Cosmologies with quasiregular singularities. II. Stability considerations

D. A. Konkowski
Center for Relativity, Department of Physics, University of Texas, Austin, Texas 78712 and Department of Physics and Astronomy, University of Maryland, College Park, Maryland 20742

T. M. Helliwell
Department of Physics, Harvey Mudd College, Claremont, California 91711
(Received 21 September 1984)

The stability properties of a class of spacetimes with quasiregular singularities is discussed. Quasiregular singularities are the end points of incomplete, inextendible geodesics at which the Riemann tensor and its derivatives remain at least bounded in all parallel-propagated orthonormal (PPON) frames; observers approaching such a singularity would find that their world lines come to an end in a finite proper time. The Taub-NUT (Newman-Unti-Tamburino)-type cosmologies investigated are $R^1 \times T^3$ and $R^3 \times S^1$ flat Kasner spacetimes, the two-parameter family of spatially homogeneous but anisotropic Bianchi type-IX Taub-NUT spacetimes, and an infinite-dimensional family of Einstein-Rosen-Gowdy spacetimes studied by Moncrief. The behavior of matter near the quasiregular singularity in each of these spacetimes is explored through an examination of the behavior of the stress-energy tensors and scalars for conformally coupled and minimally coupled, massive and massless scalar waves as observed in both coordinate and PPON frames. A conjecture is postulated concerning the stability of the nature of the singularity in these spacetimes. The conjecture for a Taub-NUT-type background spacetime is that if a test-field stress-energy tensor evaluated in a PPON frame mimics the behavior of the Riemann tensor components which indicate a particular type of singularity (quasiregular, nonscalar curvature, or scalar curvature), then a complete nonlinear backreaction calculation, in which the fields are allowed to influence the geometry, would show that this type of singularity actually occurs. Evidence supporting the conjecture is presented for spacetimes whose symmetries are unchanged when fields with the same symmetries are added. The conjecture and the exact solutions which support it both indicate that most waves mimic scalar curvature singularities; only very special wave modes mimic nonscalar curvature or quasiregular singularities. Therefore if general fields are added to the idealized empty Taub-NUT-type cosmologies, one would expect the quasiregular singularities to be converted into scalar curvature singularities.

I. INTRODUCTION

The nature of singularities which occur in the classical solutions of Einstein's field equations is largely an unexplored area. The singularity theorems which predict the occurrence of singularities in broad classes of spacetimes give little clue to the nature of these singularities. We must therefore investigate spacetimes with all varieties of singularities together with their stability properties in order to understand more fully their relevance both within the mathematical structure of Einstein's equations and within the physical universe.

A singularity in a maximal[1] spacetime (i.e., a connected, C^∞, Hausdorff manifold M together with a Lorentzian metric $g_{\mu\nu}$) is indicated by incomplete geodesics or incomplete curves of bounded acceleration.[2] The obstacle which bars the embedding of singular spacetimes in larger nonsingular spacetimes is obvious in those cases where physical quantities (e.g., energy density and tidal forces) diverge for all observers who encounter the singularity. However, not all singularities which occur in exact solutions to Einstein's equations are of this type.

Under a classification scheme devised by Ellis and Schmidt,[3,4] singularities in maximal, four-dimensional spacetimes are divided into three basic types: quasiregular, nonscalar curvature, and scalar curvature. Exact mathematical descriptions of each type of singularity can be found in the articles by Ellis and Schmidt,[3,4] or a summary of the description may be found in the preceding paper by Konkowski, Helliwell, and Shepley[5] which we refer to as paper I. Only in the case of a scalar curvature singularity do scalar curvature invariants diverge and physical quantities become infinite as described above. The physical significance of the other two types of singularity is less obvious. For a nonscalar curvature singularity the Riemann tensor or its derivatives diverge in some, but not all, parallel-propagated orthonormal (PPON) frames; that is, some, but not all, observers feel infinite tidal forces as they approach the singularity. It is even more curious that for a quasiregular singularity the Riemann tensor and its derivatives are bounded in all PPON frames, so no observers see physical quantities diverge as they approach the singularity.

When we discuss the stability of a spacetime (or class of spacetimes) with one of these types of singularity, we are concerned with whether the nature of the singularity is a stable feature of the spacetime (or class of spacetimes). In particular, we are concerned with the likelihood of the different singularity types, both as classes within all mathematical solutions of the field equations and in astrophysically relevant situations. No single technique is used to test the stability of such a singular spacetime; it may be stable in many different and not necessarily equivalent ways. It may satisfy a mathematical "genericness" condition; it may be unaffected by a linear perturbation; it may appear unchanged if test (classical or quantum) matter is added; or it may be that a similar exact solution, usually one with the same symmetries but containing different source fields, behaves in the same manner. Since general relativity is a nonlinear theory and since no unambiguous back-reaction scheme exists, all one can do is take the results of these analyses to give some indication of the likelihood of the different kinds of singular spacetimes.

In this paper we are concerned with the stability properties of spacetimes with quasiregular singularities. We continue the discussion of the global properties of such spacetimes and the test-field behavior within them which was begun in paper I. There we focus upon the known exact cosmological solutions to Einstein's field equations which possess quasiregular singularities. All such models we term Taub-NUT (Newman-Unti-Tamburino)-type[6] since, in common with the original Taub-NUT spacetime, they are characterized by incomplete geodesics which spiral infinitely around a topologically closed spatial dimension. One or more quasiregular singularities, each of which appears as a topological defect in the spacetime, is present in all maximal, Hausdorff and non-Hausdorff extensions of each Taub-NUT-type cosmology.[5] As in paper I, we focus upon one maximal Hausdorff extension for each spacetime, and the spacetimes discussed are each a member of one of three classes of Taub-NUT-type cosmologies: $R^1\times T^3$ and $R^3\times S^1$ flat Kasner spacetimes,[7] the two-parameter family of spatially homogeneous but anisotropic Bianchi type-IX Taub-NUT spacetimes,[1,8-10] or the infinite-dimensional family of spatially inhomogeneous Moncrief universes[11] which are a subclass of Einstein-Rosen-Gowdy spacetimes.[12-14]

Here we investigate the likelihood of quasiregular singularities by studying their stability in each of the Taub-NUT-type cosmologies. Within spatially homogeneous cosmologies, quasiregular singularities are generally considered to be unstable (see, e.g., the reviews by Ellis and Schmidt[4] and by Tipler, Clarke, and Ellis[15]). Within spatially inhomogeneous spacetimes, however, the situation is less clear, and the existence of infinite-dimensional families of spacetimes[11,16,17] with quasiregular singularities increases one's curiosity about the general stability properties of such spacetimes.

As discussed by Clarke[18,19] and in related review papers,[3,15] quasiregular singularities (holes and primeval quasiregular singularities) cannot be ruled out in spacetime models on physical or mathematical grounds; however, specialized quasiregular singularities (which can only occur if the spacetime is Petrov type D, O, or N, and electrovac or with "unrealistic" negative pressure or density) are considered somewhat unlikely in realistic cosmological models because of their specialized nature. Those quasiregular singularities corresponding to imprisoned incompleteness[2] are necessarily specialized. Finally, since specialized quasiregular singularities can be considered the milder vacuum analog of a particular nonscalar curvature singularity, the whimper,[20] a result by Siklos[21] implies the unlikelihood of these cosmologies among all spatially homogeneous cosmologies.

These general considerations are supported by studies of individual classes of spacetimes, especially by studies of Taub-NUT-type cosmologies. In $R^1\times T^3$ and $R^3\times S^1$ flat Kasner spacetimes, test fields[5,6] and vacuum polarization effects[22] appear to have a destabilizing influence on the nature of the singularity. The quasiregular singularities of Taub-NUT spacetimes also appear to be unstable: Test fields[5,6,23] and test matter[1,23] both classical and quantum build up near the singularity, and the addition of matter as a source for spacetimes with similar symmetries[25-27] appears to produce, in general, a curvature singularity. In addition, in the Moncrief universes, a linear perturbation[28] appears to change the nature of the singularity, and a quantum Moncrief universe[29] appears to be unstable.

Here we shall take a slightly different approach to look at the stability of the singularity structure in Taub-NUT-type cosmologies. Our approach will utilize the test fields whose behavior in these cosmologies was studied in paper I. Instead of simply taking the divergence of a field amplitude to indicate an instability in one of these spacetimes,[30] we examine the behavior of stress-energy tensor components since they play the role of the source in Einstein's field equations. In particular, we make predictions about the stability of Taub-NUT-type quasiregular singularities under back-reaction, based on the behavior of stress-energy scalars and tensors calculated in PPON frames.

We say that a test field mimics a particular type of singularity if the stress-energy scalars and tensors behave in a way analogous to the behavior of the curvature scalars and tensors used in the Ellis and Schmidt classification scheme.[4] We then conjecture that if a test field mimics a particular type of singularity on a Taub-NUT-type background spacetime, then a full back-reaction calculation would convert the Taub-NUT-type singularity present into the type of singularity mimicked. A similar hypothesis was advanced in a previous paper[31] for two-dimensional models of these cosmologies, but in two dimensions there is no unambiguous way to couple matter and geometry, and there is no way to test the back-reaction conjecture. Here, on the other hand, we can test our conjecture by using field modes with symmetries identical to or easily generalized from the symmetries of the background cosmology.

The plan of this paper is as follows. In Sec. II we briefly summarize those properties of Taub-NUT-type cosmologies with massive and massless, conformally and minimally coupled scalar test fields derived in paper I which are needed for the discussion to follow. In Sec. III we discuss the divergence properties of the corresponding stress-energy tensors and scalars. In Sec. IV we present a stability conjecture regarding the effect of the back-reaction of these fields on the singularity structure of the

Taub-NUT-type spacetimes. In Sec. V we test this conjecture using exact spatially homogeneous and spatially inhomogeneous solutions of the Einstein-Maxwell-scalar field equations. In Sec. VI we discuss our results and list avenues for future research.

II. TAUB-NUT-TYPE COSMOLOGIES WITH SCALAR TEST FIELDS

In this section we briefly summarize those results of paper I which are needed for the discussion to follow. Here, as in paper I, we restrict detailed discussion to one maximal Hausdorff extension for each of the three classes of Taub-NUT-type spacetimes:

1. Flat Kasner spacetime. One maximal Hausdorff extension of the flat Kasner universe[7] on the manifolds $R^1 \times T^3$ or $R^3 \times S^1$ is defined by the metric

$$ds^2 = 2d\psi\, dt + 2t\, d\psi^2 + d\theta^2 + d\phi^2 . \tag{2.1}$$

The coordinate ranges for the $R^1 \times T^3$ cosmology are $-\infty < t < \infty$, $0 \le \psi < 2\pi$, $0 \le \theta < 2\pi$, and $0 \le \phi < 2\pi$. For the $R^3 \times S^1$ cosmology, the ψ coordinate is periodic with $0 \le \psi < 2\pi$, while $-\infty < t < \infty$, $-\infty < \theta < \infty$, and $-\infty < \phi < \infty$. In either cosmology the hypersurface $t = 0$ is a null hypersurface which separates a noncausal spatially inhomogeneous region $(t < 0)$ from a causal spatially homogeneous region $(t > 0)$. The null hypersurface at $t = 0$ is both a Cauchy horizon and a Killing horizon; in addition, it contains a quasiregular singularity in the sense that incomplete geodesics hit $t = 0$ at $\psi = \infty$. The metric of Eq. (2.1) is that for one of the two maximal Hausdorff extensions of the original flat Kasner metric

$$ds^2 = -dt^2 + t^2 d\psi^2 + d\theta^2 + d\phi^2 \;(t > 0) .$$

The extension corresponding to the metric of Eq. (2.1) is based upon Regions I + III of the Minkowski

$$[ds^2 = -(dx^0)^2 + (dx^1)^2 + (dx^2)^2 + (dx^3)^2]$$

covering space shown in Fig. 1. The half-line $x^1 = -x^0$ $(x^0 > 0)$, together with the boundary line $x^1 = x^0$, maps into the $t = 0$ null hypersurface in the extended flat Kasner geometry; the boundary line (which corresponds to $\psi = +\infty$) is the site of the quasiregular singularity. The alternate Hausdorff extension, which is based upon Regions I + II rather than I + III, has the metric of Eq. (2.1) except that the first term is negative; the quasiregular singularity in that case arises from the boundary line $x^1 = -x^0$. A non-Hausdorff extension is also possible, which includes all four regions of the Minkowski covering space, modulo the action of the isometry group of this spacetime, but which excludes the point $P = (x^0, x^1) = (0,0)$. This point is an essential quasiregular singularity, because it is singular in all of the extensions; the other boundary points in the Hausdorff extensions form a nonessential quasiregular singularity.

