmatrix}\mu\\ \rho\sigma\end{Bmatrix}-2a^{\kappa\rho}U^\sigma\begin{Bmatrix}\mu\\ \rho\sigma\end{Bmatrix}-a^{\rho\sigma}U^\mu\begin{Bmatrix}\kappa\\ \rho\sigma\end{Bmatrix}\right]\delta_{,\kappa}(x-\xi)ds +(2\pi/l)ql^2\int[-a^{\kappa\lambda}U^\mu]\delta_{,\kappa\lambda}(x-\xi)ds\ . \tag{4.1}$$

Making use of (3.59)–(3.62), (3.70), and the fact that $\tilde{s}^{[\mu\nu]}$ vanishes, we find from (3.51) that

$$\begin{aligned}\mathbf{T}^{\mu\nu}=8\pi\int\Bigg[&mU^\mu U^\nu+\frac{1}{2}\frac{Ds^{\mu\rho}}{Ds}U_\rho U^\nu+\frac{1}{2}\frac{Ds^{\nu\rho}}{Ds}U_\rho U^\mu+\left[\frac{1}{4l}\right]ql^2(-a^{\mu\rho}\gamma^{*[\nu\sigma]}{}_{;\rho}U_\sigma-a^{\nu\rho}\gamma^{*[\mu\sigma]}{}_{;\rho}U_\sigma)\\ &+\tfrac{1}{2}s^{\mu\rho}U^\sigma\begin{Bmatrix}\nu\\ \rho\sigma\end{Bmatrix}+\tfrac{1}{2}s^{\nu\rho}U^\sigma\begin{Bmatrix}\mu\\ \rho\sigma\end{Bmatrix}+\left[\frac{1}{4l}\right]ql^2\left[a^{\rho\sigma}\gamma^{*[\mu\kappa]}U_\kappa\begin{Bmatrix}\nu\\ \rho\sigma\end{Bmatrix}+a^{\rho\sigma}\gamma^{*[\nu\kappa]}U_\kappa\begin{Bmatrix}\mu\\ \rho\sigma\end{Bmatrix}-\gamma^{*[\mu\rho]}U^\sigma\begin{Bmatrix}\nu\\ \rho\sigma\end{Bmatrix}\right.\\ &\left.-\gamma^{*[\nu\rho]}U^\sigma\begin{Bmatrix}\mu\\ \rho\sigma\end{Bmatrix}\right]\Bigg]\delta(x-\xi)ds\\ &+8\pi\int\Bigg[\tfrac{1}{2}s^{\mu\kappa}U^\nu+\tfrac{1}{2}s^{\nu\kappa}U^\mu+\left[\frac{1}{4l}\right]ql^2(a^{\mu\kappa}\gamma^{*[\nu\rho]}U_\rho+a^{\nu\kappa}\gamma^{*[\mu\rho]}U_\rho)\\ &-\gamma^{*[\mu\kappa]}U^\nu-\gamma^{*[\nu\kappa]}U^\mu-a^{\mu\nu}\gamma^{*[\kappa\rho]}U_\rho\Bigg]\delta_{,\kappa}(\kappa-\xi)ds\ .\end{aligned} \tag{4.2}$$

V. COMPARISON WITH THE RESULTS OF THE APPROXIMATION PROCEDURE

If we make use of the power-series expansion in κ,[21]

$$g_{\mu\nu}=\eta_{\mu\nu}+\sum_{k=1}^{\infty}\kappa^k{}_{(k)}g_{\mu\nu} \tag{5.1}$$

and regard $\gamma^*_{[\mu\nu]}$ and q as possessing only terms of odd order in κ, and $a_{\mu\nu}$ and m as possessing only terms of even order in κ (that there is no loss in generality in making such an assumption is shown in paper RVIII of Ref. 6), and treat the spin $s^{\mu\nu}$ as negligible, we find from the equations of motion (3.73)—(3.75)

