Test-particle motion in Einstein's unified field theory. I. General theory and application to neutral test particles

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(Received 8 September 1983)

We develop a method for finding the exact equations of structure and motion of multipole test particles in Einstein's unified field theory—the theory of the nonsymmetric field. The method is also applicable to Einstein's gravitational theory. Particles are represented by singularities in the field. The method is covariant at each step of the analysis. We also apply the method and find both in Einstein's unified field theory and in Einstein's gravitational theory the equations of structure and motion of neutral pole-dipole test particles possessing no electromagnetic multipole moments. In the case of Einstein's gravitational theory the results are the well-known equations of structure and motion of a neutral pole-dipole test particle in a given background gravitational field. In the case of Einstein's unified field theory the results are the same, providing we identify a certain symmetric second-rank tensor field appearing in Einstein's theory with the metric and gravitational field. We therefore discover not only the equations of structure and motion of a neutral test particle in Einstein's unified field theory, but we also discover what field in Einstein's theory plays the role of metric and gravitational field.

I. INTRODUCTION

In this paper we develop a method for finding the exact equations of structure and motion of multipole test particles in Einstein's unified field theory—the theory of the nonsymmetric field.[1] The method is also applicable to Einstein's gravitational theory, as that theory is a special case of the unified field theory. The method is covariant at each step of the analysis. In this paper, in addition to developing the method, we shall apply the method and find the equations of structure and motion of neutral pole-dipole test particles in both Einstein's unified field theory and Einstein's gravititational theory. In finding the equations of structure and motion of a neutral test particle in Einstein's unified field theory, we discover not only the equations of structure and motion of the particle, but we shall also discover what field in the theory is the natural choice for the metric and gravitational field.

However, our reason for developing the method goes beyond the above simple applications. We wish to investigate in Einstein's unified field theory the interaction of particles over microsopic distances (atomic and molecular distances) where the fields involved are relatively strong and where the approximation procedure which has been used previously to investigate the interaction of particles over macroscopic distances is difficult to apply.[2,3] An investigation of the motion of a charged test particle in a relatively strong external field can be regarded as a first step in such an investigation. In papers II and III of this series of papers we shall find the exact equations of structure and motion of charged test particles in Einstein's theory. The important role a test particle can play in the study of the interaction of particles over microscopic distances in Einstein's unified field theory will become clear through the following discussion.

In several earlier papers the author developed an approximation method for finding the Lorentz-covariant equations of structure and motion of particles in general-relativistic field theories.[4] In the earlier papers the approximation method was applied both in Einstein's unified field theory—the theory of the nonsymmetric field—and in Einstein's gravitational theory. The approximation method will allow one to find the equations of structure and motion of particles step by step with respect to the powers of a parameter κ which measures the strength of the field associated with the particles. In the approximation method the procedure for finding the field associated with the particles can only be expected to be valid at points which are sufficiently far from each particle, so that one finds meaningful equations of structure and motion only for particles which are not too near one another. The order of magnitude of these distances have been discussed in the earlier papers. When dealing with Einstein's unified field theory and with particles having a microscopic mass and a charge whose magnitude is of the order of that of the electron charge, the interaction distances at which the approximation method breaks down have been estimated to be of the order of the electron Compton wavelength.[2]

Using the author's approximation method, one does not introduce phenomenological source terms into the general-relativistic field equations one is studying. Particles are represented by regions of space-time in which the field is very strong (i.e., the regions are very nonflat[5]). For example, in Einstein's gravitational theory an isolated particle might be represented through the Schwarzschild solution[6] (we shall call such a particle a Schwarzschild particle), or through the Kerr solution[7] (a Kerr particle), or perhaps through some other solution to the field equations. In Einstein's unified field theory an isolated particle might be represented through the Wyman solution[8] (a Wyman particle), or through the Bandyopadhyay-

Vanstone solution[9] (a Bandyopadhyay-Vanstone particle), or perhaps through some other solution to the field equations.

If sufficiently far from each particle the field one is studying can be expanded in a power series in the parameter κ, then the author's method for finding the equations of structure and motion of the particles is expected to be applicable as long as the particles are sufficiently far from each other. This is true whether the particles are represented through singular or nonsingular solutions to the field equations. However, at each order of approximation the approximate solution obtained using the author's approximation method when analytically continued into the strong-field region associated with a particle will generally no longer approximate the exact solution to the field equations and will become singular along a world line associated with the particle. This world line can be considered to define the position of the particle. We shall, in fact, adopt this definition of position. The exact solution of the field equations may or may not be singular along this world line.

Using the author's approximation method, one finds that particles in both Einstein's unified field theory and Einstein's gravitational theory are characterized by a series of multipole moments, and that the particles interact with each other through what can be interpreted as forces and torques. The multipole moments associated with a particle can be related to the particle's mass, charge, magnetic monopole moment, spin, and higher mass and electromagnetic moments. Both the force and torque acting on a particle, in addition to depending on the particle's kinematic properties, are found to depend on the particle's multipole moments and an external field in the vicinity of the particle. This external field can be regarded as produced by the other particles which interact with the particle under consideration.

If one is willing to neglect the self-interaction terms in the force and torque acting on a particle, (i.e., those terms in the force and torque which are nonlinear in the multipole moments which characterize the particle), then the equations of structure and motion of the particle can also be found to any order of approximation desired through the use of certain conservation laws which follow from the author's approximation method. This is discussed in the author's papers.[10] The conservation laws do not determine all the terms in the equations of structure and motion of a particle, but do determine those which survive in the test-particle limit, that is, in the limit where one neglects self-interaction and the effects of the particle on the external field in its vicinity.

Through the use of the conservation laws one finds that one can obtain the equations of structure and motion of test particles step by step to any order of approximation desired through only algebraic manipulation and differentiation. No partial differential equations need be solved. This suggests that one might also be able to obtain the exact equations of structure and motion of test particles through only algebraic manipulation and differentiation. In this paper we shall show that this is so providing the test particles are described through a finite number of multipole moments. We call such test particles multipole test particles.

In this paper we derive a general method for finding the exact equations of structure and motion of multipole test particles in Einstein's unified field theory and in Einstein's gravitational theory. The method we derive is closely related to a method developed earlier by Tulczyjew[11] for finding the equations of structure and motion of multipole test particles in Einstein's gravitational theory. Tulczyjew's method is based on previous work by Mathisson.[12]

The principle advantage of our method over that of Tulczyjew is that we derive it from the field equations of the general-relativistic theories in which we are interested without having to introduce phenomenological source terms into the fundamental field equations of the theories. Particles in the theories are represented by regions of strong field. This distinction between the two methods is important when dealing with Einstein's unified field theory as Einstein regarded his unified field theory as a complete theory of nature and thus in a fundamental investigation of the theory phenomenological source terms should be avoided. This distinction is also important when dealing with Einstein's gravitational theory if one regards the gravitational theory as a special limiting case of Einstein's unified field theory.

Although we derive our method for finding the equations of structure and motion of multipole test particles from the general-relativistic field theories in which we are interested without introducing phenomenological source terms into the fundamental field equations of the theories, we do introduce and use, in finding the equations of structure and motion of multipole test particles, an auxiliary set of field equations involving source terms. However, these auxiliary field equations, which are useful for finding the equations of structure and motion of multipole test particles, do not replace the fundamental field equations of the theories. What we show is that through the proper use of the auxiliary field equations one obtains the same equations of structure and motion for multipole test particles in an external field as one would obtain from the fundamental field equations of the theories under consideration, under the condition that the particles involved are interacting over distances sufficiently great so that the approximation method developed by the author and described earlier in this introduction is valid. The equations of structure and motion obtained from the auxiliary field equations are not expected to be valid under more general conditions. Thus, if one uses the author's method for finding the equations of structure and motion of multipole test particles in Einstein's unified field theory and in Einstein's gravitational theory, one can see under what conditions the test-particle concept is physically meaningful in these theories, and one can also relate the parameters characterizing a multipole test particle to the fields associated with the particle when the test-particle limit is not taken.

The method described in this paper when applied to Einstein's gravitational theory gives the same results as a previous method for finding the equations of structure and motion of multipole test particles developed by Papapetrou.[13] With respect to our needs, however, the Papapetrou method has two disadvantages which the method described in this paper does not possess. If one uses the Papapetrou method one must introduce phenomenological source terms into the fundamental field equations of the theories one is investigating, and the Papapetrou method is noncovariant.

The method described in this paper for finding the equations of structure and motion of multipole test particles is easy to apply. In this paper, paper I of a series of papers, we use the method to find the equations of structure and motion of neutral pole-dipole test particles[14] in both Einstein's unified field theory and Einstein's gravitational theory. This is the first time this has been done in Einstein's unified field theory. The results in the case of Einstein's gravitational theory are of course the same equations of structure and motion as previously obtained by Papapetrou[13] and by Tulczyjew.[11] In papers II and III of this series of papers, we shall use the method to find the equations of structure and motion of a charged test particle in Einstein's unified field theory. In paper II we find the equations of structure and motion of a charged test particle possessing no magnetic monopole moment, and in paper III we find the equations of structure and motion of a charged test particle possessing both an electric charge and a magnetic monopole moment.

Why are we interested in the motion of test particles in Einstein's unified field theory? First, as mentioned earlier, until now there has been no satisfactory way for deciding what field in the theory is the natural choice for the physical metric and gravitational field. The natural choice for the electromagnetic field is known.[2] In investigating the motion of a neutral test particle in Einstein's unified field theory, we find that there is a symmetric second-rank tensor field with respect to which a neutral pole test particle posssessing no electromagnetic multipole moments moves along a geodesic. This field is therefore a natural choice for the physical metric and gravitational field in Einstein's unified field theory. In the case where the antisymmetric part of the fundamental field in Einstein's unified field theory vanishes, and thus Einstein's unified field theory reduces to Einstein's gravitational theory, this symmetric second-rank tensor field reduces to the metric of the gravitational theory. The choice is therefore consistent with the choice of metric in Einstein's gravitational theory. With this choice of a metric one can say that in both Einstein's unified field theory and Einstein's gravitational theory a neutral pole test particle possessing no electromagnetic multipole moments travels along a geodesic of the metric of the background field.

However, the principle reason that we are interested in the equations of motion of test particles in Einstein's unified field theory is that a study of the motion of test particles may give us insight into the interaction among particles over microscopic distances in Einstein's theory. Using the approximation method developed in earlier papers by the author one can easily investigate the interaction of particles over macroscopic distances (laboratory and astronomical distances) since, in this case, one needs only keep terms of lowest nontrivial order in the approximation method. Higher-order terms can be neglected. One finds that charged particles in Einstein's theory interact over laboratory and moderate astronomical distances through the conventional classical electromagnetic interaction. This has been discussed in the literature.[2] However, over atomic and molecular distances we do not yet know the form taken by the interaction between charged particles in Einstein's theory. Although the approximation method used to investigate interactions over macroscopic distances is still expected to be valid over atomic and molecular distances,[15] higher-order terms in the interaction among the particles are not expected to be negligible over such distances and thus the approximation method is very difficult to apply. Many higher-order terms must be evaluated and studied. However, if one is willing to neglect the self-interaction acting on a particle and the effect of the particle on the external field in its vicinity, then one can obtain the exact equations of motion satisfied by the interacting particle. One needs only find the equations of motion in the test-particle limit. A knowledge of the motion of a test particle in an external field can thus be regarded as a first step in the investigation of the interaction of particles over microscopic distances.

In seeking the equations of motion of a particle in the test-particle limit, we are neglecting the self-interaction terms acting on the particle and the effect of the particle on the other particles with which it interacts. We are also assuming the particle is sufficiently far from those other particles so that the test-particle limit is physically meaningful,[16] and since we are restricting our study to multipole test particles we are also assuming the particle can be characterized with sufficient accuracy by a finite number of multipole moments. It is because of these idealizations and assumptions that finding the equations of motion in the test-particle limit can only be regarded as a first step in the investigation of the interaction of particles over microscopic distances. The validity of the above idealizations and assumptions when applied to particles interacting over microscopic distances will be studied in future papers.

We must next ask the question, can one understand the interaction of particles over atomic and molecular distances by means of Einstein's theory without transforming the theory into a quantum field theory, that is without "quantizing" the theory. Einstein believed that this must be so if the theory is valid. Einstein believed that if the interaction among particles as derived from the theory was treated statistically, then the theory, if correct, should give results that are in a first approximation identical to those obtained from conventional quantum theory. In other words, he believed one should be able to derive in a first approximation the statistical results of conventional quantum theory from the unified field theory. Finding the equations of motion of a test particle in a given background field can be regarded as a first step in an effort to test Einstein's hope. Only after obtaining the interaction among particles over microscopic interaction distances can we treat the interaction statistically and see if in a first approximation the results of conventional quantum theory follow from Einstein's theory. If this turns out to be so, Einstein's ideas will have proved fruitful and his years of labor justified.

The organization of the paper is as follows. Section I of the paper introduces the subject. In Sec. II we describe the space-time manifold in Einstein's unified field theory and also introduce the concepts of particle and physical field in the theory. In Sec. III we discuss the contracted "Bianchi" identities. In Sec. IV we introduce a set of auxiliary field equations which will be useful for obtaining the equations of motion of test particles. In Sec. V we develop the method for obtaining the equations of motion of test particles. In Sec. VI we apply the method to the case of neutral test particles possessing no electromagnetic multipole moments in Einstein's unified field theory and in Einstein's gravitational theory. In particular, we find the equations of motion of a neutral pole-dipole particle and a neutral pole particle. In finding the equations of motion of these neutral test particles in an external field in Einstein's unified field theory, we discover what field in the theory is the natural choice for the metric and gravitational field.