2. Taub-NUT spacetimes. The metric for one family of maximal Hausdorff extensions of this two-parameter family of spatially homogeneous, anisotropic Bianchi type -IX solutions[1] to the vacuum Einstein equations is

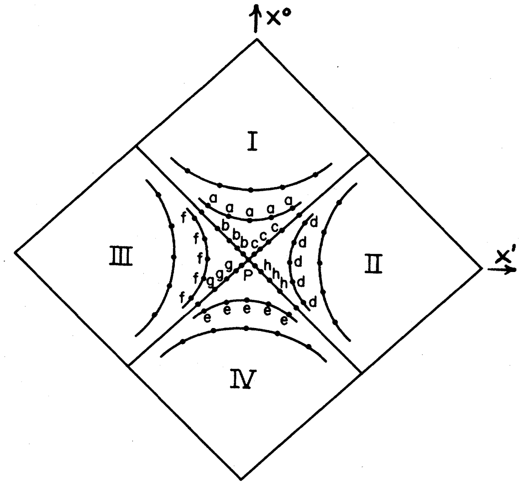

FIG. 1. Penrose diagram of the covering space of a (t, ψ) slice of the flat Kasner universes. The R^4 flat Kasner spacetime is only a portion (region I) of Minkowski spacetime (regions I + II + III + IV). For $R^1 \times T^3$ or $R^3 \times S^1$ flat Kasner spacetimes, the points a are identified, the points b are identified, etc., and the original spacetimes can be extended in two inequivalent Hausdorff ways across the null hypersurfaces shown: One extension is given by regions I + II, and the other by regions I + III. A maximally extended non-Hausdorff spacetime consists of regions I + II + III + IV with the point P removed.

$$ds^2 = 2(2l)(d\psi + \cos\theta\, d\phi)dt + U(t)(2l)^2(d\psi + \cos\theta\, d\phi)^2 + (t^2 + l^2)(d\theta^2 + \sin^2\theta\, d\phi^2) , \tag{2.2}$$

where

$$U(t) = -1 + \frac{2(mt + l^2)}{t^2 + l^2} \tag{2.3}$$

and m and l are constants. The manifold for these cosmologies is $R^1 \times S^3$ where $-\infty < t < \infty$, and ψ, θ, ϕ are Euler angles on the three-sphere with $0 \le \psi < 4\pi$, $0 \le \theta \le \pi$, and $0 \le \phi < 2\pi$. The null hypersurfaces[10] at $U(t) = 0$ [i.e., at $t = t_\pm = m \pm (m^2 + l^2)^{1/2}$] are also Cauchy and Killing horizons; they separate the causal spatially homogeneous Taub universe[8] $(t_- < t < t_+)$ from the noncausal spatially inhomogeneous Newman-Unti-Tamburino (NUT) cosmologies[9] $(t < t_-$ or $t > t_+)$. Each $t = t_\pm$ null hypersurface contains a quasiregular singularity in the same sense as the $R^1 \times T^3$ and $R^3 \times S^1$ flat Kasner spacetimes.

3. Moncrief spacetimes. The metric[11] for one family of maximal Hausdorff extensions of this infinite-dimensional family of inhomogeneous vacuum solutions to the field equations is

$$ds^2 = -\frac{1}{4t}(e^{2a} - e^{2b})(dt)^2 + \tfrac{1}{2} e^{2b} d\psi\, dt + \frac{t}{4} e^{2b}(d\psi)^2 + e^{2a}(d\theta)^2 + e^{-2b}(d\phi)^2 , \tag{2.4}$$

where

$$b(t,\theta)=\sum_{n=1}^{\infty} a_n J_0(nt^{1/2})\sin(n\theta+\gamma_n)\ , \tag{2.5}$$

$$a(t,\theta)=b(t,\theta)+\frac{1}{2}\int_0^t ds\left[4s\left[\frac{\partial b(s,\theta)}{\partial s}\right]^2+\left[\frac{\partial b(s,\theta)}{\partial \theta}\right]^2\right]. \tag{2.6}$$

Here a_n and γ_n are constants and J_0 is a regular Bessel function of zeroth order. The manifold is $R^1\times T^3$ with $-\infty<t<\infty$ and $0\le\psi<2\pi$, $0\le\theta<2\pi$, and $0\le\phi<2\pi$. To avoid the question of convergence of the series for $b(t,\theta)$, only a finite number of coefficients $\{a_n\}$ are assumed to be nonzero. If all the a_n are zero, the resulting spatially homogeneous spacetime is an $R^1\times T^3$ flat Kasner spacetime.

The Moncrief cosmologies are Einstein-Rosen-Gowdy spacetimes[12–14] which do not have any curvature singularities. Each Moncrief universe does have a quasiregular singularity located in the $t=0$ null hypersurface (at $\psi=\infty$); the null hypersurface is also a Cauchy horizon and a Killing horizon.

In later sections we will find it useful to let T be the length of time away from the null hypersurfaces which occur in each of these spacetimes. In the flat Kasner and Moncrief cosmologies there is one null hypersurface at $T=t=0$, while in the Taub-NUT cosmologies two null hypersurfaces occur, at $T=t-t_\pm=0$.

In paper I, we showed the similarities in wave behavior near the null hypersurface at $T=0$ in all three classes of universes. The complete wave solution to the massive ($M\neq0$) or massless ($M=0$), conformally coupled ($\xi=\frac{1}{6}$) or minimally coupled ($\xi=0$) scalar wave equation,

$$(\Box-\xi R-M^2)\Phi=0\ , \tag{2.7}$$

where $\Box$ is the Laplace-Beltrami operator, was given as the sum

$$\Phi=\Phi^{(0)}+\Phi^{(1)}+\Phi^{(2)}\ , \tag{2.8}$$

where

$$\Phi^{(0)}=\ln T\sum_{\lambda,\mu}\left[\sum_n a_n^{0\lambda\mu}T^n\right]f_{0\mu}^{\lambda}(\theta,\phi)\ , \tag{2.8a}$$

$$\Phi^{(1)}=\sum_{\kappa,\lambda,\mu}\left[\sum_n b_n^{\kappa\lambda\mu}T^n\right]e^{i\kappa\alpha\ln T}f_{\kappa\mu}^{\lambda}(\psi,\theta,\phi)\ , \tag{2.8b}$$

$$\Phi^{(2)}=\sum_{\kappa,\lambda,\mu}\left[\sum_n a_n^{\kappa\lambda\mu}T^n\right]f_{\kappa\mu}^{\lambda}(\psi,\theta,\phi)\ . \tag{2.8c}$$

Here the constant α and the functions $f_{\kappa\mu}^{\lambda}$ depend on the spacetime:

$$\alpha=\begin{cases}1 \ \ \text{(flat Kasner)}\ ,\\ 2 \ \ \text{(Moncrief)}\ ,\\ t_\pm/l \ \ \text{(Taub-NUT)}\end{cases} \tag{2.9}$$

and

$$f_{\kappa\mu}^{\lambda}(\psi,\theta,\phi)=\begin{cases}e^{i(\kappa\psi+\lambda\theta+\mu\phi)} \ \ \text{(flat Kasner or Moncrief)}\ ,\\ e^{i\kappa\psi}d_{\kappa\mu}^{\lambda}(\theta)e^{i\mu\phi} \ \ \text{(Taub-NUT)}\ ,\end{cases} \tag{2.10}$$

where the $d_{\kappa\mu}^{\lambda}$ are Wigner functions.[32]

For $R^1\times T^3$ flat Kasner and the Moncrief universes the summation sign denotes

$$\sum_{\kappa,\lambda,\mu}=\sum_{\kappa=-\infty}^{\infty}\ \sum_{\lambda=-\infty}^{\infty}\ \sum_{\mu=-\infty}^{\infty}\ ; \tag{2.11}$$

for $R^3\times S^1$ flat Kasner, where ψ is the only periodic coordinate, it includes two integrations

$$\sum_{\kappa,\lambda,\mu}=\sum_{\kappa=-\infty}^{\infty}\int_{-\infty}^{\infty}d\lambda\int_{-\infty}^{\infty}d\mu\ ; \tag{2.12}$$

and for the Taub-NUT cosmologies it signifies

$$\sum_{\kappa,\lambda,\mu}=\sum_{\lambda=0}^{\infty}\ \sum_{\mu=-\lambda}^{\lambda}\ \sum_{\kappa=-\lambda}^{\lambda}+\sum_{\lambda=1}^{\infty}\ \sum_{\substack{\mu=-\lambda\\(\mu\neq0)}}^{\lambda}\ \sum_{\kappa=-|\mu|+\frac{1}{2}}^{|\mu|-\frac{1}{2}}+\sum_{\lambda=\frac{1}{2}}^{\infty}\ \sum_{\kappa=-\lambda}^{\lambda}\ \sum_{\mu=-|\kappa|+\frac{1}{2}}^{|\kappa|-\frac{1}{2}}\ , \tag{2.13}$$

where μ is always an integer, λ has integer values in the first two terms and half (odd) integer values in the third term, and κ has integer values in the first term and half (odd) integer values in the second and third terms.

The coefficients $a_n^{\kappa\lambda\mu}$ and $b_n^{\kappa\lambda\mu}$ are different in each cosmology. Simple recursion relations for the a_n's and b_n's exist for both the flat Kasner and Taub-NUT cases, since the wave equation is separable. The procedure used to determine the coefficients for waves on the Moncrief universes is more complicated, however, since the wave equation is not separable in the coordinates t and θ. In fact, for Moncrief waves, terms of fixed λ (i.e., individual modes of Φ) are not generally solutions of the wave equation; only for massless waves with no ψ or ϕ dependence are individuals modes also solutions. For more details see paper I.

Finally, it is important to note the behavior of the different types of modes in Eq. (2.8) near the null hypersurface at $T=0$: The $\Phi^{(0)}$ modes diverge in amplitude, the $\Phi^{(1)}$ modes diverge in phase, but the $\Phi^{(2)}$ modes do not diverge in any way. Since the only singularities in these maximal Hausdorff spacetimes are quasiregular ones at $T=0$, one might expect that a generic wave perturbation would destroy the mild nature of these singularities and turn them into curvature singularities. This supposition is considered in more detail in the following sections.

III. DIVERGENCE OF THE STRESS-ENERGY TENSORS

The wave equation for a massive scalar field Φ with coupling ξ is given by Eq. (2.7),

$$(\Box-M^2-\xi R)\Phi=0\ .$$

Since each Taub-NUT-type cosmology under consideration is a vacuum solution of the Einstein equations and thus Ricci flat, Eq. (2.7) and its solutions are independent of curvature coupling. The corresponding stress-energy tensor[33,34] is

$$T_{\mu\nu}=\frac{1}{4\pi}[(1-2\xi)\Phi_{,\mu}\Phi_{,\nu}+(2\xi-\tfrac{1}{2})g_{\mu\nu}(S+M^2\Phi^2)$$
$$-2\xi\Phi\Phi_{;\mu\nu}+\xi(R_{\mu\nu}-\tfrac{1}{2}g_{\mu\nu}R+2\xi g_{\mu\nu}R)\Phi^2]\ , \tag{3.1}$$

where $S=g^{\alpha\beta}\Phi_{,\alpha}\Phi_{,\beta}$. $T_{\mu\nu}$ is not independent of the curvature coupling ξ even though, since $R=0$, the last three terms are absent. We will investigate separately the divergence properties of the minimally coupled ($\xi=0$) tensor $T^m_{\mu\nu}$ and the conformally coupled ($\xi=\frac{1}{6}$) tensor $T^c_{\mu\nu}$.