$$
\begin{aligned}
m\dot{u}^{\mu}= \left[\frac{1}{4l}\right] & [\, q(\gamma^{*[\rho\mu]}u_\rho+\tfrac{1}{2}\gamma_{(\rho\sigma)}\gamma^{*[\mu\kappa]}u^\rho u^\sigma u_\kappa+\gamma^{(\rho\sigma)}\gamma^*_{[\rho\kappa]}u_\sigma u^\kappa u^\mu-\gamma^{(\mu\rho)}\gamma^*_{[\rho\sigma]}u^\sigma+\tfrac{1}{4}\gamma\gamma^{*[\mu\rho]}u_\rho) \\
& -ql^2(\,\Box^2\gamma^{*[\rho\mu]}u_\rho+\tfrac{1}{2}\gamma_{(\rho\sigma)}\Box^2\gamma^{*[\mu\kappa]}u^\rho u^\sigma u_\kappa+\gamma^{(\rho\sigma)}\Box^2\gamma^*_{[\rho\kappa]}u_\sigma u^\kappa u^\mu-\gamma^{(\mu\rho)}\Box^2\gamma^*_{[\rho\sigma]}u^\sigma \\
& -\gamma^{(\rho\sigma)}\gamma^{*[\mu\kappa]}{}_{,\rho\sigma}u_\kappa+\tfrac{3}{4}\gamma\Box^2\gamma^{*[\mu\rho]}u_\rho-\gamma^{(\mu\rho),\sigma}\gamma^*_{[\rho\kappa],\sigma}u^\kappa \\
& +\gamma^{(\mu\rho),\sigma}\gamma^*_{[\sigma\kappa],\rho}u^\kappa-\gamma^{(\rho\sigma),\mu}\gamma^*_{[\rho\kappa],\sigma}u^\kappa-\gamma_{(\rho\sigma),\kappa}\gamma^{*[\mu\rho],\kappa}u^\sigma-\gamma_{(\rho\sigma),\kappa}\gamma^{*[\mu\rho],\sigma}u^\kappa \\
& +\gamma_{(\rho\sigma),\kappa}\gamma^{*[\mu\kappa],\rho}u^\sigma-\gamma_{(\rho\sigma)}{}^{,\sigma}\gamma^{*[\mu\kappa],\rho}u_\kappa+\gamma_{,\rho}\gamma^{*[\mu\sigma],\rho}u_\sigma-\tfrac{1}{2}\gamma^{,\mu}\gamma^*_{[\rho\sigma]}{}^{,\sigma}u^\rho-\tfrac{1}{2}\gamma_{,\rho}\gamma^{*[\mu\rho],\sigma}u_\sigma \\
& +\tfrac{1}{2}\gamma^{,\rho}\gamma^{*[\mu\sigma]}{}_{,\sigma}u_\rho-\tfrac{1}{2}\gamma_{,\rho}\gamma^{*[\rho\sigma],\mu}u_\sigma-\gamma^{(\mu\rho),\sigma\kappa}\gamma^*_{[\rho\sigma]}u_\kappa+\gamma^{(\rho\sigma),\mu\kappa}\gamma^*_{[\rho\kappa]}u_\sigma \\
& -\gamma^{(\rho\sigma),\mu}{}_{,\sigma}\gamma^*_{[\rho\kappa]}u^\kappa-\gamma_{(\rho\sigma)}{}^{,\sigma\kappa}\gamma^{*[\mu\rho]}u_\kappa+\tfrac{1}{2}\gamma^{,\mu\rho}\gamma^*_{[\rho\sigma]}u^\sigma+\tfrac{1}{2}\gamma_{,\rho\sigma}\gamma^{*[\mu\rho]}u^\sigma)] \\
& +[m(-\tfrac{1}{2}\gamma_{(\rho\sigma),\kappa}u^\rho u^\sigma u^\kappa u^\mu+\gamma^{(\mu\rho),\sigma}u_\rho u_\sigma-\tfrac{1}{2}\gamma^{(\rho\sigma),\mu}u_\rho u_\sigma-\tfrac{1}{4}\gamma_{,\rho}u^\rho u^\mu+\tfrac{1}{4}\gamma^{,\mu})]+O(\kappa^6)\ ,
\end{aligned}
\tag{5.2}
$$

where in (5.2) all indices are raised and lowered with the Minkowski metric $\eta_{\mu\nu}=\eta^{\mu\nu}$ and

$$\gamma^*_{[\mu\nu]}=\tfrac{1}{2}\epsilon_{\mu\nu\rho\sigma}\mathbf{g}^{[\sigma\rho]}\ , \tag{5.3}$$

$$\gamma^{(\mu\nu)}=\mathbf{g}^{(\mu\nu)}-\eta^{\mu\nu},\quad \gamma=\eta_{\rho\sigma}\gamma^{(\rho\sigma)}\ , \tag{5.4}$$

and

$$u^\mu=\frac{d\xi^\mu}{d\tau},\quad \dot{u}^\mu=\frac{du^\mu}{d\tau}\ , \tag{5.5}$$

$$d\tau^2=\eta_{\mu\nu}d\xi^\mu d\xi^\nu\ . \tag{5.6}$$

The above equations of motion should be identical to the equations of motion to fourth order of a simple charged test particle possessing no magnetic monopole moment and spin obtained using the authors approximation method described in papers RI—RVIII of Ref. 6. A long calculation using that method shows that this is indeed so.