II. SPACE-TIME MANIFOLD

A. Field equations

In Einstein's theory of the nonsymmetric field nature is regarded as a four-dimensional space-time manifold whose structure is described through a second-rank tensor field $g_{\mu\nu}$. The fundamental field $g_{\mu\nu}$ satisfies the general-relativistic field equations[17]

$$\Gamma^{\rho}_{[\mu\rho]}=0\ , \tag{2.1a}$$

$$R_{[\mu\nu,\rho]}=0\ , \tag{2.1b}$$

$$R_{(\mu\nu)}=0\ , \tag{2.1c}$$

where the displacement field $\Gamma^{\rho}_{\mu\nu}$ and the contracted curvature tensor $R_{\mu\nu}$ are defined through the equations

$$g_{\mu^+\nu^-;\rho}(=g_{\mu\nu,\rho}-g_{\sigma\nu}\Gamma^{\sigma}_{\mu\rho}-g_{\mu\sigma}\Gamma^{\sigma}_{\rho\nu})=0\ , \tag{2.2}$$

$$R_{\mu\nu}=\Gamma^{\rho}_{\mu\nu,\rho}-\Gamma^{\rho}_{\mu\rho,\nu}-\Gamma^{\rho}_{\mu\sigma}\Gamma^{\sigma}_{\rho\nu}+\Gamma^{\rho}_{\mu\nu}\Gamma^{\sigma}_{\rho\sigma}\ . \tag{2.3}$$

B. Particles and physical fields

A region of the space-time manifold is regarded as flat if a coordinate system can be found in the region so that the fundamental tensor field is equal to the Minkowski metric throughout the region, that is,

$$g_{\mu\nu}=\eta_{\mu\nu}\ . \tag{2.4}$$

Particles are limited portions of the manifold—limited at least in the spatial directions— which have a very nonflat structure. Portions of the manifold between the particles and possessing a nearly flat structure are known as empty space or vacuum. The slight deviations from flatness in such portions of space-time indicate the presence of an electromagnetic field if $g_{[\mu\nu]}\neq 0$, and the presence of a gravitational field if $g_{(\mu\nu)}\neq\eta_{\mu\nu}$. Nearer the particles, where the deviations from flatness are larger, the field $g_{\mu\nu}$ may also be associated with weak and strong interactions.

III. CONTRACTED BIANCHI IDENTITIES

Before we discuss the motion of particles in Einstein's theory, we shall investigate certain identities which are satisfied by the fundamental field $g_{\mu\nu}$. Making use of the definition given in (2.2) of the displacement field $\Gamma^{\rho}_{\mu\nu}$ in terms of the field $g_{\mu\nu}$, it is not difficult to show that the following contracted "Bianchi" identities are satisfied by $g_{\mu\nu}$:[18]

$$g^{\rho\sigma}(R_{\rho^+\sigma^-;\mu}-R_{\rho^+\mu^+;\sigma}-S_{\mu^-\sigma^-;\rho})=0\ , \tag{3.1}$$

where

$$\begin{aligned}R_{\rho^+\sigma^-;\mu}&=R_{\rho\sigma,\mu}-R_{\kappa\sigma}\Gamma^{\kappa}_{\rho\mu}-R_{\rho\kappa}\Gamma^{\kappa}_{\mu\sigma}\ ,\\ R_{\rho^+\mu^+;\sigma}&=R_{\rho\mu,\sigma}-R_{\kappa\mu}\Gamma^{\kappa}_{\rho\sigma}-R_{\rho\kappa}\Gamma^{\kappa}_{\mu\sigma}\ ,\\ S_{\mu^-\sigma^-;\rho}&=S_{\mu\sigma,\rho}-S_{\kappa\sigma}\Gamma^{\kappa}_{\rho\mu}-S_{\mu\kappa}\Gamma^{\kappa}_{\rho\sigma}\ .\end{aligned} \tag{3.2}$$

The tensor $R_{\mu\nu}$ has been defined in (2.3), and the tensor $S_{\mu\nu}$ is defined through the equations

$$S_{\mu\nu}=\Gamma^{\rho}_{\mu\nu,\rho}-\Gamma^{\rho}_{\rho\nu,\mu}-\Gamma^{\rho}_{\sigma\nu}\Gamma^{\sigma}_{\mu\rho}+\Gamma^{\rho}_{\mu\nu}\Gamma^{\sigma}_{\sigma\rho}\ . \tag{3.3}$$

The contracted Bianchi identities will now be put into a form which we shall find convenient for later use.

A contravariant tensor density $\mathbf{g}^{\mu\nu}$ associated with the fundamental field $g_{\mu\nu}$ can be defined through the equations

$$\mathbf{g}^{\mu\nu}=(-g)^{1/2}g^{\mu\nu}\ , \tag{3.4}$$

where $g^{\mu\nu}$ is defined through

$$g_{\mu\rho}g^{\nu\rho}=g_{\rho\mu}g^{\rho\nu}=\delta^{\nu}_{\mu}\ , \tag{3.5}$$

and g denotes the determinant of $g_{\mu\nu}$. We are assuming

$$g<0\ . \tag{3.6}$$

It is easy to show from (2.2) and the definition of $\mathbf{g}^{\mu\nu}$ that

$$\mathbf{g}^{\rho\sigma}{}_{,\mu}+\mathbf{g}^{\kappa\sigma}\Gamma^{\rho}_{\kappa\mu}+\mathbf{g}^{\rho\kappa}\Gamma^{\sigma}_{\mu\kappa}-\mathbf{g}^{\rho\sigma}\Gamma^{\kappa}_{(\mu\kappa)}=0\ , \tag{3.7}$$

$$\mathbf{g}^{[\mu\rho]}{}_{,\rho}-\mathbf{g}^{(\mu\rho)}\Gamma^{\sigma}_{[\rho\sigma]}=0\ , \tag{3.8}$$

$$\mathbf{g}_{,\mu}-\mathbf{g}\Gamma^{\sigma}_{(\mu\rho)}=0\ , \tag{3.9}$$

where

$$\mathbf{g}=(-g)^{1/2}\ . \tag{3.10}$$

From (3.7) one also finds

$$\mathbf{g}^{\rho\sigma}{}_{,\sigma}-\mathbf{g}^{\rho\sigma}\Gamma^{\kappa}_{[\sigma\kappa]}+\mathbf{g}^{\kappa\lambda}\Gamma^{\rho}_{\kappa\lambda}=0\ , \tag{3.11}$$

$$\mathbf{g}^{\sigma\rho}{}_{,\sigma}+\mathbf{g}^{\sigma\rho}\Gamma^{\kappa}_{[\sigma\kappa]}+\mathbf{g}^{\kappa\lambda}\Gamma^{\rho}_{\kappa\lambda}=0\ , \tag{3.12}$$

and from (3.9),

$$\Gamma^{\rho}_{(\mu\rho),\nu}=\Gamma^{\rho}_{(\nu\rho),\mu}\ . \tag{3.13}$$

We introduce some additional notation. A field $A\ldots{}_{\mu\nu}\ldots$ which is a function of $g_{\rho\sigma}$ is defined as being transposition symmetric with respect to the indices μ and ν if with the replacement of $g_{\rho\sigma}$ by $g_{\sigma\rho}$ and the simultaneous interchange of the indices μ and ν of $A\ldots{}_{\mu\nu}\ldots$ the field is transformed into itself. If the field is transformed into the negative of itself, the field is defined as being transposition antisymmetric. It is clear that any second-rank tensor which is a function of $g_{\mu\nu}$ can be decomposed uniquely into a transposition-symmetric part and a transposition-antisymmetric part. It also follows from its definition in (2.2) that the displacement field $\Gamma^{\rho}_{\mu\nu}$ is transposition symmetric with respect to its indices μ and ν.

We shall decompose both the contracted curvature tensor $R_{\mu\nu}$ and the tensor $S_{\mu\nu}$ into parts which are transposition symmetric with respect to the indices μ and ν and parts which are transposition antisymmetric. One finds from (2.3) and (3.3)

$$R_{\mu\nu}=R^{S}_{\mu\nu}+R^{A}_{\mu\nu}\ , \tag{3.14}$$

$$S_{\mu\nu}=R^{S}_{\mu\nu}-R^{A}_{\mu\nu}\ , \tag{3.15}$$

where

$$R^{S}_{\mu\nu}=\Gamma^{\rho}_{\mu\nu,\rho}-\Gamma^{\rho}_{\mu\rho,\nu}-\Gamma^{\rho}_{\mu\sigma}\Gamma^{\sigma}_{\rho\nu}+\Gamma^{\rho}_{\mu\nu}\Gamma^{\sigma}_{\rho\sigma} +\tfrac{1}{2}(\Gamma^{\rho}_{[\mu\rho],\nu}+\Gamma^{\rho}_{[\nu\rho],\mu}-2\Gamma^{\rho}_{\mu\nu}\Gamma^{\sigma}_{[\rho\sigma]})\ , \tag{3.16}$$

$$R^{A}_{\mu\nu}=-\tfrac{1}{2}(\Gamma^{\rho}_{[\mu\rho],\nu}+\Gamma^{\rho}_{[\nu\rho],\mu}-2\Gamma^{\rho}_{\mu\nu}\Gamma^{\sigma}_{[\rho\sigma]})\ . \tag{3.17}$$

The tensor $R^{S}_{\mu\nu}$ is transposition symmetric with respect to the indices μ and ν, and the tensor $R^{A}_{\mu\nu}$ is transposition antisymmetric.

From (3.11) and (3.12) one finds

$$\mathbf{g}^{\rho\sigma}{}_{,\sigma}R^{S}_{\rho\mu}+\mathbf{g}^{\sigma\rho}{}_{,\sigma}R^{S}_{\mu\rho}+2\mathbf{g}^{\kappa\sigma}R^{S}_{(\rho\mu)}\Gamma^{\rho}_{\kappa\sigma} =\mathbf{g}^{\rho\sigma}(R^{S}_{\rho\mu}\Gamma^{\kappa}_{[\sigma\kappa]}-R^{S}_{\mu\sigma}\Gamma^{\kappa}_{[\rho\kappa]})\ . \tag{3.18}$$

If we place (3.2) in (3.1) and make use of (3.14)–(3.17) and the definition of $\mathbf{g}^{\mu\nu}$ given in (3.4), we obtain

$$\mathbf{g}^{\rho\sigma}R^{S}_{\rho\mu,\sigma}+\mathbf{g}^{\sigma\rho}R^{S}_{\mu\rho,\sigma}-2\mathbf{g}^{\rho\sigma}R^{S}_{(\mu\kappa)}\Gamma^{\kappa}_{\rho\sigma} =\mathbf{g}^{\rho\sigma}R^{S}_{\rho\sigma,\mu}+\mathbf{g}^{\rho\sigma}(R^{A}_{\rho\sigma,\mu}+R^{A}_{\mu\sigma,\rho}-R^{A}_{\rho\mu,\sigma}-2R^{A}_{\kappa\sigma}\Gamma^{\kappa}_{\rho\mu} +2R^{A}_{[\kappa\mu]}\Gamma^{\kappa}_{\rho\sigma})\ . \tag{3.19}$$

From (3.18) and (3.19) we see that the contracted Bianchi identities (3.1) can be written in the form

$$(\mathbf{g}^{\rho\sigma}R^{S}_{\rho\mu}+\mathbf{g}^{\sigma\rho}R^{S}_{\mu\rho})_{,\sigma}-\mathbf{g}^{\rho\sigma}R^{S}_{\rho\sigma,\mu} -\mathbf{g}^{\rho\sigma}(R^{S}_{\rho\mu}\Gamma^{\kappa}_{[\sigma\kappa]}-R^{S}_{\mu\sigma}\Gamma^{\kappa}_{[\rho\kappa]}) =\mathbf{g}^{\rho\sigma}(R^{A}_{\rho\sigma,\mu}+R^{A}_{\mu\sigma,\rho}-R^{A}_{\rho\mu,\sigma}-2R^{A}_{\kappa\sigma}\Gamma^{\kappa}_{\rho\mu} +2R^{A}_{[\kappa\mu]}\Gamma^{\kappa}_{\rho\sigma})\ . \tag{3.20}$$

If we decompose the fields $\mathbf{g}^{\mu\nu}$ and $R^{S}_{\mu\nu}$ appearing on the left-hand side of (3.20) into their symmetric and antisymmetric parts, the Bianchi identities (3.20) will take a form which we will find especially convenient later. Decomposing the fields $\mathbf{g}^{\mu\nu}$ and $R^{S}_{\mu\nu}$ as mentioned, and making use of (3.8), one finds from (3.20),

$$(\mathbf{g}^{(\rho\sigma)}R^{S}_{(\rho\mu)}-\tfrac{1}{2}\delta^{\sigma}_{\mu}\mathbf{g}^{(\rho\kappa)}R^{S}_{(\rho\kappa)})_{,\sigma}+\tfrac{1}{2}\mathbf{g}^{(\rho\kappa)}{}_{,\mu}R^{S}_{(\rho\kappa)}=C_{\mu}\ , \tag{3.21}$$

where

$$C_{\mu}=C^{E}_{\mu}+C^{M}_{\mu}\ , \tag{3.22a}$$

$$C^{E}_{\mu}=\tfrac{1}{2}\mathbf{g}^{[\rho\sigma]}R^{S}_{[\rho\sigma,\mu]}\ , \tag{3.22b}$$

$$C^{M}_{\mu}=\mathbf{g}^{[\rho\sigma]}R^{S}_{(\mu\rho)}\Gamma^{\kappa}_{[\sigma\kappa]} +\tfrac{1}{2}\mathbf{g}^{\rho\sigma}(R^{A}_{\rho\sigma,\mu}+R^{A}_{\mu\sigma,\rho}-R^{A}_{\rho\mu,\sigma}-2R^{A}_{\kappa\sigma}\Gamma^{\kappa}_{\rho\mu} +2R^{A}_{[\kappa\mu]}\Gamma^{\kappa}_{\rho\sigma})\ . \tag{3.22c}$$