A. The minimally coupled scalar field stress-energy tensor

The stress-energy tensor in the minimally coupled case is

$$T^m_{\mu\nu}=\frac{1}{4\pi}[\Phi_{,\mu}\Phi_{,\nu}-\tfrac{1}{2}g_{\mu\nu}(S+M^2\Phi^2)]\ . \tag{3.2}$$

The divergent terms in this tensor can be found by first writing Φ^2 and the derivatives $\Phi_{,\mu}$ as sums of terms of decreasingly divergent order in T. Since Φ can be separated into three mode types $\Phi=\Phi^{(0)}+\Phi^{(1)}+\Phi^{(2)}$ for each cosmology, the derivatives are then

$$\Phi_{,0}=\left[\Phi_0^0\left[\frac{1}{T}\right]+\Phi_0^1\left[\frac{1}{T}\right]\right]+\Phi_0^0(\ln T)$$
$$+[\Phi_0^0(1)+\Phi_0^1(1)+\Phi_0^2(1)]$$
$$+\Phi_0^0(T\ln T)+\cdots\ , \tag{3.3a}$$

$$\Phi_{,1}=[\Phi_1^1(1)+\Phi_1^2(1)]+[\Phi_1^1(T)+\Phi_1^2(T)]+\cdots\ , \tag{3.3b}$$

$$\Phi_{,2}=\Phi_2^0(\ln T)+[\Phi_2^1(1)+\Phi_2^2(1)]+\Phi_2^0(T\ln T)+\cdots\ , \tag{3.3c}$$

$$\Phi_{,3}=\Phi_3^0(\ln T)+[\Phi_3^1(1)+\Phi_3^2(1)]+\Phi_3^0(T\ln T)+\cdots\ , \tag{3.3d}$$

where each symbol Φ_j^i refers to a specific set of terms. The superscript indicates which set of modes $\Phi^{(0)}$, $\Phi^{(1)}$, or $\Phi^{(2)}$ has been differentiated; the subscript $(0,1,2,3)=(t,\psi,\theta,\phi)$ indicates which derivative has been taken; and the argument indicates the time dependence of these terms. For example,

$$\Phi_0^0\left[\frac{1}{T}\right]=\frac{1}{T}\sum_{\lambda,\mu}a_0^{0\lambda\mu}f_{0\mu}^{\lambda}(\theta,\phi)\ ,$$

which are the terms of $\Phi^{(0)}_{,0}$ whose time dependence is $1/T$.

From the derivatives of Φ we find that S may be written

$$S=S_1/T+S_2\ln^2T+S_3\ln T+\text{convergent terms}\ , \tag{3.4}$$

where the first term is

$$S_1/T=-2T\left[\Phi_0^0\left[\frac{1}{T}\right]+\Phi_0^1\left[\frac{1}{T}\right]\right]^2$$
$$+2\left[\Phi_0^0\left[\frac{1}{T}\right]+\Phi_0^1\left[\frac{1}{T}\right]\right][\Phi_1^1(1)+\Phi_1^2(1)] \tag{3.5}$$

in the flat Kasner case. Similar expressions for $S_2\ln^2T$ and $S_3\ln T$ are easily found. The results are slightly different in the Taub-NUT and Moncrief geometries, because of the metric tensor in the definition of S.

Finally, we need the divergent terms of Φ^2, which are logarithmic terms arising from the divergence of $\Phi^{(0)}$. That is,

$$\Phi^2=B_1\ln^2T+B_2\ln T+\text{convergent terms}\ , \tag{3.6}$$

where

$$B_1=\left[\sum_{\lambda,\mu}a_0^{0\lambda\mu}f_{0\mu}^{\lambda}(\theta,\phi)\right]^2$$

and

$$B_2=2\left[\sum_{\lambda,\mu}a_0^{0\lambda\mu}f_{0\mu}^{\lambda}(\theta,\phi)\right]\times\left[\sum_{\kappa,\lambda,\mu}(a_0^{\kappa\lambda\mu}+b_0^{\kappa\lambda\mu}e^{i\kappa\alpha\ln T})\right]f_{\kappa\mu}^{\lambda}(\psi,\theta,\phi)\ .$$

From these results it is easy to find the most divergent terms in each element of the symmetric stress-energy tensor $T^m_{\mu\nu}$ in all of the Taub-NUT-type cosmologies:

$$T^m_{\mu\nu}\sim\begin{bmatrix}T^{-2} & T^{-1} & T^{-1}\ln T & T^{-1}\ln T\\ & T^0 & \ln T & \ln T\\ & & T^{-1} & \ln^2T\\ & & & T^{-1}\end{bmatrix}\ . \tag{3.7}$$

Of more interest is the behavior of the stress-energy tensor in a parallel-propagated orthonormal (PPON) frame $e^{\mu}_{(\alpha)}$ carried by a generic freely falling observer approaching the null hypersurface. Here by a generic observer we mean one whose path does not end at $T=0$ in the essential quasiregular singularity but rather an observer whose path is extendible across $T=0$ (i.e., a pass-through observer) or one whose path ends at the nonessential quasiregular singularity at $T=0$ (i.e., a spiraling observer). Results for nongeneric observers will be stated in our final theorems, but for simplicity are not derived here. In a PPON frame carried by a generic observer, the stress-energy tensor is[35]

$$T^m_{(\alpha\beta)}=e^{\mu}_{(\alpha)}e^{\nu}_{(\beta)}T^m_{\mu\nu} \tag{3.8}$$

$$=\frac{1}{4\pi}[(e_{(\alpha)}{}^{\mu}\Phi_{,\mu})(e^{\nu}_{(\beta)}\Phi_{,\nu})$$
$$-\tfrac{1}{2}(e^{\mu}_{(\alpha)}e_{(\beta)\mu})(S+M^2\Phi^2)]\ , \tag{3.9}$$

where $e^{\mu}_{(\alpha)}e_{(\beta)\mu}=-1$ if $\alpha=\beta=0$; $=+1$ if $\alpha=\beta=1$, 2, or 3; and $=0$ if $\alpha\neq\beta$. The frame vectors $e^{\mu}_{(\alpha)}$ differ for generic spiraling and pass-through observers; they are listed for the two cases in the Appendix, for the flat Kasner, Taub-NUT, and Moncrief spacetimes. In the Moncrief case, because of the relative difficulty in finding geodesic paths near the singularity, we have found explicitly only the vector $e^{\mu}_{(0)}$, including only terms of order T^{-1} and of order unity.

The energy density in a PPON frame is

$$T^{m}_{(00)}=\frac{1}{4\pi}[(e^{\mu}_{(0)}\Phi_{,\mu})^2+\tfrac{1}{2}(S+M^2\Phi^2)]\ , \tag{3.10}$$

and the frame vector $e^{\mu}_{(0)}$ can be expanded in decreasingly divergent powers of time as follows:

$$e^{\mu}_{(0)}=\begin{bmatrix} e^0_0(1)+e^0_0(T)+\cdots \\ e^1_0(1/T)+e^1_0(1)+e^1_0(T)+\cdots \\ e^2_0(1) \\ e^3_0(1)\end{bmatrix}\ , \tag{3.11}$$

where the superscripts refer to the component of $e^{\mu}_{(0)}$, and the argument refers to the term's time dependence. The value of each term can be found by referring to the Appendix. In particular, the quantity $e^1_0(1/T)=0$ in a pass-through frame.

In order to find all divergent terms of $T^{m}_{(00)}$, we must find all divergent terms in $(e^{\mu}_{(0)}\Phi_{,\mu})^2$; therefore because the most divergent term of $e^{\mu}_{(0)}\Phi_{,\mu}$ is of order $1/T$, we must keep terms up to order $T\ln T$ in this quantity. The result is

$$(e^{\mu}_{(0)}\Phi_{,\mu})^2=A_1/T^2+A_2(\ln T)/T+A_3/T +A_4\ln^2T+A_5\ln T+\text{convergent terms}\ , \tag{3.12}$$

where, for example,

$$A_1/T^2=\left\{e^0_0(1)\left[\Phi^0_0\left[\frac{1}{T}\right]+\Phi^1_0\left[\frac{1}{T}\right]\right] +e^1_0\left[\frac{1}{T}\right][\Phi^1_1(1)+\Phi^2_1(1)]\right\}^2 . \tag{3.13}$$

Similar expressions for the other divergent terms are readily found. The energy density therefore becomes

$$T^{m}_{(00)}=\frac{1}{4\pi}[A_1/T^2+A_2\ln T/T+(A_3+\tfrac{1}{2}S_1)/T+(A_4+\tfrac{1}{2}S_2-\tfrac{1}{2}M^2B_1)\ln^2T+(A_5+\tfrac{1}{2}S_3-\tfrac{1}{2}M^2B_2)\ln T] +\text{convergent terms}\ , \tag{3.14}$$

which converges if and only if each of the divergent terms of given time order vanishes. In particular, A_1/T^2 must vanish; this condition constrains which modes of Φ are allowed. From Eq. (3.13) and the fact that the modes of Φ are linearly independent, it follows that $A_1/T^2=0$ implies the three conditions:

(1) $\Phi^0_0\left[\frac{1}{T}\right]=0$,

(2) $e^0_0(1)\Phi^1_0\left[\frac{1}{T}\right]+e^1_0\left[\frac{1}{T}\right]\Phi^1_1(1)=0$,

(3) $e^1_0\left[\frac{1}{T}\right]\Phi^2_1(1)=0$.

Condition (1) forces all coefficients $a_0^{0\lambda\mu}=0$. For pass-through observers, where $e^1_0(1/T)=0$, condition (3) imposes no constraints, but condition (2) forces all coefficients $b_0^{\kappa\lambda\mu}=0$ (unless $\kappa=0$). For spiraling observers, condition (2) imposes no constraints because it is satisfied identically; condition (3) forces all coefficients $a_0^{\kappa\lambda\mu}=0$ (unless $\kappa=0$). In summary, the vanishing of A_1/T^2 implies that only $\Phi^{(1)}$ modes are permitted for spiraling observers, and only $\Phi^{(2)}$ modes are permitted for pass-through observers. This result is both valid and easily obtained in all of the Taub-NUT-type cosmologies. Here we have used the fact that if the coefficients $a_0^{0\lambda\mu}=0$, then the $\kappa=0$ modes within $\Phi^{(1)}$ and $\Phi^{(2)}$ have the same functional form.

It is then straightforward to show in the flat Kasner and Taub-NUT universes that if $A_1/T^2=0$, i.e., if only the modes listed above are permitted, then all other potentially divergent tems in $T^{m}_{(00)}$ vanish as well. That is, the terms $A_2(\ln T)/T,\ldots,\ (A_5+S_3-M^2B_2)\ln T$ are also all zero. The proof simply requires expressions analogous to Eq. (3.13) for A_1/T^2, and the observation that $A_2\ldots A_5$, $S_1\ldots S_3$, B_1, and B_2 all vanish if the above mode constraints are applied. The conclusion is that $T^{m}_{(00)}$ converges if and only if $A_1/T^2=0$. Because of the comparative difficulty of finding the frame vectors for the general Moncrief cosmologies, we have not proved the result in this case, although we believe it is valid. Subsequent results in this section have also been established only in the flat Kasner and Taub-NUT cases.

The next step is to investigate the convergence of $T^{m}_{(\alpha\beta)}$ in general, using Eq. (3.9) for $T^{m}_{(\alpha\beta)}$. If only the above modes are permitted, then we already know that S and $M^2\Phi^2$ converge. In the case of pass-through observers, the quantities $e^{\mu}_{(\alpha)}$ and $\Phi_{,\mu}$ all converge, so clearly $T^{m}_{(\alpha\beta)}$ converges. In the case of spiraling observers, the quantity

$$e^{\mu}_{(1)}\Phi_{,\mu}=e^0_{(1)}\Phi_{,0}+e^1_{(1)}\Phi_{,1}+\text{convergent terms} \tag{3.15}$$

converges because the divergent parts of the first two terms on the right cancel. The quantities $e^{\mu}_{(2)}\Phi_{,\mu}$ and $e^{\mu}_{(3)}\Phi_{,\mu}$ also converge, so $T^{m}_{(\alpha\beta)}$ converges for spiraling observers as well. Therefore, in general, $T^{m}_{(\alpha\beta)}$ converges if and only if

(a) $a_0^{\kappa\lambda\mu}=0$ for a spiraling frame, for all κ,λ,μ;

(b) $a_0^{0\lambda\mu}=0$ and $b_0^{\kappa\lambda\mu}=0$ for a pass-through frame, for all κ,λ,μ except for $\kappa=0$.