According to (5.2), the equations of motion to second order are

$$m\dot{u}^\mu=\left[\frac{1}{4l}\right][q\gamma^{*[\rho\mu]}u_\rho-ql^2\Box^2\gamma^{*[\rho\mu]}u_\rho]\ . \tag{5.7}$$

This is in agreement with the result obtained in the author's earlier papers using his approximation method.[22]

If we make use of the above power-series expansion in κ in the expressions for the electromagnetic current density $\mathbf{s}^\mu$ given through (4.1) and the energy-momentum tensor density $\mathbf{T}^{\mu\nu}$ given through (4.2), we find

$$\mathbf{s}^\mu=(2\pi/l)\left[\int[qu^\mu]\delta(x-\xi)d\tau-\int[ql^2u^\mu]\Box^2\delta(x-\xi)d\tau\right]+O(\kappa^3)\ , \tag{5.8}$$

$$
\begin{aligned}
\mathbf{T}^{\mu\nu}= 8\pi & \left[\int[mu^\mu u^\nu]\delta(x-\xi)d\tau\right] \\
& +(2\pi/l)\Big\{\int[ql^2(\gamma^{*[\sigma\nu],\mu}u_\sigma+\gamma^{*[\sigma\mu],\nu}u_\sigma)]\delta(x-\xi)d\tau \\
& +\int[\,ql^2(\eta^{\mu\kappa}\gamma^{*[\nu\rho]}u_\rho+\eta^{\nu\kappa}\gamma^{*[\mu\rho]}u_\rho-\eta^{\mu\nu}\gamma^{*[\kappa\rho]}u_\rho+\gamma^{*[\kappa\mu]}u^\nu+\gamma^{*[\kappa\nu]}u^\mu)]\delta_{,\kappa}(x-\xi)d\tau\Big\}+O(\kappa^4).
\end{aligned}
\tag{5.9}
$$

If we then make use of the relationship of s^{kin}_μ and s^h_μ to $\mathbf{s}^\mu$, given by (5.10) of paper I, and the relationship of $t^{\text{kin}}_{\mu\nu}$ and $t^h_{\mu\nu}$ to $\mathbf{T}^{\mu\nu}$, given by (5.11) of the same paper, we see from (5.8) and (5.9) that

$$s_\mu^{\rm kin}+s_\mu^h=(4\pi/l)\left[\int[qu_\mu]\delta(x-\xi)d\tau-\int[ql^2u_\mu]\Box^2\delta(x-\xi)d\tau\right]+O(\kappa^3)\ , \tag{5.10}$$

$$\begin{aligned} t_{\mu\nu}^{\rm kin}+t_{\mu\nu}^h= {} & 16\pi\left[\int[mu_\mu u_\nu]\delta(x-\xi)d\tau\right] \\ & +(4\pi/l)\left\{\int[ql^2(\gamma^*_{[\sigma\nu],\mu}u^\sigma+\gamma^*_{[\sigma\mu],\nu}u^\sigma)]\delta(x-\xi)d\tau\right. \\ & \left.+\int[ql^2(\eta_{\mu\kappa}\gamma^*_{[\nu\rho]}u^\rho+\eta_{\nu\kappa}\gamma^*_{[\mu\rho]}u^\rho-\eta_{\mu\nu}\gamma^*_{[\kappa\rho]}u^\rho+\gamma^*_{[\kappa\mu]}u_\nu+\gamma^*_{[\kappa\nu]}u_\mu)]\delta^{,\kappa}(x-\xi)d\tau\right\}+O(\kappa^4)\ . \end{aligned} \tag{5.11}$$

Finally, making use of the expressions for $s_\mu^{\rm kin}$ and $t_{\mu\nu}^{\rm kin}$ given by (2.11) and (2.12), we find from (5.10) and (5.11) that[23]

$$s_\mu^h=O(\kappa^3)\ , \tag{5.12}$$

$$\begin{aligned} t_{\mu\nu}^h=(4\pi/l) & \left\{\int[ql^2(\gamma^*_{[\sigma\nu],\mu}u^\sigma+\gamma^*_{[\sigma\mu],\nu}u^\sigma)]\delta(x-\xi)d\tau\right. \\ & \left.+\int[ql^2(\eta_{\mu\kappa}\gamma^*_{[\nu\rho]}u^\rho+\eta_{\nu\kappa}\gamma^*_{[\mu\rho]}u^\rho-\eta_{\mu\nu}\gamma^*_{[\kappa\rho]}u^\rho+\gamma^*_{[\kappa\mu]}u_\nu+\gamma^*_{[\kappa\nu]}u_\mu)]\delta^{,\kappa}(x-\xi)d\tau\right\}+O(\kappa^4)\ . \end{aligned} \tag{5.13}$$

The expressions (5.12) and (5.13) for s_μ^h and $t_{\mu\nu}^h$, respectively, are identical to those obtained earlier in paper RVII of Ref. 6 using the author's approximation procedure.