IV. AUXILIARY FIELD EQUATIONS

A. Preliminaries

In seeking the equations of motion of test particles in Einstein's theory, we shall find it convenient to supplement the fundamental field equations (2.1) with a set of auxiliary field equations. In preparation for this, we first define in Einstein's theory a covariant metric tensor $a_{\mu\nu}$ and a contravariant metric tensor $a^{\mu\nu}$ through the equations

$$(-a)^{1/2}a^{\mu\nu}=\mathbf{g}^{(\mu\nu)}\ , \tag{4.1}$$

$$a^{\mu\rho}a_{\nu\rho}=\delta^{\mu}_{\nu}\ , \tag{4.2}$$

where a denotes the determinant of $a_{\mu\nu}$.[19] We also define an electromagnetic field $\gamma^{*}_{[\mu\nu]}$ as follows:

$$\gamma^{*}_{[\mu\nu]}=\tfrac{1}{2}\epsilon_{\mu\nu\rho\sigma}\mathbf{g}^{[\sigma\rho]}\ . \tag{4.3}$$

The field $\gamma^{*}_{[\mu\nu]}$ is an oriented second-rank tensor field. We shall show later that $a_{\mu\nu}$ is the natural choice for the metric tensor and gravitational field in Einstein's theory. That $\gamma^{*}_{[\mu\nu]}$ is the natural choice for the electromagnetic field has been discussed earlier.[2]

We next define a vector field i_{μ},

$$i_{\mu}=\Gamma^{\rho}_{[\mu\rho]}\ , \tag{4.4}$$

an oriented vector field s_{μ},

$$s_{\mu}=\tfrac{1}{6}(-a)^{-1/2}a_{\mu\kappa}\epsilon^{\rho\sigma\lambda\kappa}R^{S}_{[\rho\sigma,\lambda]}\ , \tag{4.5}$$

and a symmetric tensor field $T_{\mu\nu}$,

$$T_{\mu\nu}=R^{S}_{(\mu\nu)}-\tfrac{1}{2}a_{\mu\nu}a^{\rho\sigma}R^{S}_{(\rho\sigma)}\ . \tag{4.6}$$

The indices on i_{μ}, s_{μ}, and $T_{\mu\nu}$ are understood as raised and lowered through $a^{\mu\nu}$ and $a_{\mu\nu}$. Thus, we have

$$i^{\mu}=a^{\mu\rho}i_{\rho}\ , \tag{4.7}$$

$$s^{\mu}=a^{\mu\rho}s_{\rho}\ , \tag{4.8}$$

$$T^{\mu\nu}=a^{\mu\rho}a^{\nu\sigma}T_{\rho\sigma},\quad T^{\mu}{}_{\nu}=T_{\nu}{}^{\mu}=a^{\mu\rho}T_{\rho\nu}\ . \tag{4.9}$$

We also define a vector density $\mathbf{i}_{\mu}$ associated with i_{μ}, an oriented vector density $\mathbf{s}_{\mu}$ associated with s_{μ}, and a tensor density $\mathbf{T}_{\mu\nu}$ associated with $T_{\mu\nu}$;

$$\mathbf{i}_{\mu}=(-a)^{1/2}i_{\mu}\ , \tag{4.10}$$

$$\mathbf{s}_{\mu}=(-a)^{1/2}s_{\mu}\ , \tag{4.11}$$

$$\mathbf{T}_{\mu\nu}=(-a)^{1/2}T_{\mu\nu}\ . \tag{4.12}$$

The indices on $\mathbf{i}_{\mu}$, $\mathbf{s}_{\mu}$, and $\mathbf{T}_{\mu\nu}$ are also understood as raised and lowered through $a^{\mu\nu}$ and $a_{\mu\nu}$. Thus, we have

$$\mathbf{i}^{\mu}=a^{\mu\rho}\mathbf{i}_{\rho}\ , \tag{4.13}$$

$$\mathbf{s}^{\mu}=a^{\mu\rho}\mathbf{s}_{\rho}\ , \tag{4.14}$$

$$\mathbf{T}^{\mu\nu}=a^{\mu\rho}a^{\nu\sigma}\mathbf{T}_{\rho\sigma},\quad \mathbf{T}^{\mu}{}_{\nu}=\mathbf{T}_{\nu}{}^{\mu}=a^{\mu\rho}\mathbf{T}_{\rho\nu}\ . \tag{4.15}$$

Finally, we discuss some notation. In the case of the covariant differentiation of a tensor where the displacement field is given through $\Gamma^{\rho}_{\mu\nu}$, we have associated a + or − sign with the tensor indices. See, for example, (3.2), and for a discussion of this notation see Ref. 20. In the case of the covariant differentiation of a tensor where the displacement field is given through the Christoffel symbol $\{{}^{\rho}_{\mu\nu}\}$, where we define

$$\begin{Bmatrix}\rho\\ \mu\nu\end{Bmatrix}=\tfrac{1}{2}a^{\rho\sigma}(a_{\sigma\mu,\nu}+a_{\sigma\nu,\mu}-a_{\mu\nu,\sigma})\ , \tag{4.16}$$

we will associate no sign with the tensor indices. Thus, we shall use the notation

$$\mathbf{i}^{\mu}{}_{;\lambda}=\mathbf{i}^{\mu}{}_{,\lambda}+\mathbf{i}^{\rho}\begin{Bmatrix}\mu\\ \rho\lambda\end{Bmatrix}-\mathbf{i}^{\mu}\begin{Bmatrix}\rho\\ \rho\lambda\end{Bmatrix}\ , \tag{4.17}$$

$$\mathbf{s}^{\mu}{}_{;\lambda}=\mathbf{s}^{\mu}{}_{,\lambda}+\mathbf{s}^{\rho}\begin{Bmatrix}\mu\\ \rho\lambda\end{Bmatrix}-\mathbf{s}^{\mu}\begin{Bmatrix}\rho\\ \rho\lambda\end{Bmatrix}\ , \tag{4.18}$$

$$\mathbf{T}^{\mu\nu}{}_{;\lambda}=\mathbf{T}^{\mu\nu}{}_{,\lambda}+\mathbf{T}^{\rho\nu}\begin{Bmatrix}\mu\\ \rho\lambda\end{Bmatrix}+\mathbf{T}^{\mu\rho}\begin{Bmatrix}\nu\\ \rho\lambda\end{Bmatrix}-\mathbf{T}^{\mu\nu}\begin{Bmatrix}\rho\\ \rho\lambda\end{Bmatrix}\ , \tag{4.19a}$$

$$\mathbf{T}^{\mu}{}_{\nu;\lambda}=\mathbf{T}_{\nu}{}^{\mu}{}_{;\lambda}=\mathbf{T}_{\nu}{}^{\mu}{}_{,\lambda}-\mathbf{T}_{\rho}{}^{\mu}\begin{Bmatrix}\rho\\ \nu\lambda\end{Bmatrix}+\mathbf{T}_{\nu}{}^{\rho}\begin{Bmatrix}\mu\\ \rho\lambda\end{Bmatrix}-\mathbf{T}_{\nu}{}^{\mu}\begin{Bmatrix}\rho\\ \rho\lambda\end{Bmatrix}\ . \tag{4.19b}$$

B. Identities

We shall show that $\mathbf{i}^{\mu}$, $\mathbf{s}^{\mu}$, and $\mathbf{T}^{\mu\nu}$ satisfy certain identities. Let us first look at $\mathbf{i}^{\mu}$ and $\mathbf{s}^{\mu}$. From (3.8), (4.1), (4.4), (4.10), (4.13), and (4.17), and from (4.5), (4.11), (4.14), and (4.18), we find

$$\mathbf{i}^{\mu}{}_{;\mu}=\mathbf{i}^{\mu}{}_{,\mu}=0\ , \tag{4.20}$$

$$\mathbf{s}^{\mu}{}_{;\mu}=\mathbf{s}^{\mu}{}_{,\mu}=0\ . \tag{4.21}$$

Next let us look at $\mathbf{T}^{\mu\nu}$. From (4.15), (4.16), (4.19b), and (4.2) we find

$$\begin{aligned}\mathbf{T}^{\mu\nu}{}_{;\nu}=a^{\mu\rho}\mathbf{T}_{\rho}{}^{\nu}{}_{;\nu}&=a^{\mu\rho}\left[\mathbf{T}_{\rho}{}^{\nu}{}_{,\nu}-\mathbf{T}_{\sigma}{}^{\nu}\begin{Bmatrix}\sigma\\ \rho\nu\end{Bmatrix}\right]\\ &=a^{\mu\rho}(\mathbf{T}_{\rho}{}^{\nu}{}_{,\nu}-\tfrac{1}{2}a_{\kappa\lambda,\rho}\mathbf{T}^{\kappa\lambda})\\ &=a^{\mu\rho}(\mathbf{T}_{\rho}{}^{\nu}{}_{,\nu}+\tfrac{1}{2}a^{\kappa\lambda}{}_{,\rho}\mathbf{T}_{\kappa\lambda})\ .\end{aligned} \tag{4.22}$$

From the definition of $a^{\kappa\lambda}$ we have

$$\mathbf{g}^{(\kappa\lambda)}{}_{,\rho}\mathbf{T}_{\kappa\lambda}=[(-a)^{1/2}{}_{,\rho}a^{\kappa\lambda}+(-a)^{1/2}a^{\kappa\lambda}{}_{,\rho}]\mathbf{T}_{\kappa\lambda}\ , \tag{4.23}$$

and from (4.12) and (4.23) we see that

$$a^{\kappa\lambda}{}_{,\rho}\mathbf{T}_{\kappa\lambda}=\mathbf{g}^{(\kappa\lambda)}{}_{,\rho}T_{\kappa\lambda}-(-a)^{1/2}{}_{,\rho}a^{\kappa\lambda}T_{\kappa\lambda}\ . \tag{4.24}$$

Making use of

$$a_{,\rho}=-aa_{\kappa\lambda}a^{\kappa\lambda}{}_{,\rho}\ , \tag{4.25}$$

and the definition of $a^{\kappa\lambda}$, it can also be shown that

$$(-a)^{1/2}{}_{,\rho}=\tfrac{1}{2}a_{\kappa\lambda}\mathbf{g}^{(\kappa\lambda)}{}_{,\rho}\ . \tag{4.26}$$

Making use of (4.2), (4.6), and (4.26) in (4.24), we find

$$a^{\kappa\lambda}{}_{,\rho}\mathbf{T}_{\kappa\lambda}=\mathbf{g}^{(\kappa\lambda)}{}_{,\rho}\overset{S}{R}_{(\kappa\lambda)}\ , \tag{4.27}$$

so that from the contracted Bianchi identities (3.21), the definition of $T_{\rho}{}^{\sigma}$, and from (4.1) we have

$$\mathbf{T}_{\rho}{}^{\nu}{}_{,\nu}=-\tfrac{1}{2}\mathbf{g}^{(\kappa\lambda)}{}_{,\rho}\overset{S}{R}_{(\kappa\lambda)}+C_{\rho}\ . \tag{4.28}$$

Placing (4.27) and (4.28) in (4.22), we obtain the identities

$$\mathbf{T}^{\mu\nu}{}_{;\nu}=a^{\mu\rho}C_{\rho}\ . \tag{4.29}$$

Making use of (3.22), the definition of $\gamma^{*}_{[\mu\nu]}$, and the definitions of $T_{\mu\nu}$, i_{μ}, $\mathbf{i}^{\mu}$, and $\mathbf{s}^{\mu}$, we also find

$$C_{\rho}=C^{E}_{\rho}+C^{M}_{\rho}\ , \tag{4.30}$$

where

$$C^{E}_{\rho}=\gamma^{*}_{[\nu\rho]}\mathbf{s}^{\nu}\ , \tag{4.31}$$

$$\begin{aligned}C^{M}_{\rho}=\;&\gamma^{[\kappa\nu]}(T_{\kappa\rho}-\tfrac{1}{2}a_{\kappa\rho}a^{\lambda\tau}T_{\lambda\tau})\mathbf{i}_{\nu}+(-\tfrac{1}{2}a^{\kappa\lambda}R^{\nu}{}_{\kappa\rho\lambda}-\tfrac{1}{2}a^{\kappa\nu}R_{\kappa\rho}-\tfrac{1}{2}\gamma^{[\kappa\lambda]}R^{\nu}{}_{\kappa\rho\lambda}-\tfrac{1}{2}\gamma^{[\kappa\gamma]}R_{\kappa\rho})\mathbf{i}_{\nu}\\ &+(-\tfrac{1}{2}\gamma^{[\kappa\nu]}{}_{;\rho}+\tfrac{1}{2}a_{\lambda\sigma}\gamma^{[\kappa\nu]}\gamma^{[\lambda\tau]}\Gamma^{\sigma}_{[\rho\tau]})\mathbf{i}_{\nu;\kappa}-\tfrac{1}{2}\gamma^{[\kappa\nu]}\mathbf{i}_{\nu;\kappa\rho}\ .\end{aligned} \tag{4.32}$$