B. The conformally coupled scalar field stress-energy tensor

The stress-energy tensor in the conformally coupled case is

$$T^c_{\mu\nu}=\frac{1}{4\pi}[\tfrac{2}{3}\Phi_{,\mu}\Phi_{,\nu}-\tfrac{1}{6}g_{\mu\nu}(S+M^2\Phi^2)-\tfrac{1}{3}\Phi\Phi_{;\mu\nu}] \,. \tag{3.16}$$

The final term involves the quantity

$$\Phi_{;\mu\nu}=\Phi_{,\mu\nu}-\Gamma^{\lambda}_{\mu\nu}\Phi_{,\lambda} \tag{3.17}$$

which requires an analysis of the convergence properties of the second derivatives of Φ. For example, $\Phi_{,00}$ may be written

$$\Phi_{,00}=[\Phi^0_{00}(T^{-2})+\Phi^1_{00}(T^{-2})] +[\Phi^0_{00}(T^{-1})+\Phi^1_{00}(T^{-1})]+\Phi^0_{00}(\ln T) \,, \tag{3.18}$$

in a notation similar to that used in Sec. II A. When the new final term $-\frac{1}{3}\Phi\Phi_{;\mu\nu}$ is combined with the previous results for $T^m_{\mu\nu}$, one finds that the most divergent terms of the elements of the symmetric tensor $T^c_{\mu\nu}$ are as follows, in all of the Taub-NUT-type cosmologies:

$$T^c_{\mu\nu}\sim\begin{bmatrix} T^{-2}\ln T & T^{-1}\ln T & T^{-1}\ln T & T^{-1}\ln T \\ & \ln T & \ln T & \ln T \\ & & T^{-1} & \ln^2 T \\ & & & T^{-1}\end{bmatrix} . \tag{3.19}$$

Of more interest is the tensor evaluated in a PPON frame $e^{\mu}_{(\alpha)}$:

$$\begin{aligned}T^c_{(\alpha\beta)}&=e^{\mu}_{(\alpha)}e^{\nu}_{(\beta)}T^c_{\mu\nu}\\ &=\frac{1}{4\pi}[\tfrac{2}{3}(e^{\mu}_{(\alpha)}\Phi_{,\mu})(e^{\nu}_{(\beta)}\Phi_{,\nu})-\tfrac{1}{6}(e^{\mu}_{(\alpha)}e_{(\beta)\mu})(S+M^2\Phi^2)-\tfrac{1}{3}\Phi(e^{\mu}_{(\alpha)}e^{\nu}_{(\beta)})\Phi_{;\mu\nu}] \,.\end{aligned} \tag{3.20}$$

In particular, the energy density is

$$T^c_{(00)}=\frac{1}{4\pi}[\tfrac{2}{3}(e^{\mu}_{(0)}\Phi_{,\mu})^2+\tfrac{1}{6}(S+M^2\Phi^2)-\tfrac{1}{3}Z] \,, \tag{3.21}$$

where

$$\begin{aligned}Z&=\Phi(e^{\mu}_{(0)}e^{\nu}_{(0)})\Phi_{;\mu\nu}\\ &=Z_1(\ln T)/T^2+Z_2/T^2+Z_3(\ln^2 T)/T+Z_4\ln T/T+Z_5/T+Z_6\ln^2 T+Z_7\ln T+\text{convergent terms} \,.\end{aligned} \tag{3.22}$$

Here, for example,

$$Z_1\ln T/T^2=\Phi(\ln T)\{[e^0_0(1)]^2\Phi_{;00}(T^{-2})+2e^0_0(1)e^1_0(T^{-1})\Phi_{;01}(T^{-1})+[e^1_0(T^{-1})]^2\Phi_{;11}(1)\} \,, \tag{3.23}$$

where $\Phi(\ln T)$ consists of all modes of order $\ln T$ in Φ, $\Phi_{;00}(T^{-2})$ consists of all terms in $\Phi_{;00}$ of order T^{-2}, etc. The energy density Eq. (3.21) is then

$$\begin{aligned}T^c_{(00)}=\frac{1}{4\pi}[&-\tfrac{1}{3}Z_1(\ln T)/T^2+(\tfrac{2}{3}A_1-\tfrac{1}{3}Z_2)T^{-2}-\tfrac{1}{3}Z_3(\ln^2 T)/T+(\tfrac{2}{3}A_2-\tfrac{1}{3}Z_4)(\ln T)/T+(\tfrac{2}{3}A_3+\tfrac{1}{6}S_1-\tfrac{1}{3}Z_6)/T\\ &+(\tfrac{2}{3}A_4+\tfrac{1}{6}S_2+\tfrac{1}{6}M^2B_1-\tfrac{1}{3}Z_6)\ln^2 T+(\tfrac{2}{3}A_5+\tfrac{1}{6}S_3+\tfrac{1}{6}M^2B_2-\tfrac{1}{3}Z_7)\ln T+\text{convergent terms}] \,.\end{aligned} \tag{3.24}$$

We now seek the implications of a convergent $T^c_{(00)}$. In particular, it is straightforward to show that the two most divergent terms (or order $T^{-2}\ln T$ and of order T^{-2}) vanish if an only if

(a) $a_0^{\kappa\lambda\mu}=0$ for a spiraling frame, for all κ,λ,μ;

(b) $a_0^{0\lambda\mu}=0$ and $b_0^{\kappa\lambda\mu}=0$ (*except* $\kappa=0$) for a pass-through frame.

These results are easily established for any of the Taub-NUT-type cosmologies. Then one can show in the flat Kasner and Taub-NUT cases that these conditions, which are precisely the same conditions which lead to convergence of the minimally coupled stress-energy tensor, make all other potentially divergent terms $-\frac{1}{3}Z_3(\ln^2 T)/T$, etc., vanish in $T^c_{(00)}$; they also make all other elements $T^c_{(\alpha\beta)}$ of the stress-energy tensor vanish as well. As in the minimally coupled case, we have not completed the proof for the Moncrief cosmologies, because of the comparatively extensive calculations needed to obtain the frame vectors.

C. Stress-energy scalars

In this section we establish the convergence properties of certain scalars constructed from the stress-energy tensor. In particular, we examine the scalar quantities T^{μ}_{μ} and $T_{\mu\nu}T^{\mu\nu}$.

For a minimally coupled massive scalar field, the trace of the stress-energy tensor is

$$(T^m)^\mu_\mu = -\frac{1}{4\pi}(S+2M^2\Phi^2) \tag{3.25}$$

and the scalar product is

$$T^m_{\mu\nu}T^{m\mu\nu} = \frac{1}{16\pi^2}[S^2+M^2\Phi^2(S+M^2\Phi^2)] \ . \tag{3.26}$$

For a conformally coupled massive scalar field, these quantities are

$$(T^c)^\mu_\mu = -\frac{1}{4\pi}M^2\Phi^2 \tag{3.27}$$

and

$$T^c_{\mu\nu}T^{c\mu\nu} = \frac{1}{16\pi^2}[\tfrac{1}{3}S^2+\tfrac{1}{9}SM^2\Phi^2+\tfrac{2}{9}(M^2\Phi^2)^2 + \tfrac{1}{9}\Phi\Phi^{;\mu\nu}(\Phi\Phi_{;\mu\nu}-4\Phi_{,\mu}\Phi_{,\nu})] \ . \tag{3.28}$$

For a massless field, the trace $(T^c)^\mu_\mu=0$ and so obviously converges; for a massive field $(T^c)^\mu_\mu$ converges if and only if Φ converges, i.e., if and only if $a_0^{0\lambda\mu}=0$ for all λ,μ. For the other scalar quantities $(T^m)^\mu_\mu$, $T^m_{\mu\nu}T^{m\mu\nu}$, and $T^c_{\mu\nu}T^{c\mu\nu}$, it is straightforward to show that they converge if and only if

(a) $a_0^{0\lambda\mu}=0$ (for all λ,μ), and also

(b) either $a_0^{\kappa\lambda\mu}=0$ (for $\kappa\neq 0$) or $b_0^{\kappa\lambda\mu}=0$ (for $\kappa\neq 0$) or both.

D. Summary

We can condense the results of the preceding sections into two theorems.

Theorem I. The stress-energy tensor $T_{(\alpha\beta)}$ for a minimally coupled or conformally coupled massless or massive scalar test field Φ on a Taub-NUT-type cosmology converges at the singularity at $T=0$ if and only if the following conditions hold.

(1) If $T_{(\alpha\beta)}$ is evaluated in a PPON pass-through frame, then Φ must be restricted to the $\Phi^{(2)}$ modes of Eq. (2.8c),

$$\Phi = \sum_{\kappa,\lambda,\mu}\left[\sum_{n=0}^{\infty} a_n^{\kappa\lambda\mu}T^n\right] f^\lambda_{\kappa\mu}(\psi,\theta,\phi) \ .$$

(2) If $T_{(\alpha\beta)}$ is evaluated in a PPON spiraling frame, then Φ must be restricted to the $\Phi^{(1)}$ modes of Eq. (2.8b),

$$\Phi = \sum_{\kappa,\lambda,\mu}\left[\sum_{n=0} b_n^{\kappa\lambda\mu}T^n\right] e^{i\kappa\alpha\ln T} f^\lambda_{\kappa\mu}(\psi,\theta,\phi) \ .$$

The above results are for frames carried by geodesics which do not end at the essential singularity. A similar analysis of $T_{(\alpha\beta)}$ convergence properties for the additional geodesics has the following more restrictive result.

(3) If $T_{(\alpha\beta)}$ is evaluated in a PPON (necessarily spiraling) frame for a geodesic which approaches the essential singularity, then Φ must be restricted to the ψ-independent, convergent modes

$$\Phi = \sum_{\lambda,\mu}\left[\sum_{n=0} b_n^{0\lambda\mu}T^n\right] f^\lambda_{0\mu}(\theta,\phi) \ .$$

The sums over κ,λ,μ extend over all permitted values for the cosmology in question. The only modes in common are those with $\kappa=0$; that is, ψ-independent, convergent modes are allowed for both pass-through and spiraling frames and for geodesics which do and do not end at the essential singularity. Only such modes have a convergent stress-energy tensor in the PPON frames of all geodesics.

We have proved the theorem for the flat Kasner and Taub-NUT cosmologies. We believe that it is also correct for the Moncrief cosmologies but have not proved it, because we have not calculated all of the necessary frame vectors.

Theorem II. The divergence properties of stress-energy scalars can be summarized as follows:

(1) The trace $(T^c)^\mu_\mu$ of the stress-energy tensor of a conformally coupled scalar test field Φ on a Taub-NUT-type cosmology (a) always converges at the singularity at $T=0$ if the field is massless; (b) converges at the singularity at $T=0$ for a massive field if and only if there are no $\Phi^{(0)}$ modes.

(2) For a massless or massive scalar test field Φ on a Taub-NUT-type cosmology, the scalar quantities $T^c_{\mu\nu}(T^c)^{\mu\nu}$ (for a conformally coupled field), and $(T^m)^\mu_\mu$ and $T^m_{\mu\nu}(T^m)^{\mu\nu}$ (for a minimally coupled field), converge at the singularity at $T=0$ if and only if the only modes present are $\Phi^{(1)}$ modes or $\Phi^{(2)}$ modes, but not both; an exception is the $\kappa=0$ modes of $\Phi^{(1)}$ and $\Phi^{(2)}$ which are identical in form (since $\Phi^{(0)}=0$) and may therefore both be present.

Theorem II has been proven for the flat Kasner, Taub-NUT, and Moncrief cosmologies, and is independent of whether the geodesic is pass-through or spiraling, or whether or not it ends at the essential singularity.

IV. A CONJECTURE ON THE STABILITY OF SINGULARITIES

We present in this section a method for testing the stability of the quasiregular singularities in a Taub-NUT-type universe. We would like to be able to place fields in such a universe and find their effect upon the singularities. However, such a back-reaction program is very difficult to carry out in general. We have placed scalar test fields on the Taub-NUT-type spacetimes, but the quasiregular singularities remain quasiregular; we have not allowed the test fields to influence the geometries. However, we can speculate about the effect of various fields if the full problem could be solved.