APPENDIX A: $\tilde{s}^{\mu\kappa\lambda}$

We shall show that there is no loss in generality in choosing the oriented tensor $\tilde{s}^{\mu\kappa\lambda}$, appearing in (3.2), to be of the form

$$\tilde{s}^{\mu\kappa\lambda}=\tilde{s}^{(\mu\kappa)\lambda}+\tilde{s}^{[\mu\kappa]\lambda}\ , \tag{A1}$$

where

$$\tilde{s}^{(\mu\kappa)\lambda}+\tilde{s}^{(\kappa\lambda)\mu}+\tilde{s}^{(\lambda\mu)\kappa}=0\ . \tag{A2}$$

However, we shall first show that the requirement

$$\int C^{(\rho\kappa\lambda)}(s)\delta_{,\rho\kappa\lambda}(x-\xi)ds+\int C^{(\kappa\lambda)}(s)\delta_{,\kappa\lambda}(x-\xi)ds+\int C^\lambda(s)\delta_{,\lambda}(x-\xi)ds+\int C(s)\delta(x-\xi)ds=0\ , \tag{A3}$$

supplemented by the conditions

$$C^{(\rho\kappa\lambda)}U_\lambda=0\ , \tag{A4}$$

implies

$$C^{(\rho\kappa\lambda)}=0\ . \tag{A5}$$

If we evaluate the integrals appearing in (A3), we find (A3) takes the form[24]

$$\bar{C}^{(rkl)}\delta_{,rkl}(\vec{x}-\vec{\xi})+\bar{C}^{(kl)}\delta_{,kl}(\vec{x}-\vec{\xi})+\bar{C}^l\delta_{,l}(\vec{x}-\vec{\xi})+\bar{C}\delta(\vec{x}-\vec{\xi})=0\ , \tag{A6}$$

where

$$\begin{aligned} \bar{C}^{(rkl)}=\frac{1}{U^4}[& C^{(rkl)}-\beta^rC^{(4kl)}-\beta^kC^{(4lr)}-\beta^lC^{(4rk)} \\ & +\beta^r\beta^kC^{(44l)}+\beta^k\beta^lC^{(44r)} \\ & +\beta^l\beta^rC^{(44k)}-\beta^r\beta^k\beta^lC^{(444)}]\ ,\quad \beta^r=U^r/U^4\ . \end{aligned} \tag{A7}$$

The explicit form of $\bar{C}^{(kl)}$, $\bar{C}^l$, and $\bar{C}$ will not be needed in what follows. Condition (A3) requires

$$\bar{C}^{(rkl)}=0\ . \tag{A8}$$

From (A4), (A7), and (A8) one finds

$$C^{(\rho\kappa\lambda)}=0\ . \tag{A9}$$

Thus, the requirement (A3) supplemented by the conditions (A4) implies (A5).

Now we are ready to investigate $\tilde{s}^{\mu\kappa\lambda}$. We can always write $\tilde{s}^{\mu\kappa\lambda}$ in the form

$$\tilde{s}^{\mu\kappa\lambda}=\tilde{s}'^{(\mu\kappa)\lambda}+\tilde{s}'^{[\mu\kappa]\lambda}\ . \tag{A10}$$

Using arguments similar to those applied to $\tilde{T}^{(\mu\nu)\kappa}$ in Appendix B of paper I (see Ref. 1), one finds that with no loss in generality one can always write $\tilde{s}'^{(\mu\kappa)\lambda}$ in the form

$$\tilde{s}'^{(\mu\kappa)\lambda}={}^*\tilde{s}^{(\mu\kappa)\lambda}+\tfrac{1}{2}A^{\mu\lambda}U^\kappa+\tfrac{1}{2}A^{\kappa\lambda}U^\mu\ , \tag{A11}$$

where

$${}^*\tilde{s}^{(\mu\kappa)\lambda}U_\kappa=0,\quad {}^*\tilde{s}^{(\mu\kappa)\lambda}U_\lambda=0\ . \tag{A12}$$

The quantities ${}^*\tilde{s}^{(\mu\kappa)\lambda}$ and $A^{\mu\lambda}$ appearing in (A11) are oriented tensors. The above results mean that with no loss in generality we can always write $\tilde{s}^{\mu\kappa\lambda}$ in the form

$$\tilde{s}^{\mu\kappa\lambda}={}^*\tilde{s}^{(\mu\kappa)\lambda}+{}^*\tilde{s}^{[\mu\kappa]\lambda}+\Delta\tilde{s}^{\mu\kappa\lambda}\ , \tag{A13}$$

where

$${}^*\tilde{s}^{[\mu\kappa]\lambda}=\tilde{s}'^{[\mu\kappa]\lambda}+\tfrac{1}{2}A^{\kappa\lambda}U^\mu-\tfrac{1}{2}A^{\mu\lambda}U^\kappa\ , \tag{A14}$$