The curvature tensor $R^{\kappa}{}_{\mu\nu\lambda}$ and the "dual" electromagnetic field tensor $\gamma^{[\mu\nu]}$ are defined through

$$R^{\kappa}{}_{\mu\nu\lambda}=\Gamma^{\kappa}_{\mu\nu,\lambda}-\Gamma^{\kappa}_{\mu\lambda,\nu}-\Gamma^{\kappa}_{\sigma\nu}\Gamma^{\sigma}_{\mu\lambda}+\Gamma^{\kappa}_{\sigma\lambda}\Gamma^{\sigma}_{\mu\nu}\ , \tag{4.33}$$

$$\gamma^{[\mu\nu]}=(-a)^{-1/2}\mathbf{g}^{[\mu\nu]}\ . \tag{4.34}$$

C. Field equations

From the analysis contained in Secs. IV A and IV B, we see that if through Eqs. (2.2) one defines the displacement field $\Gamma^{\rho}_{\mu\nu}$ in terms of $g_{\mu\nu}$ and the derivatives of $g_{\mu\nu}$, then the field $g_{\mu\nu}$ will satisfy identically the field equations[21]

$$\Gamma^{\rho}_{[\mu\rho]}=a_{\mu\rho}i^{\rho}\ , \tag{4.35}$$

$$\overset{S}{R}_{[\mu\nu,\rho]}=(-a)^{1/2}\epsilon_{\mu\nu\rho\sigma}s^{\sigma}\ , \tag{4.36}$$

$$\overset{S}{R}_{(\mu\nu)}-\tfrac{1}{2}a_{\mu\nu}a^{\rho\sigma}\overset{S}{R}_{(\rho\sigma)}=a_{\mu\rho}a_{\nu\sigma}T^{\rho\sigma}\ , \tag{4.37}$$

where the vector density $\mathbf{i}^\mu$, the oriented vector density $\mathbf{s}^\mu$, and the tensor density $\mathbf{T}^{\mu\nu}$, defined in terms of i^μ, s^μ, and $T^{\mu\nu}$, respectively, through

$$\mathbf{i}^\mu=(-a)^{1/2}i^\mu \,, \tag{4.38}$$

$$\mathbf{s}^\mu=(-a)^{1/2}s^\mu \,, \tag{4.39}$$

$$\mathbf{T}^{\mu\nu}=(-a)^{1/2}T^{\mu\nu} \,, \tag{4.40}$$

satisfy the identities

$$\mathbf{i}^\mu{}_{;\mu}=0 \,, \tag{4.41}$$

$$\mathbf{s}^\mu{}_{;\mu}=0 \,, \tag{4.42}$$

$$\begin{aligned}\mathbf{T}^{\mu\nu}{}_{;\nu}= {}& a^{\mu\rho}\gamma^*_{[\nu\rho]}\mathbf{s}^\nu+a^{\mu\rho}\gamma^{[\kappa\nu]}(T_{\kappa\rho}-\tfrac{1}{2}a_{\kappa\rho}a^{\lambda\tau}T_{\lambda\tau})\mathbf{i}_\nu \\ &+(\tfrac{1}{2}a^{\mu\kappa}a^{\rho\lambda}R^{\ \nu}_{\rho\kappa\lambda}-\tfrac{1}{2}a^{\mu\kappa}a^{\rho\nu}R_{\rho\kappa}-\tfrac{1}{2}a^{\mu\kappa}\gamma^{[\rho\lambda]}R^{\ \nu}_{\rho\kappa\lambda}-\tfrac{1}{2}a^{\mu\kappa}\gamma^{[\rho\nu]}R_{\rho\kappa})\mathbf{i}_\nu \\ &+(-\tfrac{1}{2}a^{\mu\lambda}\gamma^{[\kappa\nu]}{}_{;\lambda}+\tfrac{1}{2}a^{\mu\lambda}a_{\rho\sigma}\gamma^{[\kappa\nu]}\gamma^{[\rho\tau]}\Gamma^\sigma_{[\lambda\tau]})\mathbf{i}_{\nu;\kappa}+(-\tfrac{1}{2}a^{\mu\lambda}\gamma^{[\kappa\nu]})\mathbf{i}_{\nu;\kappa\lambda} \,.\end{aligned} \tag{4.43}$$

In the special case where one chooses $i^\mu=0$, $s^\mu=0$, and $T^{\mu\nu}=0$, Eqs. (4.35)–(4.37) are Einstein's field equations (2.1).

The auxiliary field equations which we shall find useful for investigating the motion of particles in Einstein's theory are obtained from Eqs. (4.35)–(4.37) by setting

$$\mathbf{i}^\mu=\sum_p\int{}^{(p)}[\tilde{i}^\mu(x)\delta(x-\xi)]d^{(p)}s+\sum_p\sum_i\int{}^{(p)}[\tilde{i}^{\mu\lambda_1\cdots\lambda_i}(x)\delta(x-\xi)]_{;\lambda_1\cdots\lambda_i}d^{(p)}s \,, \tag{4.44}$$

$$\mathbf{s}^\mu=\sum_p\int{}^{(p)}[\tilde{s}^\mu(x)\delta(x-\xi)]d^{(p)}s+\sum_p\sum_i\int{}^{(p)}[\tilde{s}^{\mu\lambda_1\cdots\lambda_i}(x)\delta(x-\xi)]_{;\lambda_1\cdots\lambda_i}d^{(p)}s \,, \tag{4.45}$$

$$\mathbf{T}^{\mu\nu}=\sum_p\int{}^{(p)}[\tilde{T}^{(\mu\nu)}(x)\delta(x-\xi)]d^{(p)}s+\sum_p\sum_i\int{}^{(p)}[\tilde{T}^{(\mu\nu)\lambda_1\cdots\lambda_i}(x)\delta(x-\xi)]_{;\lambda_1\cdots\lambda_i}d^{(p)}s \,. \tag{4.46}$$

In (4.44)–(4.46) a superscript (p) to the left of an expression means that those quantities in the expression which are associated with a particle are to be associated with the pth particle. The ${}^{(p)}\xi$ in (4.44)–(4.46) are the coordinates of points along the world line of the pth particle,[22] and ${}^{(p)}s$ is a parameter defined along the world line through the equations

$$d^{(p)}s^2={}^{(p)}a_{\mu\nu}d^{(p)}\xi^\mu d^{(p)}\xi^\nu \,. \tag{4.47}$$

The field ${}^{(p)}a_{\mu\nu}$ appearing in (4.47) is understood to be the external metric field, appropriately defined,[23] at the position of the pth particle. In the case of a test particle, ${}^{(p)}a_{\mu\nu}$ represents the background field at the position of the test particle. The tensor fields ${}^{(p)}\tilde{i}^\mu$, ${}^{(p)}\tilde{i}^{\mu\lambda_1\cdots\lambda_i}$, ${}^{(p)}\tilde{T}^{(\mu\nu)}$, and ${}^{(p)}\tilde{T}^{(\mu\nu)\lambda_1\cdots\lambda_i}$ and the oriented tensor fields ${}^{(p)}\tilde{s}^\mu$ and ${}^{(p)}\tilde{s}^{\mu\lambda_1\cdots\lambda_i}$ are defined in the vicinity of the pth particle and depend on the structure of the pth particle, on its kinematic properties, and on the external field in its vicinity. Their explicit form will be discussed later. The quantity $\delta(x-\xi)$ represents the four-dimensional Dirac delta function. The indices on both x^μ and ξ^μ have been suppressed in (4.44)–(4.46).

V. EQUATIONS OF MOTION

A. Auxiliary field equations

We shall investigate in more detail the auxiliary field equations discussed in Sec. IV, that is, Eqs. (4.35)–(4.37), where $\mathbf{i}^\mu$, $\mathbf{s}^\mu$, and $\mathbf{T}^{\mu\nu}$ are given through (4.44)–(4.46). We wish to show why and under what circumstances the auxiliary field equations can be used to find the equations of structure and motion of particles in Einstein's unified field theory.

If we assume the expansion

$$g_{\mu\nu}=\sum_{k=0}^\infty \kappa^k{}_{(k)}g_{\mu\nu},\quad {}_{(0)}g_{\mu\nu}=\eta_{\mu\nu} \,, \tag{5.1}$$

for the fundamental field $g_{\mu\nu}$ (κ is the expansion parameter), the auxiliary field equations can be put into the form[24]

$$\gamma^*_{[\mu\nu,\rho]}=\eta^{\kappa\sigma}\epsilon_{\mu\nu\rho\kappa}i^{\text{aux}}_\sigma \,, \tag{5.2a}$$

$$\Box^2\gamma^*_{[\mu\nu]}{}^{,\nu}=-s^{\text{aux}}_\mu \,, \tag{5.2b}$$

$$\Box^2\gamma_{(\mu\nu)}-\gamma_{(\mu\rho),\nu}{}^{,\rho}-\gamma_{(\nu\rho),\mu}{}^{,\rho}+\eta_{\mu\nu}\gamma_{(\rho\sigma)}{}^{,\rho\sigma}=t^{\text{aux}}_{\mu\nu} \,, \tag{5.2c}$$

where $\gamma^*_{[\mu\nu]}$ is defined as in (4.3), and

$$\gamma_{(\mu\nu)}=\eta_{\mu\rho}\eta_{\nu\sigma}\mathbf{g}^{(\rho\sigma)}-\eta_{\mu\nu} \,, \tag{5.3}$$

and

$$i^{\text{aux}}_\mu=\eta_{\mu\rho}\mathbf{i}^\rho \,, \tag{5.4}$$

$$s^{\text{aux}}_\mu=2\eta_{\mu\rho}\mathbf{s}^\rho+\tfrac{1}{3}\eta_{\mu\rho}\epsilon^{\rho\sigma\kappa\lambda}\overset{N}{R}_{[\kappa\lambda,\sigma]} \,, \tag{5.5}$$

$$\begin{aligned}t^{\text{aux}}_{\mu\nu}= {}& 2a_{\mu\rho}a_{\nu\sigma}T^{\rho\sigma}+(a_{\mu\nu}a^{\rho\sigma}-\eta_{\mu\nu}\eta^{\rho\sigma})\overset{S}{R}_{(\rho\sigma)} \\ &-2(\overset{N}{R}_{(\mu\nu)}-\tfrac{1}{2}\eta_{\mu\nu}\eta^{\rho\sigma}\overset{N}{R}_{(\rho\sigma)}) \,.\end{aligned} \tag{5.6}$$

The field $\overset{N}{R}_{\mu\nu}$ is that part of the tensor $\overset{S}{R}_{\mu\nu}$ which is nonlinear in $\mathbf{g}^{\mu\nu}-\eta^{\mu\nu}$. On the left-hand side of Eqs. (5.2b) and (5.2c) the indices have been raised with the Minkowski metric. From Eqs. (4.37) we find

$$\overset{S}{R}_{(\mu\nu)}=T_{\mu\nu}-\tfrac{1}{2}a_{\mu\nu}a^{\rho\sigma}T_{\rho\sigma} \,, \tag{5.7}$$

so that the field $t^{\text{aux}}_{\mu\nu}$ defined in (5.6) can also be written

$$t_{\mu\nu}^{\rm aux}=2T_{\mu\nu}+(a_{\mu\nu}a^{\rho\sigma}-\eta_{\mu\nu}\eta^{\rho\sigma})(T_{\rho\sigma}-\tfrac{1}{2}a_{\rho\sigma}a^{\kappa\lambda}T_{\kappa\lambda})-2(R^N_{(\mu\nu)}-\tfrac{1}{2}\eta_{\mu\nu}\eta^{\rho\sigma}R^N_{\rho\sigma})\ . \quad (5.8)$$

Because of the general form of $\mathbf{i}^\mu$, $\mathbf{s}^\mu$, and $\mathbf{T}^{\mu\nu}$ as given in Eqs. (4.44), (4.45), and (4.46) respectively, it follows, assuming the expansion (5.1), that we can always write

$$\eta_{\mu\rho}\mathbf{i}^\rho=i_\mu^{\rm kin}+i_\mu^h\ , \quad (5.9)$$

$$2\eta_{\mu\rho}\mathbf{s}^\rho=s_\mu^{\rm kin}+s_\mu^h\ , \quad (5.10)$$

$$2a_{\mu\rho}a_{\nu\sigma}\mathbf{T}^{\rho\sigma}+(a_{\mu\nu}a^{\rho\sigma}-\eta_{\mu\nu}\eta^{\rho\sigma})(\mathbf{T}_{\rho\sigma}-\tfrac{1}{2}a_{\rho\sigma}a^{\kappa\lambda}\mathbf{T}_{\kappa\lambda})=(-a)^{1/2}(t_{\mu\nu}^{\rm kin}+t_{\mu\nu}^h)\ , \quad (5.11)$$

where

$$i_\mu^{\rm kin}=\frac{4\pi}{c^2}\sum_p\int{}^{(p)}[e^Mu_\mu]\delta(x-{}^{(p)}\xi)d^{(p)}\tau+\frac{4\pi}{c^2}\sum_p\sum_i\int{}^{(p)}e^M_{[\mu\sigma_1]\cdots\sigma_i}\delta^{,\sigma_1\cdots\sigma_i}(x-{}^{(p)}\xi)d^{(p)}\tau\ , \quad (5.12)$$

$$s_\mu^{\rm kin}=\frac{4\pi}{c^2}\sum_p\int{}^{(p)}[e^Eu_\mu]\delta(x-{}^{(p)}\xi)d^{(p)}\tau+\frac{4\pi}{c^2}\sum_p\sum_i\int{}^{(p)}e^E_{[\mu\sigma_1]\cdots\sigma_i}\delta^{,\sigma_1\cdots\sigma_i}(x-{}^{(p)}\xi)d^{(p)}\tau\ , \quad (5.13)$$

$$t_{\mu\nu}^{\rm kin}=\frac{4\pi}{c^2}\sum_p\int{}^{(p)}[m^Gu_\mu u_\nu+\tfrac{1}{2}\dot S^G_{\mu\rho}u^\rho u_\nu+\tfrac{1}{2}\dot S^G_{\nu\rho}u^\rho u_\mu]\delta(x-{}^{(p)}\xi)d^{(p)}\tau$$
$$+\frac{4\pi}{c^2}\sum_p\int{}^{(p)}[\tfrac{1}{2}S^G_{\mu\rho}u_\nu+\tfrac{1}{2}S^G_{\nu\rho}u_\mu]\delta^{,\rho}(x-{}^{(p)}\xi)d^{(p)}\tau+\frac{4\pi}{c^2}\sum_p\sum_i\int{}^{(p)}m^G_{[\mu\sigma_1][\nu\sigma_2]\cdots\sigma_i}\delta^{,\sigma_1\sigma_2\cdots\sigma_i}(x-{}^{(p)}\xi)d^{(p)}\tau\ , \quad (5.14)$$

and

$$i_\mu^h=\frac{4\pi}{c^2}\sum_p\int{}^{(p)}c_\mu\delta(x-{}^{(p)}\xi)d^{(p)}\tau+\frac{4\pi}{c^2}\sum_p\sum_i\int{}^{(p)}c_{\mu\rho_1\cdots\rho_i}\delta^{,\rho_1\cdots\rho_i}(x-{}^{(p)}\xi)d^{(p)}\tau\ , \quad (5.15)$$

$$s_\mu^h=\frac{4\pi}{c^2}\sum_p\int{}^{(p)}b_\mu\delta(x-{}^{(p)}\xi)d^{(p)}\tau+\frac{4\pi}{c^2}\sum_p\sum_i\int{}^{(p)}b_{\mu\rho_1\cdots\rho_i}\delta^{,\rho_1\cdots\rho_i}(x-{}^{(p)}\xi)d^{(p)}\tau\ , \quad (5.16)$$

$$t_{\mu\nu}^h=\frac{4\pi}{c^2}\sum_p\int{}^{(p)}a_{(\mu\nu)}\delta(x-{}^{(p)}\xi)d^{(p)}\tau+\frac{4\pi}{c^2}\sum_p\sum_i\int{}^{(p)}a_{(\mu\nu)\rho_1\cdots\rho_i}\delta^{,\rho_1\cdots\rho_i}(x-{}^{(p)}\xi)d^{(p)}\tau\ . \quad (5.17)$$