At a quasiregular singularity the Riemann tensor in a PPON frame is bounded and all scalars constructed from the Riemann tensor are also bounded. But if there is at least one PPON frame, such that one or more Riemann tensor elements diverge, but all scalars constructed from the Riemann tensor remain bounded, then the singularity is a nonscalar curvature singularity. Finally, at a scalar curvature singularity, some scalar quantity constructed from the Riemann tensor diverges. Because of the coupling of the stress-energy tensor with the curvature tensor through the Einstein field equations, it is not unreasonable to conjecture that the behavior of the scalar-test-field stress-energy tensor on a given background Taub-NUT-type spacetime can be used to predict the nature of the singularity which would occur if the corresponding scalar field modes were allowed to influence the geometry. Specifically, our conjecture is

Stability conjecture. For all maximally extended Hausdorff spacetimes with Taub-NUT-type quasiregular singularities, the back-reaction due to a field (whose test-field stress energy tensor is $T_{\mu\nu}$) will affect the singularity structure in the following manner:

(1) If both T^μ_μ and $T_{\mu\nu}T^{\mu\nu}$ are finite and if the $T_{(\alpha\beta)}$ in all PPON frames are finite, then the singularity will remain quasiregular.

(2) If both T^μ_μ and $T_{\mu\nu}T^{\mu\nu}$ are finite but $T_{(\alpha\beta)}$ diverges in either a pass-through or a spiraling PPON frame, but not both, then the singularity will be nonscalar curvature.

(3) If either T^μ_μ or $T_{\mu\nu}T^{\mu\nu}$ diverges, then the singularity will be scalar curvature.

Using Theorems I and II from Sec. III D regarding the behavior of the stress-energy tensor elements in PPON frames and the behavior of the scalars T^μ_μ and $T_{\mu\nu}T^{\mu\nu}$, we can translate the conjecture into a statement about the expected back-reaction effects of the various scalar field modes.

In particular, according to the conjecture, if the $\kappa=0$ $\Phi^{(1)}$ modes

$$\Phi=\sum_{\lambda,\mu}\left[\sum_n b_n^{0\lambda\mu}T^n\right]f^\lambda_{0\mu}(\theta,\phi)$$

are added to a Taub-NUT-type spacetime, the singularity remains quasiregular, because these modes (and only these modes) leave $T_{(\alpha\beta)}$ finite in all PPON frames and leave both stress-energy scalars finite as well. This holds for massive or massless, minimally coupled or conformally coupled scalar fields. If instead the only modes present are $\Phi^{(2)}$ modes

$$\Phi=\sum_{\kappa,\lambda,\mu}\left[\sum_n a_n^{\kappa\lambda\mu}T^n\right]f^\lambda_{\kappa\mu}(\psi,\theta,\phi)$$

including at least one mode with $\kappa\neq 0$, *or* are $\Phi^{(1)}$ modes

$$\Phi=\sum_{\kappa,\lambda,\mu}\left[\sum_n b_n^{\kappa\lambda\mu}T^n\right]e^{i\kappa\alpha\ln T}f^\lambda_{\kappa\mu}(\psi,\theta,\phi)$$

including at least one mode with $\kappa\neq 0$, but *not both* (i.e., a_n modes and b_n modes are not both present), then the singularity becomes a nonscalar curvature singularity, according to the conjecture. Finally, if there are $\Phi^{(0)}$ modes

$$\Phi=\ln T\sum_{\lambda,\mu}\left[\sum_n a_n^{0\lambda\mu}T^n\right]f^\lambda_{0\mu}(\theta,\phi)$$

and/or both $\Phi^{(1)}$ and $\Phi^{(2)}$ modes, including in both cases at least one mode with $\kappa\neq 0$, then the singularity is converted into a scalar curvature singularity, according to the conjecture: This is because at least one of the stress-energy scalars diverges.

Our stability conjecture therefore states that if test-field behavior on a background spacetime mimics the wave behavior expected in the vicinity of a particular type of spacetime singularity, then that same type of singularity would be expected to occur if a complete back-reaction computation could be carried out. However, the mechanism for such a back-reaction is poorly understood given the nonlinear nature of Einstein's equations; therefore, in the following section, the back-reaction scheme we employ assumes that the nonlinearly perturbed spacetimes have exactly the same symmetries as the original spacetimes.

V. TESTS OF THE STABILITY CONJECTURE

The stability conjecture introduced in the previous section allows one to predict how a given field mode will affect the structure of a singularity, if the field is allowed to influence the geometry, based on the behavior of the mode when treated as a test field on a given Taub-NUT-type cosmology. We can test the conjecture using exact solutions of the coupled Einstein-Maxwell-scalar field equations, which reduce to the appropriate background cosmology when the field is turned off. Then a direct comparison is possible between predictions of the conjecture and the actual singularities which occur in the field-containing cosmologies. We present here two examples of such universes, one which reduces to Taub-NUT spacetime and one which reduces to Moncrief spacetime, if the fields are turned off.

A. The Batakis-Cohen and Brill cosmologies

A class of homogeneous but anisotropic cosmologies due to Batakis and Cohen[36] contain lowest-mode scalar and electromagnetic fields; these cosmologies reduce to those of Brill[37] if the scalar fields are turned off, and to Taub-NUT spacetime if the electromagnetic fields are turned off as well. Batakis-Cohen models can be expressed in terms of the metric

$$ds^2=-(dt')^2+d\sigma^2\ , \tag{5.1}$$

where

$$d\sigma^2=B^2(t')(\sigma_1{}^2+\sigma_2{}^2)+A^2(t')\sigma_3{}^2\ , \tag{5.2}$$

in the case that two of the principal directions are equivalent. The forms σ_i are

$$\sigma_1=\sin\psi\, d\theta-\cos\psi\sin\theta\, d\phi\ , \tag{5.3a}$$

$$\sigma_2=\cos\psi\, d\theta+\sin\psi\sin\theta\, d\phi\ , \tag{5.3b}$$

$$\sigma_3=-d\psi-\cos\theta\, d\psi\ , \tag{5.3c}$$

in terms of Euler angles, and $A(t')$ and $B(t')$ can be found from Einstein's equations, for given scalar and electromagnetic field strengths.

The scalar field is chosen to be a solution of the minimally coupled massless scalar wave equation; its lowest mode depends upon t' alone and obeys $d\phi/dt'=a/AB^2$, where a is a constant. The lowest-mode sourceless electromagnetic field also depends upon t' alone; from Maxwell's equations one has the field tensor

$$f_{\mu\nu}=\begin{bmatrix}0&0&0&e_3\\0&0&-h_3&0\\0&h_3&0&0\\e_3&0&0&0\end{bmatrix}, \tag{5.4}$$

where

$$\sqrt{16\pi}e_3=(b/B^2)\sin(t_3'-\alpha)\ ,$$
$$\sqrt{16\pi}h_3=(b/B^2)\cos(t_3'-\alpha)\ .$$

Here α is a constant and t_3' is the integral of $dt_3'=(A/B^2)dt'$. The combined minimally coupled scalar field and electromagnetic field stress-energy tensor is diagonal, with elements

$$8\pi T_{00}=8\pi T_{11}=8\pi T_{22}=(a/AB^2)^2+(b/B^2)^2\ , \tag{5.5a}$$

$$8\pi T_{33}=(a/AB^2)^2-(b/B^2)^2\ , \tag{5.5b}$$

where the terms containing a are due to the scalar field and those containing b are due to the electromagnetic field. The trace

$$T=\frac{1}{4\pi}(a/AB^2)^2 \tag{5.6}$$

is independent of the electromagnetic field, as expected.

If one introduces a new time coordinate τ by defining $dt'=AB^2d\tau$, the metric becomes

$$ds^2=-(AB^2)^2d\tau^2+A^2(d\psi+\cos\theta\,d\phi)^2+B^2(d\theta^2+\sin^2\theta\,d\phi^2) \tag{5.7}$$

and the field equations give

$$A^2=2\omega^2/[b^2+(\omega^2+b^4)^{1/2}\cosh(2\omega\tau+2\mu)] \tag{5.8}$$

and

$$B^2=\Omega^2[b^2+(\omega^2+b^4)^{1/2}\cosh(2\omega\tau+2\mu)]/2\omega^2\cosh^2(\Omega\tau+\lambda) \tag{5.9}$$

as shown by Batakis and Cohen.[36] Here ω, Ω, λ, and μ are constants, with $\Omega^2=\omega^2+a^2$, and we choose $\Omega\geq\omega>0$. The metric is singular as $\tau\to\pm\infty$.

The scalar field in this metric is

$$\Phi=\Phi_0+a\tau\ ; \tag{5.10}$$

expressions for the electric and magnetic fields can also be found.[36]

We now introduce a time coordinate

$$t=m+\frac{\Omega}{2l}\tanh(\Omega\tau+\lambda) \tag{5.11}$$

so that the singularities are brought from $\tau=\pm\infty$ into $t=t_\pm=m\pm\Omega/2l$; the metric then takes on the form

$$ds^2=-U^{-1}dt^2+(2l)^2U(d\psi+\cos\theta\,d\phi)^2+B^2(d\theta^2+\sin^2\theta\,d\phi^2)\ . \tag{5.12}$$

Here m,l are constants, and

$$U=\omega^2/2l^2D(t)\ , \tag{5.13}$$

$$B^2=(\Omega^2/2\omega^2)\left[1-\left[\frac{2l}{\Omega}(t-m)\right]^2\right]D(t)\ , \tag{5.14}$$

where

$$D(t)=b^2+(\omega^2+b^4)^{1/2}\cosh\left\{\frac{2\omega}{\Omega}\left[\tanh^{-1}\left[\frac{2l}{\Omega}(t-m)\right]-\lambda\right]+2\mu\right\}. \tag{5.15}$$

If we specialize to the case $a=0$ by turning off the scalar field, the metric reduces to

$$ds^2=-U^{-1}dt^2+(2l)^2U(d\psi+\cos\theta\,d\phi)^2+(t^2+l^2)(d\theta^2+\sin^2\theta\,d\phi^2) \tag{5.16}$$

with

$$U=-(t-t_-)(t-t_+)/(t^2+l^2)\ ,$$

where

$$t_\pm=m\pm\omega/2l=m\pm(m^2+l^2-b^2)^{1/2}\ . \tag{5.17}$$

This metric is identical in form to the Taub-NUT universe, except that the singularities have been displaced by the lowest-mode electromagnetic field, as represented by the parameter b; it is the metric of the Brill universe.[37] The singularities are obviously still quasiregular, which is consistent with the fact that the contracted electromagnetic stress-energy tensor $T=T^\mu_\mu=0$, so that from the contracted Einstein equations the curvature scalar R must also be zero. Other scalars constructed from the Riemann tensor, such as $R_{\mu\nu}R^{\mu\nu}$, also remain bounded at the singularities. Finally, if the electromagnetic field is turned off as well, the metric reduces to that of Taub space.

We now investigate the general metric given by Eq. (5.12) in the vicinity of the singularities at $t_\pm=m\pm\Omega/2l$, where

$$\Omega/2l=(m^2+l^2-b^2+a^2/4l^2)^{1/2}\ . \tag{5.18}$$

Let $t=t_\pm\mp(\Omega/l)\Delta$ where Δ is small; then

$$U\simeq(\omega^2/l^2)\Delta^{\omega/\Omega}/({t_{0\pm}}^2+l^2) \tag{5.19}$$

and

$$B^2\simeq(\Omega/\omega)^2\Delta^{1-\omega/\Omega}({t_{0\pm}}^2+l^2)\ , \tag{5.20}$$

where

$$t_{0\pm}=m\pm\omega/2l=m\pm(m^2+l^2-b^2)^{1/2}\ .$$

Thus if the scalar field is turned off, so that $\Omega\to\omega$, then $U\sim\Delta$ and $B^2\sim$constant, which is the behavior of these quantities for the Taub-NUT or Brill universes. But for $a>0$, then $\omega/\Omega<1$, so that $U\sim\Delta^{\omega/\Omega}$ and $B^2\sim\Delta^{1-\omega/\Omega}\to0$ as $\Delta\to0$: the metric behavior is then quite different. Furthermore, the curvature scalar R is

$$R=-8\pi T=-2(a/AB^2)^2\sim-a^2/UB^4\sim-a^2/\Delta^{2-\omega/\Omega} \tag{5.21}$$

which diverges as $\Delta \rightarrow 0$. The addition of a generic lowest-mode scalar field causes the quasiregular singularity to be converted into a scalar curvature singularity.