$$\Delta\tilde{s}^{\mu\kappa\lambda}=A^{\mu\lambda}U^\kappa\ . \tag{A15}$$

Since one can show that

$$\int [\Delta\tilde{s}^{\mu\kappa\lambda}]\delta_{,\kappa\lambda}(x-\xi)ds+\int \left[\Delta\tilde{s}^{\mu\rho\sigma}\left\{\begin{matrix}\kappa\\ \rho\sigma\end{matrix}\right\}+\Delta\tilde{s}^{\rho\kappa\sigma}\left\{\begin{matrix}\mu\\ \rho\sigma\end{matrix}\right\}+\Delta\tilde{s}^{\rho\sigma\kappa}\left\{\begin{matrix}\mu\\ \rho\sigma\end{matrix}\right\}\right]\delta_{,\kappa}(x-\xi)ds$$

$$+\int \left[-\tfrac{1}{2}\Delta\tilde{s}^{\rho\kappa\sigma}\left\{\begin{matrix}\mu\\ \rho\kappa\end{matrix}\right\}_{,\sigma}-\tfrac{1}{2}\Delta\tilde{s}^{\rho\kappa\sigma}\left\{\begin{matrix}\mu\\ \rho\sigma\end{matrix}\right\}_{,\kappa}+\Delta\tilde{s}^{\rho\kappa\sigma}\left\{\begin{matrix}\lambda\\ \sigma\kappa\end{matrix}\right\}\left\{\begin{matrix}\mu\\ \rho\lambda\end{matrix}\right\}+\tfrac{1}{2}\Delta\tilde{s}^{\rho\kappa\sigma}\left\{\begin{matrix}\lambda\\ \rho\kappa\end{matrix}\right\}\left\{\begin{matrix}\mu\\ \sigma\lambda\end{matrix}\right\}\right.$$

$$\left.+\tfrac{1}{2}\Delta\tilde{s}^{\rho\kappa\sigma}\left\{\begin{matrix}\lambda\\ \rho\sigma\end{matrix}\right\}\left\{\begin{matrix}\mu\\ \kappa\lambda\end{matrix}\right\}+\tfrac{1}{2}\Delta\tilde{s}^{\rho\kappa\sigma}R^{*\mu}{}_{\rho\kappa\sigma}\right]\delta(x-\xi)ds=\int [\Delta\tilde{s}^{\mu\kappa}]\delta_{,\kappa}(x-\xi)ds+\int \left[\Delta\tilde{s}^{\rho\sigma}\left\{\begin{matrix}\mu\\ \rho\sigma\end{matrix}\right\}\right]\delta(x-\xi)ds\ , \tag{A16}$$

where

$$\Delta\tilde{s}^{\mu\kappa\lambda}=A^{\mu\lambda}U^{\kappa}\ ,\qquad \Delta\tilde{s}^{\mu\kappa}=A^{\mu\kappa}{}_{;\rho}U^{\rho}\ , \tag{A17}$$

we see that the term $\Delta\tilde{s}^{\mu\kappa\lambda}$ in (A13) is equivalent to a term $\Delta\tilde{s}^{\mu\kappa}$ in the oriented tensor $\tilde{s}^{\mu\kappa}$ appearing in (3.2). This follows from the form of the right-hand side of (3.2). Since the tensor $\tilde{s}^{\mu\kappa}$ in (3.2) is arbitrary at this stage of the analysis, there is no loss in generality in choosing

$$\Delta\tilde{s}^{\mu\kappa\lambda}=0\ , \tag{A18}$$

and thus $\tilde{s}^{\mu\kappa\lambda}$ can with no loss in generality always be written in the form

$$\tilde{s}^{\mu\kappa\lambda}={}^{*}\tilde{s}^{(\mu\kappa)\lambda}+{}^{*}\tilde{s}^{[\mu\kappa]\lambda}\ , \tag{A19}$$

where ${}^{*}\tilde{s}^{(\mu\kappa)\lambda}$ satisfies (A12).

We must also satisfy the equations

$$s^{\mu}{}_{;\mu}=0\ . \tag{A20}$$

We see from (3.4) and (A20) that the requirement (A3) discussed earlier in this Appendix will then be satisfied, where in this case

$$C^{(\rho\kappa\lambda)}=\tfrac{1}{3}({}^{*}\tilde{s}^{(\mu\kappa)\lambda}+{}^{*}\tilde{s}^{(\kappa\lambda)\mu}+{}^{*}\tilde{s}^{(\lambda\mu)\kappa})\ . \tag{A21}$$

It also follows from (A12) and (A21) that

$$C^{(\rho\kappa\lambda)}U_{\lambda}=0\ . \tag{A22}$$

Thus condition (A4) is satisfied. This means

$${}^{*}\tilde{s}^{(\mu\kappa)\lambda}+{}^{*}\tilde{s}^{(\kappa\lambda)\mu}+{}^{*}\tilde{s}^{(\lambda\mu)\kappa}=0\ . \tag{A23}$$

We see that there is no loss in generality in choosing the oriented tensor $\tilde{s}^{\mu\kappa\lambda}$, which appears in (3.2), to be of the form (A1), where (A2) is satisfied.