The quantities

$$ {}^{(p)}e^M,\ {}^{(p)}e^M_{[\mu\sigma_1]\cdots\sigma_i},\ {}^{(p)}e^E,\ {}^{(p)}e^E_{[\mu\sigma_1]\cdots\sigma_i}\ ,\quad {}^{(p)}m^G,\ {}^{(p)}S^G_{\mu\nu},\ {}^{(p)}m^G_{[\mu\sigma_1][\nu\sigma_2]\cdots\sigma_i} \quad (5.18)$$

in (5.12)–(5.14) can be regarded as characterizing the pth particle and are functions of a parameter ${}^{(p)}\tau$ defined along the world line of the pth particle through

$$d^{(p)}\tau^2=\eta_{\mu\nu}d^{(p)}\xi^\mu d^{(p)}\xi^\nu\ . \quad (5.19)$$

We are using the notation

$$ {}^{(p)}u^\mu={}^{(p)}\dot\xi^\mu=\frac{d^{(p)}\xi^\mu}{d^{(p)}\tau}\ . \quad (5.20)$$

A dot over a quantity associated with the pth particle means the derivative of that quantity with respect to ${}^{(p)}\tau$. We also assume with no loss in generality

$$ {}^{(p)}S^G_{\mu\nu}=-{}^{(p)}S^G_{\nu\mu}\ , \quad (5.21)$$

$$ {}^{(p)}m^G_{[\mu\sigma_1][\nu\sigma_2]\cdots\sigma_i}={}^{(p)}m^G_{[\nu\sigma_2][\mu\sigma_1]\cdots\sigma_i}\ . \quad (5.22)$$

All indices in (5.12)–(5.17) are to be understood as raised and lowered with the Minkowski metric. The quantities ${}^{(p)}c_\mu$, ${}^{(p)}c_{\mu\rho_1\cdots\rho_i}$, ${}^{(p)}b_\mu$, ${}^{(p)}b_{\mu\rho_1\cdots\rho_i}$, ${}^{(p)}a_{(\mu\nu)}$, and ${}^{(p)}a_{(\mu\nu)\rho_1\cdots\rho_i}$ in (5.15)–(5.17) will generally be functions of ${}^{(p)}\tau$.

We see from (5.4), (5.5), and (5.8)–(5.11) that we can write

$$i_\mu^{\rm aux}=i_\mu^{\rm kin}+i_\mu^{\rm int}\ , \quad (5.23)$$

$$s_\mu^{\rm aux}=s_\mu^{\rm kin}+s_\mu^{\rm int}\ , \quad (5.24)$$

$$t_{\mu\nu}^{\rm aux}=t_{\mu\nu}^{\rm kin}+t_{\mu\nu}^{\rm int}\ , \quad (5.25)$$

where

$$i_\mu^{\rm int}=i_\mu^h\ , \quad (5.26)$$

$$s_\mu^{\rm int}=\tfrac{1}{3}\eta_{\mu\rho}\epsilon^{\rho\sigma\kappa\lambda}R^N_{[\kappa\lambda,\sigma]}+s_\mu^h\ , \quad (5.27)$$

$$t_{\mu\nu}^{\rm int}=-2(R^N_{(\mu\nu)}-\tfrac{1}{2}\eta_{\mu\nu}\eta^{\rho\sigma}R^N_{(\rho\sigma)})+t_{\mu\nu}^h\ . \quad (5.28)$$

This means that the auxiliary field equations (4.35)–(4.37), where $\mathbf{i}^\mu$, $\mathbf{s}^\mu$, and $\mathbf{T}^{\mu\nu}$ are given by (4.44)–(4.46), are at each order with respect to the power series in κ entirely equivalent to the field equations used in the author's papers RVI and RVII (See Ref. 4) to find the equations of structure and motion of particles in Einstein's unified field theory.[25] The approximation method used in papers RVI and RVII to find the equations of structure and motion of particles in Einstein's theory is called the conservation law method.

In papers RVI and RVII, it is shown that the conservation law method gives at each order of approximation with respect to the powers of κ the same results for the equations of structure and motion of particles as an earlier approximation method described in the author's papers RI–RV, in which the field equations analyzed were the fundamental field equations (2.1), and particles were represented by regions of space-time in which the field was very strong, i.e., the regions were very nonflat. The earlier approximation method made use of the expansion (5.1) and thus could only be expected to be valid when investigating particles which are sufficiently far from each other. The order of magnitude of these distances are discussed in the earlier papers and in the introduction to this paper. Under the restriction that the particles are sufficiently far from each other so that the approximation method can be considered valid, it was shown that particles in Einstein's unified field theory could be regarded as characterized by the quantities, which we shall call multipole moments, given in (5.18). If the number of these moments characterizing a particle was finite the particles were called standard ideal particles.[26]

What we have found in this section of the paper is the following: if one uses the auxiliary field equations (4.35)–(4.37) and (4.44)–(4.47) to find the equations of structure and motion of particles characterized by the quantities (5.18), one obtains equations of structure and motion which, if one expands them in a power series in κ, will be identical to each order in κ with those obtained using the author's approximation method described in papers RI–RV. The auxiliary field equations can thus be used to find the equations of motion of particles in Einstein's unified theory—and also in Einstein's gravitational theory since that theory is a special case of the unified field theory—under the same conditions for which the author's earlier approximation method is valid.

However, there is a difficulty in using the method which one also finds in using the conservation law method. If one uses the auxiliary field equations one finds that certain terms appear in the equations of structure and motion of a particle whose values are not determined through the formal use of delta-function notation, and therefore one must determine the contribution of such terms to the equations of structure and motion through some other method—for example, through the method described in the author's papers RI–RV. In the case where one restricts oneself to finding the equations of structure and motion of particles in the test-particle limit, however, these terms do not appear, so that in this case the equations of structure and motion can be completely determined through the use of the auxiliary field equations. Test particles will be discussed in the next subsection.

B. Test particles

We define a test particle in the following way. If one neglects the self-interaction terms in the equations of structure and motion of a particle (i.e., those terms in the equations of structure and motion which are nonlinear in the multipole moments which characterize the particle), and if one neglects the effect of the particle on the external field in its vicinity, then one is investigating the equations of structure and motion of the particle in the test-particle limit. A particle which is being studied in the test-particle limit will be known as a test particle. If, in addition, the particle can be described through a finite number of the mulitpole moments, then it will be known as a multipole test particle.

From the analysis in the previous sections we know that the exact equations of structure and motion of a multipole test particle must be consistent with the equations

$$\mathbf{i}^{\mu}{}_{;\mu}=0\,, \tag{5.29}$$

$$\mathbf{s}^{\mu}{}_{;\mu}=0\,, \tag{5.30}$$

$$\begin{aligned}\mathbf{T}^{\mu\nu}{}_{;\nu}= {} & a^{\mu\rho}\gamma^{*}_{[\nu\rho]}\mathbf{s}^{\nu}+a^{\mu\rho}\gamma^{[\kappa\nu]}(T_{\kappa\rho}-\tfrac{1}{2}a_{\kappa\rho}a^{\lambda\tau}T_{\lambda\tau})\mathbf{i}_{\nu}\\ &+(-\tfrac{1}{2}a^{\mu\kappa}a^{\rho\lambda}R^{\nu}{}_{\rho\kappa\lambda}-\tfrac{1}{2}a^{\mu\kappa}a^{\rho\nu}R_{\rho\kappa}-\tfrac{1}{2}a^{\mu\kappa}\gamma^{[\rho\lambda]}R^{\nu}{}_{\rho\kappa\lambda}-\tfrac{1}{2}a^{\mu\kappa}\gamma^{[\rho\nu]}R_{\rho\kappa})\mathbf{i}_{\nu}\\ &+(-\tfrac{1}{2}a^{\mu\lambda}\gamma^{[\kappa\nu]}{}_{;\lambda}+\tfrac{1}{2}a^{\mu\lambda}a_{\rho\sigma}\gamma^{[\kappa\nu]}\gamma^{[\rho\tau]}\Gamma^{\sigma}_{[\lambda\tau]})\mathbf{i}_{\nu;\kappa}-\tfrac{1}{2}a^{\mu\lambda}\gamma^{[\kappa\nu]}\mathbf{i}_{\nu;\kappa\lambda}\,,\end{aligned} \tag{5.31}$$

where over a region of space-time enclosing the world line of the test particle but not enclosing the world line of any other particle,

$$\mathbf{i}^{\mu}=\int \tilde{i}^{\mu}(x)\delta(x-\xi)ds+\sum_{i}\int[\tilde{i}^{\mu\lambda_1\cdots\lambda_i}(x)\delta(x-\xi)]_{;\lambda_1\cdots\lambda_i}ds\,, \tag{5.32}$$

$$\mathbf{s}^{\mu}=\int \tilde{s}^{\mu}(x)\delta(x-\xi)ds+\sum_{i}\int[\tilde{s}^{\mu\lambda_1\cdots\lambda_i}(x)\delta(x-\xi)]_{;\lambda_1\cdots\lambda_i}ds\,, \tag{5.33}$$

$$\mathbf{T}^{\mu\nu}=\int \tilde{T}^{(\mu\nu)}(x)\delta(x-\xi)ds+\sum_{i}\int[\tilde{T}^{(\mu\nu)\lambda_1\cdots\lambda_i}(x)\delta(x-\xi)]_{;\lambda_1\cdots\lambda_i}ds\,. \tag{5.34}$$

We shall call the quantities $\mathbf{i}^{\mu}$ and $\mathbf{s}^{\mu}$ in (5.32) and (5.33) the electromagnetic current densities associated with the multipole test particle, and the quantity $\mathbf{T}^{\mu\nu}$ in (5.34) the energy-momentum tensor density associated with the particle.

Equations (5.29)–(5.34) will completely determine the equations of structure and motion of a multipole test particle in a given background field. We shall illustrate this in Sec. VI where using (5.29)–(5.34) we shall find the complete equations of structure and motion of a neutral pole-dipole test particle in both Einstein's unified field theory and Einstein's gravitational theory. In papers II and III of this series of papers we shall use Eqs. (5.29)–(5.34) to find the equations of structure and motion of charged test particles in Einstein's unified field theory.

VI. NEUTRAL TEST PARTICLES

A. Preliminaries

We shall confine our study in this section to neutral multipole test particles in Einstein's unified field theory for which all electromagnetic multipole moments vanish. Over a region of spacetime containing such a neutral test particle Eqs. (5.29) and (5.30) are identically satisfied, since

$$\mathbf{i}^{\mu}=0 \,, \tag{6.1}$$

$$\mathbf{s}^{\mu}=0 \,, \tag{6.2}$$

and Eqs. (5.31), which must also be satisfied, take the form

$$\mathbf{T}^{\mu\nu}{}_{;\nu}\left[=\mathbf{T}^{\mu\nu}{}_{,\nu}+\mathbf{T}^{\rho\nu}\begin{Bmatrix}\mu\\ \rho\nu\end{Bmatrix}\right]=0 \,, \tag{6.3}$$

where

$$\mathbf{T}^{\mu\nu}=\int \widetilde{T}^{(\mu\nu)}(x)\delta(x-\xi)ds+\sum_i \int [\widetilde{T}^{(\mu\nu)\lambda_1\cdots\lambda_i}(x)\delta(x-\xi)]_{;\lambda_1\cdots\lambda_i}ds \,, \tag{6.4}$$

and

$$\begin{Bmatrix}\kappa\\ \mu\nu\end{Bmatrix}=\tfrac{1}{2}a^{\kappa\rho}(a_{\rho\mu,\nu}+a_{\rho\nu,\mu}-a_{\mu\nu,\rho}) \,, \tag{6.5}$$

$$ds^2=a_{\mu\nu}d\xi^{\mu}d\xi^{\nu} \,. \tag{6.6}$$

The field $a_{\mu\nu}$ in (6.5) and (6.6) is the background field in the vicinity of the test particle. The coordinates of points along the world line of the particle are denoted by ξ^{μ}.

B. Pole-dipole test particles

We shall use Eqs. (6.3)–(6.6) to find in a given background field in Einstein's unified field theory the equations of structure and motion of a neutral pole-dipole test particle for which all electromagnetic multipole moments vanish. For such a particle (6.4) takes the form

$$\mathbf{T}^{\mu\nu}=\int \widetilde{T}^{(\mu\nu)}(x)\delta(x-\xi)ds+\int [\widetilde{T}^{(\mu\nu)\kappa}(x)\delta(x-\xi)]_{;\kappa}ds \,. \tag{6.7}$$

If we make use of the definition of covariant differentiation discussed in Sec. IV, and make use of the identities

$$f(x)\int g(s)\delta(x-\xi)ds=\int [f(\xi)g(s)]\delta(x-\xi)ds \,, \tag{6.8}$$

$$f(x)\int g(s)\delta_{,\rho}(x-\xi)ds=\int [f(\xi)g(s)]\delta_{,\rho}(x-\xi)ds-\int [f_{,\rho}(\xi)g(s)]\delta(x-\xi)ds \,, \tag{6.9}$$

Eqs. (6.7) can be put into the form

$$\mathbf{T}^{\mu\nu}=\int [\widetilde{T}^{(\mu\nu)\kappa}]\delta_{,\kappa}(x-\xi)ds+\int\left[\widetilde{T}^{(\mu\nu)}+\widetilde{T}^{(\mu\rho)\sigma}\begin{Bmatrix}\nu\\ \rho\sigma\end{Bmatrix}+\widetilde{T}^{(\nu\rho)\sigma}\begin{Bmatrix}\mu\\ \rho\sigma\end{Bmatrix}\right]\delta(x-\xi)ds \,, \tag{6.10}$$

where the quantities in the brackets are understood as evaluated along the world line ξ^{μ} of the test particle and are functions of s.