We can now test our stability conjecture using the Batakis-Cohen models. The lowest-mode scalar test field on Taub-NUT spacetime has the form

$$\Phi = \Phi_0 + \alpha \ln T \; , \tag{5.22}$$

where Φ_0 is the lowest mode contained in Eqs. (2.8b) and (2.8c), and $\alpha \ln T$ is the lowest-mode contribution from Eq. (2.8a), where α is a constant and $T = t - t_\pm$. For this mode, the scalars T^μ_μ and $T_{\mu\nu}T^{\mu\nu}$ constructed in Sec. III C from the minimally coupled stress-energy tensor converge at $T = 0$ if and only if $\alpha = 0$. According to the conjecture given in Sec. IV, one therefore predicts that if a lowest-mode scalar field is introduced into a cosmology of the proper symmetry, and back-reaction is fully accounted for, the resulting cosmology will have a quasiregular singularity if $\alpha = 0$ and a scalar curvature singularity if $\alpha \neq 0$. This is exactly what happens in the Batakis-Cohen universes. The scalar field is then, from Eqs. (5.10) and (5.11),

$$\begin{aligned} \Phi &= \Phi_0 + a\tau \\ &= \Phi_0 + \frac{a}{\Omega}\left[\tanh^{-1}\frac{2l}{\Omega}(t-m) - \lambda\right] \\ &\simeq \Phi_0 \mp \frac{a}{2\Omega}\ln\Delta \end{aligned} \tag{5.23}$$

near $t_\pm$, the same kind of behavior Φ has in the test-field case as given by Eq. (5.22). Furthermore, in the Batakis-Cohen models the curvature scalar converges as $\Delta \rightarrow 0$ if and only if $a = 0$, as predicted by our conjecture; other scalars constructed from the curvature tensor also converge if $a = 0$, so the singularity is quasiregular if and only if $a = 0$: otherwise it is a scalar curvature singularity.

B. Scalar fields in Moncrief universes

The conjecture introduced in Sec. IV can also be tested in the context of scalar fields in Moncrief universes. To do so, we must first exhibit classes of exact solutions of the coupled Einstein-scalar field equations which reduce to the Moncrief universes when the fields are turned off. Some of these solutions have quasiregular singularities; others have scalar curvature singularities.

Consider an Einstein-Rosen-Gowdy diagonal metric

$$\begin{aligned} ds^2 &= e^{2a}(-dt^2 + d\theta^2) + te^{2W}(d\psi)^2 \\ &\quad + te^{-2W}(d\phi)^2 \end{aligned} \tag{5.24}$$

with coordinates (t,ψ,θ,ϕ) on an $R^1 \times T^3$ spacetime. Here $a = a(t,\theta)$ and $W = W(t,\theta)$. This universe contains a minimally coupled scalar field $\Phi = \Phi(t,\theta)$ which is a solution of Eq. (2.7) and which has the stress-energy tensor

$$4\pi T^\mu{}_\nu = \Phi^{,\mu}\Phi_{,\nu} - \tfrac{1}{2}\delta^\mu{}_\nu(\Phi^{,\alpha}\Phi_{,\alpha} + M^2\Phi^2) \; ; \tag{5.25}$$

the geometry satisfies the field equations

$$R^\mu{}_\nu = 8\pi(T^\mu{}_\nu - \tfrac{1}{2}\delta^\mu{}_\nu T) \; . \tag{5.26}$$

These equations reduce to

$$\ddot{\Phi} + t^{-1}\dot{\Phi} - \Phi'' = -M^2\Phi e^{2a} \; , \tag{5.27a}$$

$$\ddot{W} + t^{-1}\dot{W} - W'' = M^2\Phi^2 e^{2a} \; , \tag{5.27b}$$

$$a' = 2t\dot{W}W' + 2t\dot{\Phi}\Phi' \; , \tag{5.27c}$$

$$\dot{a} = t(\dot{W}^2 + W'^2) + t(\dot{\Phi}^2 + \Phi'^2) - 1/4t \; , \tag{5.27d}$$

and

$$\begin{aligned} 2t\dot{W}(\ddot{W} + t^{-1}\dot{W} - W'') &+ 2t\dot{\Phi}(\ddot{\Phi} + t^{-1}\dot{\Phi} - \Phi'') \\ &= M^2\Phi^2 e^{2a} \; , \end{aligned} \tag{5.27e}$$

where overdots and primes mean partial derivatives with respect to t and θ. The first equation is the scalar field equation; the second is from the $R^2{}_2$ or the $R^3{}_3$ Einstein equation; the third is from the $R^0{}_1$ equation; the fourth is from a linear combination of the $R^0{}_0$ and $R^1{}_1$ equations; and the last is from the $R^0{}_0$ or the $R^1{}_1$ equation.

Note that in the case of a massless field there is a total decoupling of Φ from W, and the differential equations for Φ and W become identical; furthermore, these equations are the same as that of a scalar test field propagating on a Kasner background. This case of massless source fields has been discussed in detail by Carmeli, Charach, and Malin,[38] and by Charach and Malin.[39] The conventions and notation in these papers differ somewhat from those used here and by Moncrief.

With $M = 0$, the solutions of Eqs. (5.27b) and (5.27a) are

$$\begin{aligned} W = \alpha + \beta\ln t + \sum_{n=1} &[a_n J_0(nt)\sin(n\theta + \gamma_n) \\ &+ b_n N_0(nt)\sin(n\theta + \delta_n)] \end{aligned} \tag{5.28}$$

and

$$\begin{aligned} \Phi = \tilde{\alpha} + \tilde{\beta}\ln t + \sum_{n=1} &[\tilde{a}_n J_0(nt)\sin(n\theta + \tilde{\gamma}_n) \\ &+ \tilde{b}_n N_0(nt)\sin(n\theta + \tilde{\delta}_n)] \; , \end{aligned} \tag{5.29}$$

where J_0 and N_0 are Bessel and Neumann functions, and the constants α, β, a_n, b_n, γ_n, δ_n (and those with tildes) are arbitrary. It is then possible to find a general expression for $a(t,\theta)$ using Eqs. (5.27c) and (5.27d); the result is given by Carmeli *et al.*[38]

The Moncrief universes[11] result if $\Phi = 0$ and if the constants in W are chosen to be $b_n = 0$ and $\beta = \frac{1}{2}$, so that the curvature tensor converges as $t \rightarrow 0^+$. This tensor can be found from curvature forms computed by Gowdy[13,14] in the orthonormal frame

$$(\hat{e}_0, \hat{e}_1, \hat{e}_2, \hat{e}_3) = (e^{-a}\partial/\partial t, e^{-a}\partial/\partial\theta, t^{-1/2}e^{-W}\partial/\partial\psi, t^{-1/2}e^{W}\partial/\partial\phi) \tag{5.30}$$

in which one finds

$$R_{\hat{0}\hat{1}\hat{0}\hat{1}}=e^{-2a}(\ddot{a}-a'') \;, \tag{5.31a}$$

$$R_{\hat{2}\hat{0}\hat{2}\hat{0}}=e^{-2a}\left[\dot{a}\left[\dot{W}+\frac{1}{2t}\right]+a'W'+\frac{1}{2t^2}-\ddot{W}-\left[\dot{W}+\frac{1}{2t}\right]^2\right] \;, \tag{5.31b}$$

$$R_{\hat{2}\hat{0}\hat{2}\hat{1}}=e^{-2a}\left[a'\left[\dot{W}+\frac{1}{2t}\right]+\dot{a}W'-\dot{W}'-W'\left[\dot{W}+\frac{1}{2t}\right]\right] \;, \tag{5.31c}$$

$$R_{\hat{2}\hat{1}\hat{2}\hat{1}}=e^{-2a}\left[a'W'+\dot{a}\left[\dot{W}+\frac{1}{2t}\right]-W''-W'^2\right] \;, \tag{5.31d}$$

$$R_{\hat{3}\hat{0}\hat{3}\hat{0}}=e^{-2a}\left[-\dot{a}\left[\dot{W}-\frac{1}{2t}\right]-a'W'+\frac{1}{2t^2}+\ddot{W}-\left[\dot{W}-\frac{1}{2t}\right]^2\right] \;, \tag{5.31e}$$

$$R_{\hat{3}\hat{0}\hat{3}\hat{1}}=e^{-2a}\left[-a'\left[\dot{W}-\frac{1}{2t}\right]-\dot{a}W'+\dot{W}'-W'\left[\dot{W}-\frac{1}{2t}\right]\right] \;, \tag{5.31f}$$

$$R_{\hat{3}\hat{1}\hat{3}\hat{1}}=e^{-2a}\left[-a'W'-\dot{a}\left[\dot{W}-\frac{1}{2t}\right]+W''-W'^2\right] \;, \tag{5.31g}$$

$$R_{\hat{2}\hat{3}\hat{2}\hat{3}}=e^{-2a}\left[W'^2-\dot{W}^2+\frac{1}{4t^2}\right] \;, \tag{5.31h}$$

together with other nonzero elements found from the symmetry properties of $R_{\mu\nu\lambda\sigma}$. Moncrief's choice of $b_n=0$ and $\beta=\frac{1}{2}$ makes e^{-2a} converge as $t\to 0^+$, and each component of $R_{\hat{\alpha}\hat{\beta}\hat{\gamma}\hat{\delta}}$ converges there as well. Then by introducing new coordinates

$$t'=t^2 \;, \quad \psi'=2\psi-2\ln t \;, \quad \theta'=\theta \;, \quad \phi'=\phi \;, \tag{5.32}$$

Moncrief obtains a metric which can be analytically extended through the null hypersurface to negative values of t'.

We now seek a class of solutions for a nonzero Φ. We can retain Moncrief's choice for W, since Eqs. (5.27a)–(5.27e) decouple W and Φ in the massless case. It is also possible in the $M=0$ case to let $a=a_W+a_\Phi$, where

$$a'_W=2t\dot{W}W' \;, \tag{5.33a}$$

$$\dot{a}_W=t(\dot{W}^2+W'^2)-1/4t \;, \tag{5.33b}$$

$$a'_\Phi=2t\dot{\Phi}\Phi' \;, \tag{5.33c}$$

$$\dot{a}_\Phi=t(\dot{\Phi}^2+\Phi'^2) \;, \tag{5.33d}$$

so that Eqs. (5.27c) and (5.27d) are automatically satisfied. The function $a(t,\theta)$ must be continuous, so

$$\int_{-\pi}^{\pi}d\theta\frac{\partial a}{\partial\theta}=2t\int_{-\pi}^{\pi}d\theta(\dot{W}W'+\dot{\Phi}\Phi')=0 \;; \tag{5.34}$$

Moncrief's choice of W automatically makes $\int_{-\pi}^{\pi}d\theta\,\dot{W}W'=0$. The second integral is

$$\int_{-\pi}^{\pi}d\theta\,\dot{\Phi}\Phi'=\pi\sum_{n=1}\tilde{a}_n\tilde{b}_n n\sin(\tilde{\delta}_n-\tilde{\gamma}_n)[J_0(nt)\dot{N}_0(nt)-\dot{J}_0(nt)N_0(nt)] \;, \tag{5.35}$$

in which the Wronskian $J_0\dot{N}_0-\dot{J}_0N_0\neq 0$, so for each term in the sum at least one of the following must be zero: $\tilde{a}_n$, $\tilde{b}_n$, or $\sin(\tilde{\delta}_n-\tilde{\gamma}_n)$. This constraint permits divergent as well as convergent modes of Φ.