APPENDIX B: $\tilde{s}^{\mu\kappa}$

We shall show that there is no loss in generality in choosing the oriented tensor $\tilde{s}^{\mu\kappa}$, appearing in (3.2), to be of the form

$$\tilde{s}^{\mu\kappa}=\tilde{s}^{[\mu\kappa]}\ . \tag{B1}$$

First, we write $\tilde{s}^{\mu\kappa}$ in the form

$$\tilde{s}^{\mu\kappa}=\tilde{s}'^{(\mu\kappa)}+\tilde{s}'^{[\mu\kappa]}\ . \tag{B2}$$

Using arguments similar to those applied to $Y^{(\mu\nu)}$ in Appendix C of paper I (see Ref. 1), we see that with no loss in generality one can always write $\tilde{s}'^{(\mu\kappa)}$ in the form

$$\tilde{s}'^{(\mu\kappa)}={}^{*}\tilde{s}^{(\mu\kappa)}+{}^{*}B^{\mu}U^{\kappa}+{}^{*}B^{\kappa}U^{\mu}+BU^{\mu}U^{\kappa}\ , \tag{B3}$$

where ${}^{*}\tilde{s}^{(\mu\kappa)}$, ${}^{*}B^{\mu}$, and B are oriented tensors, and

$${}^{*}\tilde{s}^{(\mu\kappa)}U_{\kappa}=0\ , \tag{B4}$$

$${}^{*}B^{\kappa}U_{\kappa}=0\ . \tag{B5}$$

This means we can always write $\tilde{s}^{\mu\kappa}$ in the form

$$\tilde{s}^{\mu\kappa}={}^{*}\tilde{s}^{(\mu\kappa)}+{}^{*}\tilde{s}^{[\mu\kappa]}+\Delta\tilde{s}^{\mu\kappa}\ , \tag{B6}$$

where

$${}^{*}\tilde{s}^{[\mu\kappa]}=\tilde{s}'^{[\mu\kappa]}+{}^{*}B^{\kappa}U^{\mu}-{}^{*}B^{\mu}U^{\kappa}\ , \tag{B7}$$

$$\Delta\tilde{s}^{\mu\kappa}=(2{}^{*}B^{\mu}+BU^{\mu})U^{\kappa}\ . \tag{B8}$$

Since one can show

$$\int [\Delta\tilde{s}^{\mu\kappa}]\delta_{,\kappa}(x-\xi)ds+\int \left[\Delta\tilde{s}^{\rho\sigma}\left\{\begin{matrix}\mu\\ \rho\sigma\end{matrix}\right\}\right]\delta(x-\xi)ds=\int \Delta\tilde{s}^{\mu}\delta(x-\xi)ds\ , \tag{B9}$$

where

$$\Delta\tilde{s}^{\mu\kappa}=(2{}^{*}B^{\mu}+BU^{\mu})U^{\kappa}\ ,\qquad \Delta\tilde{s}^{\mu}=(2{}^{*}B^{\mu}+BU^{\mu})_{;\rho}U^{\rho}\ , \tag{B10}$$

we see that the term $\Delta\tilde{s}^{\mu\kappa}$ in (B6) is equivalent to a term $\Delta\tilde{s}^{\mu}$ in the oriented vector $\tilde{s}^{\mu}$ appearing in (3.2). This follows from the form of the right-hand side of (3.2). Since the oriented vector $\tilde{s}^{\mu}$ in (3.2) is arbitrary at this stage of the analysis, there is no loss in generality in choosing

$$\Delta\tilde{s}^{\mu\kappa}=0\ , \tag{B11}$$

and thus with no loss in generality $\tilde{s}^{\mu\kappa}$ can always be written in the form

$$\tilde{s}^{\mu\kappa}={}^{*}\tilde{s}^{(\mu\kappa)}+{}^{*}\tilde{s}^{[\mu\kappa]}\ , \tag{B12}$$

where ${}^{*}\tilde{s}^{(\mu\kappa)}$ satisfies (B4).

Since the equations

$$s^{\mu}{}_{;\mu}=0 \tag{B13}$$

must also be satisfied, we see from (3.8) and (B13) that conditions (D1) and (D2) of Appendix D of paper I are satisfied, where in this case

$$C^{(\mu\kappa)}={}^{*}\tilde{s}^{(\mu\kappa)}\ . \tag{B14}$$

Making use of the results of Appendix D of paper I, we see this implies

$$ {}^*\tilde{s}^{(\mu\kappa)}=0 \; . \tag{B15}$$

Thus, there is no loss in generality in choosing $\tilde{s}^{\mu\kappa}$ to be of the form

$$\tilde{s}^{\mu\kappa}=\tilde{s}^{[\mu\kappa]} \; . \tag{B16}$$

APPENDIX C: $\tilde{s}^{\mu}$

We shall show that there is no loss in generality in choosing the oriented vector $\tilde{s}^{\mu}$, which appears in (3.2), to be of the form (3.12).