If we make use of the definition of $\mathbf{T}^{\mu\nu}{}_{;\nu}$, see Eqs. (6.3), we find from (6.10) that

$$\begin{aligned}\mathbf{T}^{\mu\nu}{}_{;\nu}=&\int [\widetilde{T}^{(\mu\nu)\kappa}]\delta_{,\kappa\nu}(x-\xi)ds+\int\left[\widetilde{T}^{(\mu\nu)}+\widetilde{T}^{(\mu\rho)\sigma}\begin{Bmatrix}\nu\\ \rho\sigma\end{Bmatrix}+\widetilde{T}^{(\nu\rho)\sigma}\begin{Bmatrix}\mu\\ \rho\sigma\end{Bmatrix}+\widetilde{T}^{(\rho\sigma)\nu}\begin{Bmatrix}\mu\\ \rho\sigma\end{Bmatrix}\right]\delta_{,\nu}(x-\xi)ds\\ &+\int\left[\widetilde{T}^{(\rho\sigma)}\begin{Bmatrix}\mu\\ \rho\sigma\end{Bmatrix}-\widetilde{T}^{(\rho\sigma)\kappa}\begin{Bmatrix}\mu\\ \rho\sigma\end{Bmatrix}_{,\kappa}+2\widetilde{T}^{(\rho\sigma)\kappa}\begin{Bmatrix}\lambda\\ \rho\kappa\end{Bmatrix}\begin{Bmatrix}\mu\\ \sigma\lambda\end{Bmatrix}\right]\delta(x-\xi)ds \,,\end{aligned} \tag{6.11}$$

where the quantities in brackets are functions of s. From (6.3) and (6.11) we shall find the exact equations of structure and motion of the test particle.

From (6.11), making use of the fact that (6.3) must be satisfied, one can show that there is no loss in generality in choosing $\widetilde{T}^{(\mu\nu)\kappa}$ to be of the form[27]

$$\widetilde{T}^{(\mu\nu)\kappa}=\tfrac{1}{2}S^{\mu\kappa}U^{\nu}+\tfrac{1}{2}S^{\nu\kappa}U^{\mu} \,. \tag{6.12}$$

In (6.12) the quantity $S^{\mu\nu}$ is an antisymmetric second-rank tensor characterizing the test particle,

$$S^{\mu\nu}=-S^{\nu\mu}\ , \tag{6.13}$$

and

$$U^{\mu}=\frac{d\xi^{\mu}}{ds}\ . \tag{6.14}$$

Placing (6.12) in (6.11) one finds

$$\mathbf{T}^{\mu\nu}{}_{;\nu}=\int [\tfrac{1}{2}S^{\mu\kappa}U^{\nu}]\delta_{,\kappa\nu}(x-\xi)ds + \int \left[\tilde{T}^{(\mu\nu)}+\tfrac{1}{2}S^{\mu\rho}U^{\sigma}\left\{\begin{matrix}\nu\\ \rho\sigma\end{matrix}\right\}-\tfrac{1}{2}S^{\nu\rho}U^{\sigma}\left\{\begin{matrix}\mu\\ \rho\sigma\end{matrix}\right\}\right]\delta_{,\nu}(x-\xi)ds+\int\left[\tilde{T}^{(\rho\sigma)}\left\{\begin{matrix}\mu\\ \rho\sigma\end{matrix}\right\}-\tfrac{1}{2}S^{\rho\sigma}U^{\kappa}R^{*\mu}{}_{\kappa\rho\sigma}\right]\delta(x-\xi)ds\ . \tag{6.15}$$

We are using the notation

$$R^{*\mu}{}_{\kappa\rho\sigma}=\left\{\begin{matrix}\mu\\ \kappa\rho\end{matrix}\right\}_{,\sigma}-\left\{\begin{matrix}\mu\\ \kappa\sigma\end{matrix}\right\}_{,\rho}-\left\{\begin{matrix}\mu\\ \lambda\rho\end{matrix}\right\}\left\{\begin{matrix}\lambda\\ \kappa\sigma\end{matrix}\right\}+\left\{\begin{matrix}\mu\\ \lambda\sigma\end{matrix}\right\}\left\{\begin{matrix}\lambda\\ \kappa\rho\end{matrix}\right\}\ , \tag{6.16}$$

in (6.15). The field $R^{*\mu}{}_{\kappa\rho\sigma}$ is the curvature tensor associated with the background field $a_{\mu\nu}$.

One also has

$$\int [S^{\mu\kappa}U^{\nu}]\delta_{,\kappa\nu}(x-\xi)ds=\int\left[\frac{dS^{\mu\kappa}}{ds}\right]\delta_{,\kappa}(x-\xi)ds$$
$$=\int\left[\frac{DS^{\mu\nu}}{Ds}-S^{\mu\rho}U^{\sigma}\left\{\begin{matrix}\nu\\ \rho\sigma\end{matrix}\right\}-S^{\rho\nu}U^{\sigma}\left\{\begin{matrix}\mu\\ \rho\sigma\end{matrix}\right\}\right]\delta_{,\nu}(x-\xi)ds\ , \tag{6.17}$$

where we are using the definition

$$\frac{DS^{\mu\nu}}{Ds}=\frac{dS^{\mu\nu}}{ds}+S^{\rho\nu}\left\{\begin{matrix}\mu\\ \rho\sigma\end{matrix}\right\}U^{\sigma}+S^{\mu\rho}\left\{\begin{matrix}\nu\\ \rho\sigma\end{matrix}\right\}U^{\sigma}\ . \tag{6.18}$$

The quantity $DS^{\mu\nu}/Ds$, known as the absolute derivative of the tensor $S^{\mu\nu}$ with respect to s, is a second-rank tensor. Making use of (6.17) in (6.15) we find

$$\mathbf{T}^{\mu\nu}{}_{;\nu}=\int\left[\tilde{T}^{(\mu\nu)}+\frac{1}{2}\frac{DS^{\mu\nu}}{Ds}\right]\delta_{,\nu}(x-\xi)ds+\int\left[\tilde{T}^{(\rho\sigma)}\left\{\begin{matrix}\mu\\ \rho\sigma\end{matrix}\right\}-\tfrac{1}{2}S^{\rho\sigma}U^{\kappa}R^{*\mu}{}_{\kappa\rho\sigma}\right]\delta(x-\xi)ds\ . \tag{6.19}$$

We must next investigate further the integrals appearing in (6.19). The first integral in (6.19) can be written

$$I_1=\int Y^{\mu\nu}\delta_{,\nu}(x-\xi)ds\ , \tag{6.20}$$

where

$$Y^{\mu\nu}=Y^{[\mu\nu]}+Y^{(\mu\nu)}\ , \tag{6.21}$$

$$Y^{(\mu\nu)}=\tilde{T}^{(\mu\nu)}\ , \tag{6.22}$$

$$Y^{[\mu\nu]}=\frac{1}{2}\frac{DS^{\mu\nu}}{Ds}\ . \tag{6.23}$$

With no loss in generality, one can always write[28]

$$Y^{(\mu\nu)}={}^{*}Y^{(\mu\nu)}+X^{\mu}U^{\nu}+X^{\nu}U^{\mu}+MU^{\mu}U^{\nu}\ , \tag{6.24}$$

$$Y^{[\mu\nu]}={}^{*}Y^{[\mu\nu]}+Y^{\mu}U^{\nu}-Y^{\nu}U^{\mu}\ , \tag{6.25}$$

where M is a scalar characterizing the test particle and

$${}^{*}Y^{(\mu\nu)}U_{\nu}=0,\quad {}^{*}Y^{[\mu\nu]}U_{\nu}=0\ , \tag{6.26}$$

$$X^{\mu}U_{\mu}=0,\quad Y^{\mu}U_{\mu}=0\ , \tag{6.27}$$

where

$$U_{\mu}=a_{\mu\nu}U^{\nu}\ . \tag{6.28}$$

This means that the first integral in (6.19) can be put into the form

$$I_1=\int [{}^{*}Y^{(\mu\nu)}+{}^{*}Y^{[\mu\nu]}+X^{\nu}U^{\mu}-Y^{\nu}U^{\mu}]\delta_{,\nu}(x-\xi)ds+\int\left[\frac{d}{ds}(MU^{\mu}+X^{\mu}+Y^{\mu})\right]\delta(x-\xi)ds\ . \tag{6.29}$$

If we then make use of (6.29) in (6.19), we find

$$\mathbf{T}^{\mu\nu}{}_{;\nu}=\int [{}^{*}Y^{(\mu\nu)}+{}^{*}Y^{[\mu\nu]}+X^{\nu}U^{\mu}-Y^{\nu}U^{\mu}]\delta_{,\nu}(x-\xi)ds+\int\left[\frac{d}{ds}(MU^{\mu}+X^{\mu}+Y^{\mu})+\tilde{T}^{(\rho\sigma)}\left\{\begin{matrix}\mu\\ \rho\sigma\end{matrix}\right\}-\tfrac{1}{2}S^{\rho\sigma}U^{\kappa}R^{*\mu}{}_{\kappa\rho\sigma}\right]\delta(x-\xi)ds\ . \tag{6.30}$$

Since (6.3) must be satisfied, it follows from (6.30) that[29]

$${}^{*}Y^{(\mu\nu)}+{}^{*}Y^{[\mu\nu]}-Y^{\nu}U^{\mu}+X^{\nu}U^{\mu}=0\ , \tag{6.31}$$

which is equivalent to

$${}^{*}Y^{(\mu\nu)}=0,\quad {}^{*}Y^{[\mu\nu]}=0,\quad Y^{\mu}=X^{\mu}\ . \tag{6.32}$$

Making use of (6.32) in (6.30) we find

$$\mathbf{T}^{\mu\nu}{}_{;\nu}=\int\left[\frac{d}{ds}(MU^{\mu}+2X^{\mu})+\tilde{T}^{(\rho\sigma)}\left\{\begin{matrix}\mu\\ \rho\sigma\end{matrix}\right\}-\tfrac{1}{2}S^{\rho\sigma}U^{\kappa}R^{*\mu}{}_{\kappa\rho\sigma}\right]\delta(x-\xi)ds\ . \tag{6.33}$$

If, in addition, we make use of (6.24), (6.25), and (6.32) in (6.22) and (6.23), we find

$$\tilde{T}^{(\mu\nu)}=MU^{\mu}U^{\nu}+X^{\mu}U^{\nu}+X^{\nu}U^{\mu}\ , \tag{6.34}$$

$$\frac{DS^{\mu\nu}}{Ds}=2X^{\mu}U^{\nu}-2X^{\nu}U^{\mu}\ , \tag{6.35}$$

and from (6.35) one has

$$X^{\mu}=\frac{1}{2}\frac{DS^{\mu\rho}}{Ds}U_{\rho}\ . \tag{6.36}$$

Placing (6.36) in (6.34) we see

$$\tilde{T}^{(\mu\nu)}=MU^{\mu}U^{\nu}+\frac{1}{2}\frac{DS^{\mu\rho}}{Ds}U_{\rho}U^{\nu}+\frac{1}{2}\frac{DS^{\nu\rho}}{Ds}U_{\rho}U^{\mu}\ , \tag{6.37}$$

and placing (6.36) in (6.35) gives

$$\frac{DS^{\mu\nu}}{Ds}=\frac{DS^{\mu\rho}}{Ds}U_{\rho}U^{\nu}-\frac{DS^{\nu\rho}}{Ds}U_{\rho}U^{\mu}\ . \tag{6.38}$$

Finally, making use of (6.36) and (6.37) in (6.33) we find

$$\mathbf{T}^{\mu\nu}{}_{;\nu}=\int\left[\frac{D}{Ds}\left[MU^{\mu}+\frac{DS^{\mu\rho}}{Ds}U_{\rho}\right]-\tfrac{1}{2}S^{\rho\sigma}R^{*\mu}{}_{\kappa\rho\sigma}U^{\kappa}\right]\times\delta(x-\xi)ds\ , \tag{6.39}$$

where we have made use of the definition,

$$\frac{DA^{\mu}}{Ds}=\frac{dA^{\mu}}{ds}+\left\{\begin{matrix}\mu\\ \rho\sigma\end{matrix}\right\}A^{\rho}U^{\sigma}\ , \tag{6.40}$$

of the absolute derivative DA^{μ}/Ds of an arbitrary vector A^{μ} defined along the world line ξ^{μ}.

Since (6.3) must be satisfied, we see from (6.39) that neutral pole-dipole test particles for which all electromagnetic moments vanish obey the equations of mass and motion

$$\frac{DP^{\mu}}{Ds}-\frac{1}{2}S^{\rho\sigma}R^{*\mu}{}_{\kappa\rho\sigma}U^{\kappa}=0\ , \tag{6.41}$$

where

$$P^{\mu}=MU^{\mu}+\frac{DS^{\mu\rho}}{Ds}U_{\rho}\ . \tag{6.42}$$

From (6.38) we also see that the particles obey the equations of spin

$$\frac{DS^{\mu\nu}}{Ds}-\frac{DS^{\mu\rho}}{Ds}U_{\rho}U^{\nu}+\frac{DS^{\nu\rho}}{Ds}U_{\rho}U^{\mu}=0\ . \tag{6.43}$$

If we make use of (6.12) and (6.37) in (6.10), we find that the energy-momentum tensor density $\mathbf{T}^{\mu\nu}$ associated with these test particles is given by

$$\mathbf{T}^{\mu\nu}=\int[\tfrac{1}{2}S^{\mu\kappa}U^{\nu}+\tfrac{1}{2}S^{\nu\kappa}U^{\mu}]\delta_{,\kappa}(x-\xi)ds+\int\left[MU^{\mu}U^{\nu}+\frac{1}{2}\frac{DS^{\mu\rho}}{Ds}U_{\rho}U^{\nu}+\frac{1}{2}\frac{DS^{\nu\rho}}{Ds}U_{\rho}U^{\mu}+\tfrac{1}{2}S^{\mu\rho}\left\{\begin{matrix}\nu\\ \rho\sigma\end{matrix}\right\}U^{\sigma}+\tfrac{1}{2}S^{\nu\rho}\left\{\begin{matrix}\mu\\ \rho\sigma\end{matrix}\right\}U^{\sigma}\right]\delta(x-\xi)ds\ . \tag{6.44}$$

The particles are characterized by a mass M and a spin $S^{\mu\nu}$. The physical interpretation of M and $S^{\mu\nu}$ follows from the form of the equations of structure and motion (6.41)—(6.43).