We now investigate the nature of the singularity at $t=0$. The scalar field for small t has the form

$$\Phi=s(\theta)\ln t+u(\theta)+O(t^2\ln t) \;, \tag{5.36}$$

where

$$s(\theta)=\tilde{\beta}+\frac{2}{\pi}\sum_{n=1}\tilde{b}_n\sin(n\theta+\tilde{\delta}_n) \tag{5.37}$$

and

$$u(\theta)=\tilde{a}+\sum_{n=1}\tilde{a}_n\sin(n\theta+\tilde{\gamma}_n)+\frac{2}{\pi}\sum_{n=1}\tilde{b}_n\left[\ln\frac{n}{2}+C\right]\sin(n\theta+\tilde{\delta}_n) \;; \tag{5.38}$$

we assume the constants have been chosen to satisfy Eq. (5.34), and we have used the expansions $J_0(x)=1-\frac{1}{4}x^2+O(x^4)$ and

$$N_0(x)=\frac{2}{\pi}(\ln x+C-\ln 2)-\frac{1}{2\pi}x^2\ln x+O(x^2)\ ,$$

where C is the Euler-Mascheroni constant.[40]

Using Eqs. (5.33c) and (5.33d), we then find

$$a_\Phi=s^2(\theta)\ln t+2\int^\theta d\theta'[s(\theta')u'(\theta')]+\tfrac{1}{2}(s'(\theta))^2t^2\ln^2 t+O(t^2\ln t)\ , \tag{5.39}$$

so that the stress-energy scalar is

$$\begin{aligned}T=T^\mu{}_\mu&=-\frac{1}{4\pi}g^{\mu\nu}\Phi_{,\mu}\Phi_{,\nu}\\&=\frac{1}{4\pi}e^{-2a}(\dot\Phi^2-\Phi'^2)\\&=\frac{1}{4\pi}e^{-2a_W}t^{-2s^2(\theta)}\exp\left[-4\int^\theta d\theta' s(\theta')u'(\theta')\right][t^{-2}s^2(\theta)+O(\ln^2 t)]\ .\end{aligned} \tag{5.40}$$

Therefore a necessary condition for convergent T is $s(\theta)=0$, i.e., we must require that $\widetilde\beta=0$ and $\widetilde b_n=0$ for all n. Thus if the field Φ has any divergent mode, T also diverges. Furthermore, from the contracted field equations $R=-8\pi T$, the curvature scalar also diverges in this case. In other words, the spacetime has a scalar curvature singularity at $t=0$ if Φ has an divergent mode, i.e., if $\widetilde\beta$ or any of the $\widetilde b_n$ is not zero.

Now suppose Φ is convergent at $t=0$; then using the series expansion for $J_0(nt)$ we have

$$\Phi=\Phi_0(\theta)+\Phi_1(\theta)t^2+\Phi_2(\theta)t^4+\cdots\ , \tag{5.41}$$

where

$$\Phi_0=\widetilde\alpha+\sum_{n=1}\widetilde a_n\sin(n\theta+\widetilde\gamma_n)\ , \tag{5.42a}$$

$$\Phi_1=-\tfrac{1}{4}\sum_{n=1}\widetilde a_n n^2\sin(n\theta+\widetilde\gamma_n)\ , \tag{5.42b}$$

etc. Use of Eqs. (5.33c) and (5.33a) then gives

$$a_\Phi=c+\tfrac{1}{2}(\Phi_0')^2t^2+(\Phi_1{}^2+\tfrac{1}{2}\Phi_0'\Phi_1')t^4\cdots\ , \tag{5.43}$$

where c is a constant. In this case e^{-2a_Φ} converges, as do T, R, and all components of the Riemann tensor as given by Eqs. (5.31a)–(5.31h). Therefore, since (i) all the $R_{\hat\alpha\hat\beta\hat\gamma\hat\delta}$ components have finite limits as $t\to0^+$ along the curves of constant $\{\psi,\theta,\phi\}$, and since one can show, as Moncrief has, that (ii) each of the curves of this normal congruence has bounded acceleration as $t\to0^+$; (iii) the basis fields $\hat e_{(\alpha)}$ define a Fermi-Walker transported frame along each of these curves; and (iv) the time-dependent Lorentz transformation which carries $\hat e_{(\alpha)}$ to a parallel-propagated basis is well behaved, then $R_{\hat\alpha\hat\beta\hat\gamma\hat\delta}$ in a parallel-propagated basis has finite limits along each of the normal trajectories, and each of these solutions is extendible. In fact, one can show, in parallel with our discussion in paper I, that there are two inequivalent maximal analytic Hausdorff extensions, each containing a quasiregular Taub-NUT-type singularlity.

Therefore if only convergent modes of Φ are chosen, these spacetimes form a class of (massless minimally coupled scalar wave) solutions to the Einstein equations whose form is precisely that of the Moncrief universes, with the function $a(t,\theta)$ replaced by $a(t,\theta)+a_\Phi(t,\theta)$. Therefore the generalized models are obviously extendible and have quasiregular singularities.

In order to test the conjecture of Sec. IV, consider now a minimally coupled scalar test field of the form

$$\Phi=\sum_{\lambda=-\infty}^{\infty}\left[\sum_j b_j^{0\lambda0}T^j\right]e^{i\lambda\theta} \tag{5.44}$$

on a Moncrief universe. That is, we have set $\kappa=0$, $\mu=0$ in the most general solution of Eq. (2.8), so that $\Phi=\Phi(t,\theta)$; we have also chosen only convergent modes, so that $a_0^{\kappa\lambda\mu}=0$ for all κ,λ,μ. Therefore, according to Theorem II, the stress-energy scalars $T^\mu{}_\mu$ and $T_{\mu\nu}T^{\mu\nu}$ converge at the singularity. Furthermore, these modes for Φ lead to finite $T_{(\alpha\beta)}$ in all PPON frames, according to Theorem I. Therefore, according to the conjecture, if these convergent $\kappa=0$, $\mu=0$ modes are placed into a Moncrief universe, and back-reaction is fully accounted for, the singularity will remain quasiregular. This is exactly what was found in the previous section: The convergent modes were expressed there in the form

$$\Phi=\widetilde\alpha+\sum_{n=1}\widetilde a_n J_0(nt)\sin(n\theta+\widetilde\gamma_n) \tag{5.45}$$

which has to be rewritten in terms of the Moncrief coordinates t', ψ', θ', ϕ' of Eq. (5.32) in order to compare with Eq. (5.44). The result is simply

$$\Phi=\widetilde\alpha+\sum_{n=1}\widetilde a_n J_0(n(t')^{1/2})\sin(n\theta'+\widetilde\gamma_n) \tag{5.46}$$

which is equivalent to the Φ of Eq. (5.44): that is, the $b_0^{0\lambda0}$ can be found in terms of $\widetilde\alpha$ and the $\widetilde a_n$ and $\widetilde\gamma_n$. The spacetime containing these convergent modes we found to have a quasiregular singularity, so the conjecture is indeed correct in this case.

Suppose instead that one or more divergent test-field modes of the form

$$\Phi=\ln T\sum_\lambda\sum_j(a_j^{0\lambda0}T^j)e^{i\lambda\theta} \tag{5.47}$$

are introduced onto a background Moncrief universe.

Therefore since $a_0^{0\lambda 0}\neq 0$ for some λ, both stress-energy scalars $T^{\mu}{}_{\mu}$ and $T_{\mu\nu}T^{\mu\nu}$ diverge for this test field, according to Theorem II. Then our conjecture predicts that a scalar curvature singularity will appear if such modes are added to the universe, and the spacetime is allowed to adjust to the field. Again, this is what we have already found to be true. We expressed the divergent modes in the form

$$\Phi=\widetilde{\beta}\ln t+\sum_{n=1}\widetilde{b}_n N_0(nt)\sin(n\theta+\widetilde{\delta}_n) \tag{5.48a}$$

$$=\tfrac{1}{2}\widetilde{\beta}\ln t'+\sum_{n=1}\widetilde{b}_n N_0(n(t')^{1/2})\sin(n\theta'+\widetilde{\delta}_n) \tag{5.48b}$$

in Moncrief's coordinates. Equations (5.47) and (4.48b) are equivalent; the $a_0^{0\lambda 0}$ can be found to make the expressions the same, for given $\widetilde{\beta}$, $\widetilde{b}_n$, and $\widetilde{\delta}_n$.

The conjecture therefore survives the tests for both convergent and divergent field modes, for which back-reaction leads to quasiregular and scalar curvature singularities, respectively.

VI. CONCLUSIONS

In this paper, we have discussed the stability of Taub-NUT-type cosmologies, with particular regard to the stability of their singularity structures. The quasiregular singularities contained in these cosmologies seem unphysical, in that the world lines of freely falling observers come to an end in a finite proper time, even though the observers do not encounter unbounded tidal forces. One could forgive the presence of such singularities in exact solutions of Einstein's equations if it could be shown that they are unstable, in the sense that the addition of matter or fields to the idealized cosmologies would convert the singularity into something more physical, such as a scalar curvature singularity at which tidal forces become infinite.

We have made predictions about the stability of Taub-NUT-type quasiregular singularities under back-reaction based on the behavior of test field stress-energy scalars and tensors evaluated in parallel-propagated orthonormal frames. These predictions are useful, because the back-reaction calculations themselves are generally difficult or nearly impossible to carry out. We have said that a test field *mimics* a particular type of spacetime singularity (quasiregular, nonscalar curvature, or scalar curvature) if the stress-energy tensors and stress-energy scalars of the field behave in a way analogous to the behavior of the curvature tensor and curvature scalars in the Ellis and Schmidt singularity classification scheme.[4] We have conjectured that if a test field mimics a particular type of singularity on a Taub-NUT-type background spacetime, then a full back-reaction calculation would change the Taub-NUT-type singularity into the type of singularity mimicked.

Similar classical wave modes mimic similar singularities in each class of Taub-NUT-type cosmologies. Most waves mimic scalar curvature singularities; only very special wave modes mimic nonscalar curvature or quasiregular singularities. Comparison of these test-field results with exact solutions of the Einstein-Maxwell-scalar field equations supports the back-reaction conjecture. The exact solutions include generalized Taub-NUT spacetimes containing lowest-mode scalar and electromagnetic fields, and generalized Moncrief spacetimes containing lowest-mode scalar fields.

In further studies, one would hope to generalize these results to include all Taub-NUT-type cosmologies. In addition one would like to rest the back-reaction conjecture against Taub-NUT-type cosmologies with other source fields and source matter. There are two areas of particular concern: One is the existence of further Taub-NUT-type spatially inhomogeneous spacetimes, and the other is the existence of Taub-NUT-type spacetimes with quantum matter source fields. The latter are rather unlikely given the results of vacuum polarization calculations on $R^1\times T^3$ and $R^3\times S^1$ flat Kasner. The former are more probable especially since Moncrief has already proven the existence of large classes of vacuum inhomogeneous Taub-NUT-type models. Therefore testing the back-reaction conjecture not only involves the fundamental question of the nature of singularities; it is intimately connected with the search for inhomogeneous exact solutions and the interrelationships of gravitation with other fields, and the quantum nature of both.

ACKNOWLEDGMENTS

We wish to thank Larry Shepley and Alan Middleton for useful discussions. One of us (D.A.K.) would like to express appreciation for financial support from Zonta International and from the University of Maryland. This work was supported in part by the National Science Foundation under Grant No. PHY 82-05659. Portions of this paper form part of a dissertation by D. A. Konkowski submitted in partial fulfillment of the requirement for the degree of Ph.D. at the University of Texas at Austin.

APPENDIX: PARALLEL-PROPAGATED ORTHONORMAL FRAMES

A parallel-propagated orthonormal (PPON) frame is a set of four vectors $e^{\mu}_{(\alpha)}$ $(\alpha=0,1,2,3)$ carried along a timelike curve; these vectors satisfy the orthonormality and parallel-propagation conditions

$$e^{\mu}_{(\alpha)}e_{(\beta)\mu}=\eta_{(\alpha\beta)} \tag{A1}$$

and

$$e^{\mu}_{(\alpha);\nu}e^{\nu}_{(0)}=0\ , \tag{A2}$$

where $\eta_{(\alpha\beta)}=\mathrm{diag}(-1,1,1,1)$. The vector $e^{\nu}_{(0)}$ is tangent to the curve; Eqs. (A1) and (A2) may be solved in principle to find the other frame vectors. The result depends upon whether the geodesic spirals infinitely as it approaches the singularity or passes right through the singularity; these geodesic types are described more fully in paper I. We call the corresponding orthonormal frames *spiraling* and *pass-through*, respectively. The results given here are for the larger class of geodesics, those which do not end at the essential quasiregular singularity (for example, in the flat Kasner case the results are for geodesics with $C_1 \neq 0$, in the notation of paper I).