From (3.11), since

$$s^{\mu}{}_{;\mu}=0 \; , \tag{C1}$$

we see that

$$\int \tilde{s}'^{\mu}\delta_{,\mu}(x-\xi)ds=0 \; , \tag{C2}$$

where

$$\tilde{s}'^{\mu}=\tilde{s}^{\mu}+\tilde{s}^{\rho\kappa\sigma}R^{*\mu}{}_{[\rho\kappa]\sigma}+\tfrac{1}{3}\tilde{s}^{\rho\kappa\sigma}R^{*\mu}{}_{(\rho\kappa)\sigma} \; . \tag{C3}$$

We can always write $\tilde{s}'^{\mu}$ in the form

$$\tilde{s}'^{\mu}={}^*\tilde{s}^{\mu}+eU^{\mu} \; , \tag{C4}$$

where

$${}^*\tilde{s}^{\mu}=\tilde{s}'^{\mu}-\tilde{s}'^{\rho}U_{\rho}U^{\mu} \; , \tag{C5}$$

$$e=\tilde{s}'^{\rho}U_{\rho} \; , \tag{C6}$$

and

$$U^{\mu}=\frac{d\xi^{\mu}}{ds}, \quad U_{\mu}=a_{\mu\rho}U^{\rho} \; . \tag{C7}$$

Note

$${}^*\tilde{s}^{\mu}U_{\mu}=0 \; . \tag{C8}$$

Thus (C2) takes the form

$$\int {}^*\tilde{s}^{\mu}\delta_{,\mu}(x-\xi)ds+\int eU^{\mu}\delta_{,\mu}(x-\xi)ds=0 \; , \tag{C9}$$

so that making use of the identity

$$\int eU^{\mu}\delta_{,\mu}(x-\xi)ds=\int \frac{de}{ds}\delta(x-\xi)ds \; , \tag{C10}$$

we can write (C9) as

$$\int C^{\mu}\delta_{,\mu}(x-\xi)ds+\int C\delta(x-\xi)ds=0 \; , \tag{C11}$$

where

$$C^{\mu}={}^*\tilde{s}^{\mu}, \quad C=\frac{de}{ds} \; , \tag{C12}$$

and

$$C^{\mu}U_{\mu}=0 \; . \tag{C13}$$

From the investigation in Appendix D in paper I (see Ref. 1), we find that the above implies

$${}^*\tilde{s}^{\mu}=0 \; . \tag{C14}$$

From (C3), (C4), and (C14) we have

$$\tilde{s}^{\mu}=eU^{\mu}-\tilde{s}^{\rho\kappa\sigma}R^{*\mu}{}_{[\rho\kappa]\sigma}-\tfrac{1}{3}\tilde{s}^{\rho\kappa\sigma}R^{*\mu}{}_{(\rho\kappa)\sigma} \; . \tag{C15}$$

Thus, there is no loss in generality in choosing $\tilde{s}^{\mu}$ to be of the form (3.12).

APPENDIX D: $\tilde{T}^{(\mu\nu)\kappa}$

We shall show that there is no loss in generality in choosing the tensor $\tilde{T}^{(\mu\nu)\kappa}$, which appears in (3.17), to be of the form (3.22).

From (3.20), since (2.21) must be satisfied, we have

$$\int C^{\mu\kappa\nu}\delta_{,\kappa\nu}(x-\xi)ds+\int C^{\mu\nu}\delta_{,\nu}(x-\xi)ds + \int C^{\mu}\delta(x-\xi)ds=0 \; , \tag{D1}$$

where $C^{\mu\kappa\nu}$ can be written in the form

$$\begin{aligned} C^{\mu\kappa\nu}= {} & \tilde{T}^{(\mu\nu)\kappa}+\tfrac{1}{2}\gamma^{*[\mu}{}_{\rho]}\tilde{s}^{\rho\nu\kappa}+\tfrac{1}{2}\gamma^{*[\nu}{}_{\rho]}\tilde{s}^{\rho\mu\kappa} \\ & +\tfrac{1}{2}\gamma^{*[\mu}{}_{\rho]}\tilde{s}^{\rho\kappa\nu}+\tfrac{1}{2}\gamma^{*[\nu}{}_{\rho]}\tilde{s}^{\rho\kappa\mu} \\ & -\tfrac{1}{2}\gamma^{*[\kappa}{}_{\rho]}\tilde{s}^{\rho\nu\mu}-\tfrac{1}{2}\gamma^{*[\kappa}{}_{\rho]}\tilde{s}^{\rho\mu\nu} \\ \equiv {} & \tilde{T}'^{(\mu\nu)\kappa} \; . \end{aligned} \tag{D2}$$