In the special case of Einstein's gravitational theory, the equations of structure and motion (6.41)—(6.43) were first obtained by Papapetrou[30] using a method different from the method described in this paper but related to it. Papapetrou assumed Eqs. (6.3) in Einstein's gravitational theory and then studied the relations among integrals of the form

$$\int\mathbf{T}^{\mu\nu}d^{3}x,\quad\int(x^{\rho}-\xi^{\rho})\mathbf{T}^{\mu\nu}d^{3}x,\quad\int(x^{\rho}-\xi^{\rho})(x^{\sigma}-\xi^{\sigma})\mathbf{T}^{\mu\nu}d^{3}x,\quad\dots, \tag{6.45}$$

which are imposed by (6.3). From a study of these relations he arrived at the equations of structure and motion (6.41)—(6.43) for pole-dipole test particles in Einstein's gravitational theory.

It is clear that Papapetrou's method and the method described in this paper give identical results for the equations of structure and motion of multipole test particles in Einstein's gravitational theory. For the case of neutral pole-dipole test particles we illustrate the relationship between the two methods for finding the equations of structure and motion of test particles, by evaluating integrals (6.45) using (6.44), and then checking to see that the results obtained using the method described in this paper are identical to those obtained using Papapetrou's method. From (6.44) one finds by direct integration

$$\int\mathbf{T}^{\mu\nu}d^{3}x=\frac{1}{U^{4}}\left[MU^{\mu}U^{\nu}+\frac{1}{2}\frac{DS^{\mu\rho}}{Ds}U_{\rho}U^{\nu}+\frac{1}{2}\frac{DS^{\nu\rho}}{Ds}U_{\rho}U^{\mu}+\tfrac{1}{2}S^{\mu\rho}\left\{\begin{matrix}\nu\\ \rho\sigma\end{matrix}\right\}U^{\sigma}+\tfrac{1}{2}S^{\nu\rho}\left\{\begin{matrix}\mu\\ \rho\sigma\end{matrix}\right\}U^{\sigma}+\frac{1}{2}\frac{d}{ds}\left[S^{\mu4}\frac{U^{\nu}}{U^{4}}+S^{\nu4}\frac{U^{\mu}}{U^{4}}\right]\right]\ , \tag{6.46}$$

$$\int (x^\rho-\xi^\rho)\mathbf{T}^{\mu\nu}d^3x=\frac{1}{U^4}\left[\tfrac{1}{2}(S^{\rho\mu}U^\nu+S^{\rho\nu}U^\mu)+\tfrac{1}{2}(S^{\mu 4}U^\nu+S^{\nu 4}U^\mu)\frac{U^\rho}{U^4}\right], \tag{6.47}$$

and all other integrals of the form (6.45) vanish. From Eqs. (2.7), (3.5), (3.8)—(3.10), (4.1), (5.1), and (5.2) in Papapetrou's paper, one also obtains for $\int \mathbf{T}^{\mu\nu}d^3x$ and $\int (x^\rho-\xi^\rho)\mathbf{T}^{\mu\nu}d^3x$ the expressions given in (6.46) and (6.47), while all other integrals of the form (6.45) vanish for pole-dipole particles. The two methods thus give identical results.

C. Pole test particles

From (6.41)—(6.43) we see that in Einstein's unified field theory the equations of structure and motion of neutral pole test particles possessing no electromagnetic multipole moments are given by

$$\frac{DP^\mu}{Ds}=0, \tag{6.48}$$

where

$$P^\mu=MU^\mu. \tag{6.49}$$

The particles are characterized by a mass M. Equations (6.48) and (6.49) are equivalent to the equations

$$\frac{dU^\mu}{ds}+\begin{Bmatrix}\mu\\ \rho\sigma\end{Bmatrix}U^\rho U^\sigma=0, \tag{6.50}$$

$$\frac{dM}{ds}=0. \tag{6.51}$$

Therefore in Einstein's unified field theory, such neutral pole test particles move along geodesics in the background field $a_{\mu\nu}$. This justifies our earlier statement that $a_{\mu\nu}$ is the natural choice for the metric tensor and gravitational field in Einstein's unified field theory.

APPENDIX A: $a_{\mu\nu}$ AND $a^{\mu\nu}$

We assume that the fundamental field $g_{\mu\nu}$ exists and has a nonvanishing determinant g over the region we are studying. This means that g is finite and that $g\neq 0$. Under these conditions the field $g^{\mu\nu}$ exists and is uniquely determined through Eqs. (3.5). We also assume that Eqs. (2.2) determine, as a function of $g_{\mu\nu}$ and its derivatives, a unique and finite field $\Gamma^\rho_{\mu\nu}$ over the region. Under these conditions it follows that both the fields $g_{(\mu\nu)}$ and $g^{(\mu\nu)}$ have nonvanishing determinants over the region. That under these conditions the field $g_{(\mu\nu)}$ has over the region a nonvanishing determinant—which we denote by h—is shown in Ref. 31. Under these conditions it is also shown in Ref. 32 that

$$g^2=\tilde{h}h, \tag{A1}$$

where the quantity $\tilde{h}$ denotes the determinant of $g^{(\mu\nu)}$. From (A1) we see that the field $g^{(\mu\nu)}$ will also have a nonvanishing determinant over the region. This means that the fields $a_{\mu\nu}$ and $a^{\mu\nu}$, defined in Sec. III through Eqs. (4.1) and (4.2), exist over the region and that $a\neq 0$.

APPENDIX B: $\tilde{T}^{(\mu\nu)\kappa}$

The tensor $\tilde{T}^{(\mu\nu)\kappa}$, which appears in the equations

$$\mathbf{T}^{\mu\nu}=\int \tilde{T}^{(\mu\nu)\kappa}\delta_{,\kappa}(x-\xi)ds+\int\left[\tilde{T}^{(\mu\nu)}+\tilde{T}^{(\mu\rho)\sigma}\begin{Bmatrix}\nu\\ \rho\sigma\end{Bmatrix}+\tilde{T}^{(\nu\rho)\sigma}\begin{Bmatrix}\mu\\ \rho\sigma\end{Bmatrix}\right]\delta(x-\xi)ds, \tag{B1}$$

can with no loss in generality always be written as a function of the four-velocity U^μ,

$$U^\mu=\frac{d\xi^\mu}{ds}, \tag{B2}$$

and tensors perpendicular to U^μ. This we shall now show. If we define ${}^*T^{(\mu\nu)\kappa}$, ${}^*T^{\kappa\mu}$, ${}^*T^\kappa$, and $A^{(\mu\nu)}$ in the following ways,

$$\begin{aligned}{}^*T^{(\mu\nu)\kappa}=\;&\tilde{T}^{(\mu\nu)\kappa}-\tilde{T}^{(\mu\nu)\alpha}U_\alpha U^\kappa-\tilde{T}^{(\mu\alpha)\kappa}U_\alpha U^\nu-\tilde{T}^{(\nu\alpha)\kappa}U_\alpha U^\mu+\tilde{T}^{(\mu\alpha)\beta}U_\alpha U_\beta U^\nu U^\kappa+\tilde{T}^{(\nu\alpha)\beta}U_\alpha U_\beta U^\mu U^\kappa\\&+\tilde{T}^{(\alpha\beta)\kappa}U_\alpha U_\beta U^\mu U^\nu-\tilde{T}^{(\alpha\beta)\gamma}U_\alpha U_\beta U_\gamma U^\mu U^\nu U^\kappa,\end{aligned} \tag{B3}$$

$${}^*T^{\kappa\mu}=\tilde{T}^{(\mu\alpha)\kappa}U_\alpha-\tilde{T}^{(\mu\alpha)\beta}U_\alpha U_\beta U^\kappa-\tilde{T}^{(\alpha\beta)\kappa}U_\alpha U_\beta U^\mu+\tilde{T}^{(\alpha\beta)\gamma}U_\alpha U_\beta U_\gamma U^\mu U^\kappa, \tag{B4}$$

$${}^*T^\kappa=\tilde{T}^{(\alpha\beta)\kappa}U_\alpha U_\beta-\tilde{T}^{(\alpha\beta)\gamma}U_\alpha U_\beta U_\gamma U^\kappa, \tag{B5}$$

$$A^{(\mu\nu)}=\tilde{T}^{(\mu\nu)\alpha}U_\alpha, \tag{B6}$$

where

$$U_\mu=a_{\mu\nu}U^\nu, \tag{B7}$$

we find

$$ {}^*T^{(\mu\nu)\kappa}U_\nu=0,\quad {}^*T^{(\mu\nu)\kappa}U_\kappa=0\ , \tag{B8}$$

$$ {}^*T^{\kappa\mu}U_\kappa=0,\quad {}^*T^{\kappa\mu}U_\mu=0\ , \tag{B9}$$

$$ {}^*T^{\kappa}U_\kappa=0\ , \tag{B10}$$

and

$$\tilde{T}^{(\mu\nu)\kappa}={}^*T^{(\mu\nu)\kappa}+{}^*T^{\kappa\mu}U^\nu+{}^*T^{\kappa\nu}U^\mu+{}^*T^{\kappa}U^\mu U^\nu+A^{(\mu\nu)}U^\kappa\ . \tag{B11}$$

Furthermore, since

$$\int A^{(\mu\nu)}U^\kappa\delta_{,\kappa}(x-\xi)ds=\int [A^{(\mu\nu)}{}_{,\kappa}U^\kappa]\delta(x-\xi)ds$$
$$=\int [A^{(\mu\nu)}{}_{;\kappa}U^\kappa]\delta(x-\xi)ds+\int\left[-A^{(\sigma\nu)}\left\{\begin{matrix}\mu\\ \sigma\kappa\end{matrix}\right\}U^\kappa-A^{(\sigma\mu)}\left\{\begin{matrix}\nu\\ \sigma\kappa\end{matrix}\right\}U^\kappa\right]\delta(x-\xi)ds\ , \tag{B12}$$

we also find

$$\int \Delta\tilde{T}^{(\mu\nu)\kappa}\delta_{,\kappa}(x-\xi)ds+\int\left[\Delta\tilde{T}^{(\mu\rho)\sigma}\left\{\begin{matrix}\nu\\ \rho\sigma\end{matrix}\right\}+\Delta\tilde{T}^{(\nu\rho)\sigma}\left\{\begin{matrix}\mu\\ \rho\sigma\end{matrix}\right\}\right]\delta(x-\xi)ds=\int \Delta\tilde{T}^{(\mu\nu)}\delta(x-\xi)ds\ , \tag{B13}$$

where

$$\Delta\tilde{T}^{(\mu\nu)\kappa}=A^{(\mu\nu)}U^\kappa\ , \tag{B14}$$

$$\Delta\tilde{T}^{(\mu\nu)}=A^{(\mu\nu)}{}_{;\kappa}U^\kappa=\left[A^{(\mu\nu)}{}_{,\kappa}+A^{(\sigma\nu)}\left\{\begin{matrix}\mu\\ \sigma\kappa\end{matrix}\right\}+A^{(\sigma\mu)}\left\{\begin{matrix}\nu\\ \sigma\kappa\end{matrix}\right\}\right]U^\kappa\ . \tag{B15}$$

This means that the term $A^{(\mu\nu)}U^\kappa$ in (B11) is equivalent to a term $A^{(\mu\nu)}{}_{;\kappa}U^\kappa$ in $\tilde{T}^{(\mu\nu)}$. This follows from the form of the right-hand side of (B1). Therefore, since $\tilde{T}^{(\mu\nu)}$ is arbitrary at this stage of the analysis, there is no loss in generality in choosing

$$\Delta\tilde{T}^{(\mu\nu)\kappa}=A^{(\mu\nu)}U^\kappa=-\tfrac{1}{2}({}^*T^\mu U^\nu+{}^*T^\nu U^\mu)U^\kappa\ , \tag{B16}$$

and thus $\tilde{T}^{(\mu\nu)\kappa}$ can, with no loss in generality, always be considered to take the form

$$\tilde{T}^{(\mu\nu)\kappa}={}^*T^{(\mu\nu)\kappa}+\tfrac{1}{2}S^{\mu\kappa}U^\nu+\tfrac{1}{2}S^{\nu\kappa}U^\mu\ , \tag{B17}$$

where we are using the abbreviation

$$S^{\mu\nu}=2{}^*T^{\nu\mu}+{}^*T^\nu U^\mu-{}^*T^\mu U^\nu\ . \tag{B18}$$

We have succeeded, with no loss in generality, in writing $\tilde{T}^{(\mu\nu)\kappa}$ as a function of U^μ and the tensors ${}^*T^{(\mu\nu)\kappa}$, ${}^*T^{\mu\nu}$, and ${}^*T^\mu$ perpendicular to U^μ.