Flat Kasner spacetime. Using the geodesic equations given in paper I, one can easily write the tangent vector to timelike geodesics; the result is

$$e^{\mu}_{(0)} = \begin{bmatrix} \dot{t} \\ \dot{\psi} \\ \dot{\theta} \\ \dot{\phi} \end{bmatrix} = \begin{bmatrix} \pm(1+2\alpha t)^{1/2} \\ (2t)^{-1}[1\mp(1+2\alpha t)^{1/2}] \\ C_2 \\ C_3 \end{bmatrix}, \tag{A3}$$

where C_2, C_3, and $\alpha = 1 + C_2{}^2 + C_3{}^2$ are constants; we have chosen $C_1 = 1$. The upper and lower signs designate pass-through and spiraling geodesic segments, respectively. The other PPON frame vectors are

$$e^{\mu}_{(1)} = \begin{bmatrix} -\alpha^{-1}[1\mp(C_2{}^2+C_3{}^2)(1+2\alpha t)^{1/2}] \\ (2t)^{-1}[1\mp(1+2\alpha t)^{1/2}] \\ C_2 \\ C_3 \end{bmatrix}, \tag{A4a}$$

$$e^{\mu}_{(2)} = \begin{bmatrix} C_2\alpha^{-1}[1\pm(1+2\alpha t)^{1/2}] \\ 0 \\ 1 \\ 0 \end{bmatrix}, \tag{A4b}$$

and

$$e^{\mu}_{(3)} = \begin{bmatrix} C_3(\alpha^{-1})[1\pm(1+2\alpha t)^{1/2}] \\ 0 \\ 0 \\ 1 \end{bmatrix}. \tag{A4c}$$

Taub-NUT spacetime. The timelike frame vector in Taub-NUT spacetime is again determined from the geodesic equations:

$$e^{\mu}_{(0)} = \begin{bmatrix} \dot{t} \\ \dot{\psi} \\ \dot{\theta} \\ \dot{\phi} \end{bmatrix} = \begin{bmatrix} \pm V^{1/2} \\ (2lU)^{-1}(p_{\parallel}/2l \pm V^{1/2}) - p_{\parallel}(t^2+l^2)^{-1/2} \\ 0 \\ p(t^2+l^2)^{-1/2} \end{bmatrix}, \tag{A5}$$

where $V = [p^2(2l)^{-2} + U(p_{\perp}{}^2(t^2+l^2)^{-1}+1)]$, and l, $p_{\parallel}$, $p_{\perp}$, and $p = (p_{\parallel}{}^2 + p_{\perp}{}^2)^{1/2}$ are constants. If $p_{\parallel} > 0$, upper and lower signs correspond in the vicinity of the null hypersurfaces $t_{\pm}$ to spiraling and pass-through geodesic segments, respectively. If $p_{\parallel} < 0$, this behavior is reversed. All geodesics spiral if $p_{\parallel} = 0$.

One cannot find exact closed-form solutions of Eqs. (A1) and (A2) for the other Taub-NUT frame vectors. However, we need them only to order T in the neighborhood of $t_{\pm}$, where $T = t - t_{\pm}$. For definiteness we take $p_{\parallel} > 0$. Define $L = t_{\pm}{}^2 + l^2$ and $W = (p_{\perp}{}^2 L^{-1} + 1)$; then for pass-through PPON frames,

$$e^{\mu}_{(1)} = L^{-3/2} \begin{bmatrix} 0 \\ -l(\sin\theta\cos\theta)^{-1}T \\ L - t_{\pm}T \\ l(\sin\theta)^{-1}T \end{bmatrix}, \tag{A6a}$$

$$e^{\mu}_{(2)} = W^{-1/2} \begin{bmatrix} (2l)^{-1}p_{\parallel} \\ -\tfrac{1}{2}p_{\parallel}{}^{-1}W - [p_{\parallel}t_{\pm}L^{-2} + l^2W^2(2t_{\pm}p_{\parallel}{}^3)^{-1}]T \\ -lp_{\perp}L^{-2}T \\ pt_{\pm}L^{-2}T \end{bmatrix}, \tag{A6b}$$

$$e^{\mu}_{(3)} = W^{-1/2}(p_{\perp}{}^2\cos^2\theta + L)^{-1/2} \begin{bmatrix} p^2(2l)^{-1}\sin^2\theta\cos\theta - lt_{\pm}{}^{-1}\sin\theta\tan\theta WT \\ (1+\cos^2\theta)(2\cos\theta)^{-1}W + [l^2\sin\theta\tan\theta(2t_{\pm}p_{\parallel}{}^2)^{-1}] \\ \times[W^2 - 2t_{\pm}{}^2p^2\cot^2\theta\, l^{-2}L^{-1}(W + p_{\perp}{}^2\cos^2\theta L^{-1})]T \\ lL^{-1}\sin\theta T \\ -W + (2p_{\perp}{}^2L^{-1}+1)t_{\pm}L^{-1}T \end{bmatrix}. \tag{A6c}$$

For spiraling PPON frames,

$$e^{\mu}_{(1)}=L^{-3/2}\begin{bmatrix} 0 \\ l(1+\cos^2\theta)(2\sin\theta\cos\theta)^{-1}T \\ L-t_{\pm}T \\ -l(\sin\theta)^{-1}T \end{bmatrix}, \tag{A7a}$$

$$e^{\mu}_{(2)}=L^{-3/2}\begin{bmatrix} -lLt_{\pm}{}^{-1}\tan\theta T \\ -L\cot\theta\pm t_{\pm}(2\sin\theta\cos\theta)^{-1}(3\cos^2\theta-1)T \\ lT \\ L(\sin\theta)^{-1}-t_{\pm}(\sin\theta)^{-1}T \end{bmatrix}, \tag{A7b}$$

$$e^{\mu}_{(3)}=\begin{bmatrix} (p_{\parallel}(2l)^{-1}-lp\sin^2\theta(t_{\pm}L\cos\theta)^{-1}T \\ -p_{\parallel}t_{\pm}(2l^2T)^{-1}+[(2p_{\parallel})^{-1}W-p_{\parallel}(2l^2L)^{-1}(L+l^2)] \\ +[l^2W^2(2t_{\pm}p_{\parallel}{}^3)^{-1}+pt_{\pm}(2L^2\cos\theta)^{-1}(3\cos^2\theta-2\sin^2\theta)]T \\ 0 \\ pL^{-1}-2pt_{\pm}L^{-2}T \end{bmatrix}. \tag{A7c}$$

Moncrief universes. The tangent vectors of spiraling and pass-through $k'\neq 0$ geodesics near the $t=0$ null hypersurface in Moncrief's universes are, respectively,

$$e^{\mu}_{(0)}=\begin{bmatrix}\dot t\\ \dot\psi\\ \dot\theta\\ \dot\phi\end{bmatrix}=\begin{bmatrix} -k'e^{-2f_0} \\ 2k't^{-1}e^{-2f_0}+k'e^{-2f_0}[-4\theta_1 f_0^{(1)}-f_0^{(2)}+2\theta_1{}^2 \\ -\tfrac{1}{2}(f_0^{(1)})^2+2(k')^{-2}e^{2f_0}(k^2e^{2f_0}+1)] \\ -k'e^{-2f_0}\theta_1 \\ k\,e^{2f_0} \end{bmatrix} \tag{A8a}$$

and

$$e^{\mu}_{(0)}=\begin{bmatrix}\dot t\\ \dot\psi\\ \dot\theta\\ \dot\phi\end{bmatrix}=\begin{bmatrix} k'e^{-2f_0} \\ -k'e^{-2f_0}[2\theta_1{}^2-\tfrac{1}{2}(f_0^{(1)})^2+2(k')^{-2}e^{2f_0}(k^2e^{2f_0}+1)] \\ k'e^{-2f_0}\theta_1 \\ k\,e^{2f_0} \end{bmatrix}, \tag{A8b}$$

where k, k', and $\theta_1=d\theta/dt\,|_{t=0}$ are constants which partially define a geodesic, and

$$f(\theta)=\sum_n a_n\sin(n\theta+\gamma n)\,,$$

where the constants a_n and γ_n partially define the spacetime. If $\theta_0=\theta(t=0)$, $f_0^{(n)}=d^nf(\theta)/d\theta^n\,|_{\theta=\theta_0}$ and $f_0=f(\theta_0)$. Only terms of order t^{-1} and order t^0 are included above, although higher-order terms can be computed as well. The approximations we have used to find the geodesics complicate the calculations of the other frame vectors, but they can also be found in principle. In this paper we have restricted our use of Moncrief universe frame vectors to involve only $e^{\mu}_{(0)}$.

[1]C. W. Misner and A. H. Taub, Zh. Eksp. Teor. Fiz. **55**, 233 (1968) [Sov. Phys. JETP **28**, 122 (1969)].
[2]S. W. Hawking and G. F. R. Ellis, *The Large-Scale Structure of Spacetime* (Cambridge University Press, Cambridge, 1973).
[3]G. F. R. Ellis and B. G. Schmidt, Gen. Relativ. Gravit. **8**, 915 (1977).
[4]G. F. R. Ellis and B. G. Schmidt, Gen. Relativ. Gravit. **10**, 987 (1979).
[5]D. A. Konkowski, T. M. Helliwell, and L. C. Shepley, preceding paper, Phys. Rev. D **31**, 1178 (1985).
[6]D. A. Konkowski and T. M. Helliwell, Phys. Lett. **91A**, 149 (1982).

[7]E. Kasner, Am. J. Math. **43**, 217 (1921).
[8]A. H. Taub, Ann. Math. **53**, 472 (1951).
[9]E. T. Newman, L. Tamburino, and T. J. Unti, J. Math. Phys. **4**, 915 (1963).
[10]C. W. Misner, J. Math. Phys. **4**, 924 (1963).
[11]V. Moncrief, Phys. Rev. D **23**, 312 (1981).
[12]A. Einstein and N. Rosen, J. Franklin Inst. **223**, 43 (1937).
[13]R. Gowdy, Phys. Rev. Lett. **27**, 826 (1971).
[14]R. Gowdy, Ann. Phys. (N.Y.) **83**, 203 (1974).
[15]F. J. Tipler, C. J. S. Clarke, and G. F. R. Ellis, in *General Relativity and Gravitation,* edited by A. Held (Plenum, New York, 1980).
[16]V. Moncrief, Ann. Phys. (N.Y.) **132**, 87 (1981).
[17]V. Moncrief, Ann. Phys. (N.Y.) **141**, 83 (1982).
[18]C. J. S. Clarke, Commun. Math. Phys. **41**, 65 (1975).
[19]C. J. S. Clarke, Commun. Math. Phys. **49**, 17 (1976).
[20]G. F. R. Ellis and A. R. King, Commun. Math. Phys. **38**, 119 (1974).
[21]S. T. C. Siklos, Gen. Relativ. Gravit. **13**, 433 (1981).
[22]W. A. Hiscock and D. A. Konkowski, Phys. Rev. D **26**, 1225 (1982).
[23]A. Lapedes, Commun. Math. Phys. **51**, 121 (1976).
[24]S. Bonanos, Commun. Math. Phys. **22**, 190 (1971).
[25]L. C. Shepley, Proc. Natl. Acad. Sci. USA **52**, 1403 (1964).
[26]L. C. Shepley, Ph.D. dissertation, Princeton University, 1965.
[27]M. P. Ryan and L. C. Shepley, *Homogeneous Relativistic Cosmologies* (Princeton University Press, Princeton, New Jersey, 1975).
[28]V. Moncrief and J. Isenberg, Commun. Math. Phys. **89**, 387 (1983).
[29]V. Moncrief, Gen. Relativ. Gravit. **15**, 309 (1983).
[30]A. R. King, Phys. Rev. **D11**, 763 (1975).
[31]D. A. Konkowski and L. C. Shepley, Gen. Relativ. Gravit. **14**, 61 (1982).
[32]See, for example, A. R. Edmonds, *Angular Momentum in Quantum Mechanics* (Princeton University Press, Princeton, New Jersey, 1957); E. P. Wigner, *Group Theory and Its Application to the Quantum Mechanics of Atomic Spectra* (Academic, New York, 1959).
[33]N. A. Chernikov and E. A. Tagirov, Ann. Inst. Henri Poincaré **9**, 109 (1968).
[34]C. G. Callan, S. Coleman, and R. Jackiw, Ann. Phys. (N.Y.) **59**, 42 (1970).
[35]J. L. Synge, *Relativity: The General Theory* (North-Holland, Amsterdam, 1960).
[36]N. Batakis and J. M. Cohen, Ann. Phys. (N.Y.) **73**, 578 (1972).
[37]D. Brill, Phys. Rev. **133**, B845 (1964).
[38]M. Carmeli, Ch. Charach, and S. Malin, Phys. Rev. **76**, 79 (1981).
[39]Ch. Charach and S. Malin, Phys. Rev. D **21**, 3284 (1980).
[40]See, for example, G. Arfken, *Mathematical Methods for Physicists* (Academic, New York, 1966).