Using arguments similar to those applied to $\tilde{T}^{(\mu\nu)\kappa}$ in Appendix B of paper I, but here applied to $\tilde{T}'^{(\mu\nu)\kappa}$, we find that one can with no loss in generality always write $\tilde{T}'^{(\mu\nu)\kappa}$ in the form

$$\tilde{T}'^{(\mu\nu)\kappa}=\tfrac{1}{2}S^{\mu\kappa}U^{\nu}+\tfrac{1}{2}S^{\nu\kappa}U^{\mu} \; , \tag{D3}$$

where $S^{\mu\nu}$ is an antisymmetric second-rank tensor characterizing the particle and

$$U^{\mu}=\frac{d\xi^{\mu}}{ds} \; . \tag{D4}$$

From (D2) and (D3) we then find

$$\begin{aligned} \tilde{T}^{(\mu\nu)\kappa}= {} & \tfrac{1}{2}S^{\mu\kappa}U^{\nu}+\tfrac{1}{2}S^{\nu\kappa}U^{\mu}-\tfrac{1}{2}\gamma^{*[\mu}{}_{\rho]}\tilde{s}^{\rho\nu\kappa} \\ & -\tfrac{1}{2}\gamma^{*[\nu}{}_{\rho]}\tilde{s}^{\rho\mu\kappa}-\tfrac{1}{2}\gamma^{*[\mu}{}_{\rho]}\tilde{s}^{\rho\kappa\nu}-\tfrac{1}{2}\gamma^{*[\nu}{}_{\rho]}\tilde{s}^{\rho\kappa\mu} \\ & +\tfrac{1}{2}\gamma^{*[\kappa}{}_{\rho]}\tilde{s}^{\rho\nu\mu}+\tfrac{1}{2}\gamma^{*[\kappa}{}_{\rho]}\tilde{s}^{\rho\mu\nu} \; . \end{aligned} \tag{D5}$$

We see that there is no loss in generality in choosing the tensor $\tilde{T}^{(\mu\nu)\kappa}$, which appears in (3.17), to be of the form (3.22).

[1]C. R. Johnson, preceding paper, Phys. Rev. D **30**, 1236 (1984). This paper will be referred to as paper I.

[2]For a definition of e^M see Sec. V of Ref. 1. Particles possessing a magnetic monopole moment have been studied in Ref. 3.

[3]C. R. Johnson, Phys. Rev. D **24**, 327 (1981).

[4]Such particles are discussed in Ref. 3.

[5]C. R. Johnson and J. R. Nance, Phys. Rev. D **15**, 377 (1977); **16**, 533(E) (1977).

[6]C. R. Johnson, Phys. Rev. D **4**, 295 (1971); **4**, 318 (1971); **4**, 3555 (1971); **5**, 282 (1972); **5**, 1916 (1972); **7**, 2825 (1973); **7**, 2838 (1973); **8**, 1645 (1973). We shall refer to these papers as papers RI—RVIII, respectively.

[7]The fields $\gamma^*_{[\mu\nu]}$ and $\gamma_{(\mu\nu)}$ are defined in Ref. 1. The notation used in this paper will be the same as that used in Ref. 1. A harmonic coordinate system is defined in paper RI of Ref. 6.

[8]The universal length l is discussed in Ref. 5.

[9]The physical meaning of q is discussed in Ref. 5.

[10]The physical meaning of m is discussed in Ref. 5.

[11]The quantity $\delta(x-\xi)$ represents the four-dimensional Dirac delta function. The indices on both x^μ and ξ^μ have been suppressed.

[12]The constant c represents the speed of light.

[13]We are using the notation of Ref. 1.

[14]See Appendix A.

[15]See Appendix B.

[16]See Appendix C.

[17]See Appendix D.

[18]See Appendix C of Ref. 1.

[19]In Appendix D of Ref. 1 we show that if

$$\int C^\nu(s)\delta_{,\nu}(x-\xi)ds+\int C(s)\delta(x-\xi)ds=0 ,$$

where $C^\nu U_\nu=0$, then $C^\nu=0$, $C=0$.

[20]The equations of motion of a neutral pole test particle possessing no electromagnetic multipole moments are given in Ref. 1.

[21]This power-series expansion has been used in Refs. 1, 3, and 5. It is also used and discussed in the papers of Ref. 6.

[22]See, for example, Eqs. (41) of Ref. 5. One chooses $\epsilon=1$ in Eqs. (41) of Ref. 5 if the equations are to apply to charged particles. The radiation reaction terms found in Eqs. (41) of Ref. 5 vanish in the test-particle limit and are thus not present in Eqs. (5.7).

[23]Since we are treating the spin as negligible, we neglect the spin-dependent terms which are in (2.11) and (2.12).

[24]The quantity $\delta(\vec{x}-\vec{\xi})$ represents the three-dimensional Dirac delta function.