From (6.3) and (6.11) we see that the form of the tensor $\tilde{T}^{(\mu\nu)\kappa}$ must be such that

$$\int \tilde{T}^{(\mu\nu)\kappa}\delta_{,\kappa\nu}(x-\xi)ds+\int\left[\tilde{T}^{(\mu\nu)}+\tilde{T}^{(\mu\rho)\sigma}\left\{\begin{matrix}\nu\\ \rho\sigma\end{matrix}\right\}+\tilde{T}^{(\nu\rho)\sigma}\left\{\begin{matrix}\mu\\ \rho\sigma\end{matrix}\right\}+\tilde{T}^{(\rho\sigma)\nu}\left\{\begin{matrix}\mu\\ \rho\sigma\end{matrix}\right\}\right]\delta_{,\nu}(x-\xi)ds$$
$$+\int\left[\tilde{T}^{(\rho\sigma)}\left\{\begin{matrix}\mu\\ \rho\sigma\end{matrix}\right\}-\tilde{T}^{(\rho\sigma)\kappa}\left\{\begin{matrix}\mu\\ \rho\sigma\end{matrix}\right\}_{,\kappa}+2\tilde{T}^{(\rho\sigma)\kappa}\left\{\begin{matrix}\lambda\\ \rho\kappa\end{matrix}\right\}\left\{\begin{matrix}\mu\\ \sigma\lambda\end{matrix}\right\}\right]\delta(x-\xi)ds=0\ . \tag{B19}$$

Placing (B17) in (B19), we find the first integral on the left-hand side of (B19) takes the form

$$\int [\tfrac{1}{2}({}^*T^{(\mu\nu)\kappa}+{}^*T^{(\mu\kappa)\nu})+\tfrac{1}{2}S^{(\nu\kappa)}U^\mu]\delta_{,\kappa\nu}(x-\xi)ds+\int [\tfrac{1}{2}S^{\mu\kappa}U^\nu]\delta_{,\kappa\nu}(x-\xi)ds\ . \tag{B20}$$

Making use of (B20) and the identity

$$\int [\tfrac{1}{2}S^{\mu\kappa}U^\nu]\delta_{,\kappa\nu}(x-\xi)ds=\int [\tfrac{1}{2}S^{\mu\nu}{}_{,\kappa}U^\kappa]\delta_{,\nu}(x-\xi)ds\ , \tag{B21}$$

we see that Eqs. (B19) can be put into the form

$$\int C^{\mu(\nu\kappa)}\delta_{,\kappa\nu}(x-\xi)ds+\int C^{\mu\nu}\delta_{,\nu}(x-\xi)ds+\int C^{\mu}\delta(x-\xi)ds=0\ , \tag{B22}$$

where

$$C^{\mu(\nu\kappa)}=\tfrac{1}{2}({}^*T^{(\mu\nu)\kappa}+{}^*T^{(\mu\kappa)\nu})+\tfrac{1}{2}S^{(\nu\kappa)}U^\mu\ , \tag{B23}$$

and where

$$C^{\mu(\nu\kappa)}U_\kappa=0\ . \tag{B24}$$

Applying the theorem discussed in Appendix D, this means we must have

$$\tfrac{1}{2}({}^*T^{(\mu\nu)\kappa}+{}^*T^{(\mu\kappa)\nu})+\tfrac{1}{2}S^{(\nu\kappa)}U^\mu=0\ . \tag{B25}$$

From (B8) and (B25) it follows that

$${}^*T^{(\mu\nu)\kappa}=0,\quad S^{(\mu\nu)}=0\ . \tag{B26}$$

Therefore we see that there is no loss in generality in choosing $\widetilde{T}^{(\mu\nu)\kappa}$ to be of the form

$$\widetilde{T}^{(\mu\nu)\kappa}=\tfrac{1}{2}S^{\mu\kappa}U^\nu+\tfrac{1}{2}S^{\nu\kappa}U^\mu\ , \tag{B27}$$

where $S^{\mu\nu}$ in (B27) is an antisymmetric second-rank tensor which characterizes the test particle.

APPENDIX C: $Y^{(\mu\nu)}$ AND $Y^{[\mu\nu]}$

We first investigate the tensor $Y^{(\mu\nu)}$. Defining the tensors ${}^*Y^{(\mu\nu)}$, X^μ, and M through the equations

$$\begin{aligned}{}^*Y^{(\mu\nu)}=&\ Y^{(\mu\nu)}-Y^{(\mu\alpha)}U_\alpha U^\nu-Y^{(\nu\alpha)}U_\alpha U^\mu\\ &+Y^{(\alpha\beta)}U_\alpha U_\beta U^\mu U^\nu\ ,\end{aligned} \tag{C1}$$

$$X^\mu=Y^{(\mu\alpha)}U_\alpha-Y^{(\alpha\beta)}U_\alpha U_\beta U^\mu\ , \tag{C2}$$

$$M=Y^{(\alpha\beta)}U_\alpha U_\beta\ , \tag{C3}$$

where

$$U^\mu=\frac{d\xi^\mu}{ds},\quad U_\mu=a_{\mu\nu}U^\nu\ , \tag{C4}$$

we find

$${}^*Y^{(\mu\nu)}U_\nu=0,\quad X^\mu U_\mu=0\ , \tag{C5}$$

and

$$Y^{(\mu\nu)}={}^*Y^{(\mu\nu)}+X^\mu U^\nu+X^\nu U^\mu+MU^\mu U^\nu\ . \tag{C6}$$

We next investigate the tensor $Y^{[\mu\nu]}$. Defining the tensors ${}^*Y^{[\mu\nu]}$ and Y^μ through the equations

$${}^*Y^{[\mu\nu]}=Y^{[\mu\nu]}-Y^{[\mu\alpha]}U_\alpha U^\nu+Y^{[\nu\alpha]}U_\alpha U^\mu\ , \tag{C7}$$

$$Y^\mu=Y^{[\mu\alpha]}U_\alpha\ , \tag{C8}$$

we find

$${}^*Y^{[\mu\nu]}U_\nu=0,\quad Y^\mu U_\mu=0\ , \tag{C9}$$

and

$$Y^{[\mu\nu]}={}^*Y^{[\mu\nu]}+Y^\mu U^\nu-Y^\nu U^\mu\ . \tag{C10}$$

APPENDIX D: THEOREM

We shall show that the requirement

$$\begin{aligned}\int C^{(\kappa\lambda)}(s)\delta_{,\kappa\lambda}(x-\xi)ds+\int C^\lambda(s)\delta_{,\lambda}(x-\xi)ds&\\ +\int C(s)\delta(x-\xi)ds=0\ ,&\end{aligned} \tag{D1}$$

supplemented by the conditions

$$C^{(\kappa\lambda)}U_\lambda=0\ , \tag{D2}$$

implies

$$C^{(\kappa\lambda)}=0\ , \tag{D3}$$

and supplemented by the conditions

$$C^{(\kappa\lambda)}U_\lambda=0,\quad C^\lambda U_\lambda=0\ , \tag{D4}$$

implies

$$C^{(\kappa\lambda)}=0,\quad C^\lambda=0,\quad C=0\ . \tag{D5}$$

If we evaluate the integrals appearing in (D1), we find (D1) takes the form

$$\bar{A}^{rs}\delta_{,rs}(\mathbf{x}-\xi)+\bar{B}^r\delta_{,r}(\mathbf{x}-\xi)+\bar{C}\delta(\mathbf{x}-\xi)=0\ , \tag{D6}$$

where

$$\begin{aligned}\bar{A}^{rs}=\frac{1}{U^4}[&C^{(rs)}-(U^r/U^4)C^{(4s)}-(U^s/U^4)C^{(4r)}\\ &+(U^r/U^4)(U^s/U^4)C^{(44)}]\ ,\end{aligned} \tag{D7}$$

$$\begin{aligned}\bar{B}^r=\frac{1}{U^4}\Bigg[&C^r-(U^r/U^4)C^4+2\frac{d}{ds}(C^{(4r)}/U^4)\\ &-(C^{(44)}/U^4)\frac{d}{ds}(U^r/U^4)\\ &-2(U^r/U^4)\frac{d}{ds}(C^{(44)}/U^4)\Bigg]\ ,\end{aligned} \tag{D8}$$

$$\bar{C}=\frac{1}{U^4}\left\{C+\frac{d}{ds}\left[\left[C^4+\frac{d}{ds}(C^{(44)}/U^4)\right]/U^4\right]\right\}, \tag{D9}$$

and $\delta(\mathbf{x}-\xi)$ is the three-dimensional Dirac delta function. Condition (D1) requires

$$\bar{A}^{rs}=0\ , \tag{D10}$$

$$\bar{B}^r=0\ , \tag{D11}$$

$$\bar{C}=0\ . \tag{D12}$$

From (D7), (D10), and (D2) one finds

$$C^{(\kappa\lambda)}=0\ , \tag{D13}$$

and from (D7)—(D12) and (D4) one finds

$$C^{(\kappa\lambda)}=0,\quad C^\lambda=0,\quad C=0\ . \tag{D14}$$

Thus, the requirement (D1) supplemented by the conditions (D2) implies (D3), and the requirement (D1) supplemented by the conditions (D4) implies (D5).

[1]A. Einstein, *The Meaning of Relativity,* 5th ed. (Princeton University Press, Princeton, New Jersey, 1955), Appendix II, pp. 133–166.

[2]C. R. Johnson and J. R. Nance, Phys. Rev. D **15**, 377 (1977); **16**, 553(E) (1977).

[3]C. R. Johnson, Phys. Rev. D **24**, 327 (1981).

[4]C. R. Johnson, Phys. Rev. D **4**, 295 (1971); **4**, 318 (1971); **4**, 3555 (1971); **5**, 282 (1972); **5**, 1916 (1972); **7**, 2825 (1973); **7**, 2838 (1973); **8**, 1645 (1973). In subsequent references we shall refer to these papers as papers RI–RVIII. For a discussion of earlier work by various other authors who developed methods for finding Lorentz-covariant equations of structure and motion of particles in Einstein's gravitational theory see papers RI, RV, and RVI. See also P. Havas and J. N. Goldberg, Phys. Rev. **128**, 398 (1962).

[5]The definition of a flat region of space-time is given in Sec. II B.

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[7]R. P. Kerr, Phys. Rev. Lett. **11**, 237 (1963).

[8]M. Wyman, Can. J. Math. **2**, 427 (1950). The interaction of Wyman particles is studied in Ref. 2.

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[10]See papers RVI and RVII of Ref. 4.

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[12]M. Mathisson, Acta Phys. Polon. **6**, 163 (1937).

[13]A. Papapetrou, Proc. R. Soc. London **A209**, 248 (1951).

[14]We restrict our study to neutral particles for which all the electromagnetic multipole moments vanish.

[15]See Ref. 2. See also the Introduction to paper RI, Appendix C of paper RII, and the Introduction to paper RVIII of Ref. 4.

[16]As mentioned earlier, we shall find that the test particle limit is physically meaningful only for those particles which are sufficiently far from other particles so that the approximation method discussed earlier in this Introduction is valid.

[17]The notation

$$A_{\cdots(\mu\nu)\cdots}=\tfrac{1}{2}(A_{\cdots\mu\nu\cdots}+A_{\cdots\nu\mu\cdots})\ ,$$

$$A_{\cdots[\mu\nu]\cdots}=\tfrac{1}{2}(A_{\cdots\mu\nu\cdots}-A_{\cdots\nu\mu\cdots})\ ,$$

$$A_{[\mu\nu,\lambda]}=(A_{[\mu\nu],\lambda}+A_{[\nu\lambda],\mu}+A_{[\lambda\mu],\nu})\ ,$$

will be used in this paper. If the indices associated with any quantity are enclosed in parentheses, that quantity is understood to be symmetric with respect to the interchange of those indices. Lower case Greek indices take the values 1–4; lower case Latin indices take the values 1–3. The Levi-Civita symbols $\epsilon^{\mu\nu\rho\sigma}$ and $\epsilon_{\mu\nu\rho\sigma}$ will be chosen so that $\epsilon^{1234}=1$ and $\epsilon_{1234}=1$. The Minkowski metric $\eta_{\mu\nu}=\eta^{\mu\nu}$ will be defined through the equations $\eta_{st}=-\delta_{st}$, $\eta_{s4}=\eta_{4s}=0$, $\eta_{44}=1$.

[18]A. Einstein, Can. J. Math. **2**, 120 (1950).

[19]Sufficient conditions for the existence of $a_{\mu\nu}$ and $a^{\mu\nu}$ and for $a\neq 0$ are discussed in Appendix A.

[20]A. Einstein, *The Meaning of Relativity,* 4th ed. (Princeton University Press, Princeton, New Jersey, 1953), Appendix II, pp. 133–165.

[21]In addition to assuming the $\Gamma^{\rho}_{\mu\nu}$ are uniquely determined as a function of $g_{\mu\nu}$ and the derivatives of $g_{\mu\nu}$ through (2.2), we are also assuming that the determinate of $g_{\mu\nu}$ does not vanish. Under these conditions one can show that $g^{\mu\nu}$ and also $a_{\mu\nu}$ and $a^{\mu\nu}$ exist, and that the determinant of $a_{\mu\nu}$ does not vanish. See Appendix A.

[22]The world line of a particle is defined in the Introduction to this paper.

[23]The splitting of a field in the vicinity of a particle into a self-field and an external field is discussed in Appendix B of paper RI of Ref. 4.

[24]The procedure used to obtain Eqs. (5.2) is developed and illustrated in papers RI–RVIII of Ref. 4.

[25]Through (5.2b) and (5.24), we have defined the field $s^{\rm int}_{\mu}$ in this paper to be the negative of the field $s^{\rm int}_{\mu}$ in paper RVII of Ref. 4. Because of this, the term $\frac{1}{3}\eta_{\mu\rho}\epsilon^{\rho\sigma\kappa\lambda}R^{N}_{[\kappa\lambda,\sigma]}$ appears in $s^{\rm int}_{\mu}$ with opposite sign in the two papers.

[26]See paper RI of Ref. 4.

[27]See Appendix B.

[28]See Appendix C.

[29]In Appendix D we show that if

$$\int C^{\nu}(s)\delta_{,\nu}(x-\xi)ds+\int C(s)\delta(x-\xi)ds=0\ ,$$

where $C^{\mu}U_{\mu}=0$, then $C^{\mu}=0$, $C=0$.

[30]See Ref. 13. Earlier, Mathisson had found a particular case of Eqs. (6.41)–(6.43). See Ref. 12.

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