Cosmologies with quasiregular singularities. I. Spacetimes and test waves

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The nature of spacetimes with quasiregular singularities is discussed. Such singularities are the end points of incomplete, inextendible geodesics at which the Riemann tensor and its derivatives remain at least bounded in all parallel-propagated orthonormal frames; observers approaching such a singularity would find that their world lines come to an end in a finite proper time, without encountering infinite tidal forces. Particular attention is paid to Taub-NUT-(Newman-Unti-Tamburino) type cosmologies, which are an interesting class of spacetimes containing quasiregular singularities. These cosmologies are characterized by incomplete geodesics which spiral infinitely around a topologically closed spatial dimension: They include the $R^3\times S^1$ and $R^1\times T^3$ flat Kasner universes, the two-parameter family of Taub-NUT universes, and an infinite-dimensional subclass of Einstein-Rosen-Gowdy spacetimes studied by Moncrief. The global structure of each of these spacetimes is described. The flat Kasner and Moncrief universes both possess a null hypersurface which is a Cauchy and a Killing horizon and which contains a quasiregular singularity; the Taub-NUT universes possess two such null hypersurfaces. Timelike geodesics exhibit two sorts of behavior in the vicinity of any one of these null hypersurfaces: They may pass right through the hypersurface or they may approach it asymptotically, spiraling around a closed spatial dimension. The behavior of scalar test fields in each of the Taub-NUT-type cosmologies is also very similar in the vicinity of a singularity-containing null hypersurface. In each case there are three types of wave modes: There are modes whose amplitude diverges logarithmically in the vicinity of the null hypersurface; there are other modes whose phase diverges logarithmically; and there are modes without any divergent behavior. It is shown that generic finite data on an initial Cauchy hypersurface lead to divergent test fields at the null hypersurface. The possibility of the existence of additional Taub-NUT-type cosmologies among spatially homogeneous spacetimes of the various Bianchi types and among inhomogeneous spacetimes is also discussed. The geodesic and scalar field behavior in these spacetimes is used in a subsequent paper to investigate the stability of Taub-NUT-type cosmologies.

I. INTRODUCTION

A great deal of insight has been garnered over the years into the properties of spacetime singularities. There is no completely general definition of a singularity, but useful definitions have been formulated for many situations, and the existence of singularities in many general classes of spacetimes has been proven by using these definitions. The nature of singularities is, however, still somewhat of a mystery. Therefore, in this paper and the following paper, we will explore some of the properties of a particularly intriguing class of spacetime singularity: the *quasiregular* singularity.

A singularity in a maximal[1] spacetime (i.e., a connected, C^∞, Hausdorff manifold M together with a Lorentzian metric $g_{\mu\nu}$) is indicated by incomplete geodesics or incomplete curves of bounded acceleration.[2] One usually thinks of a singularity as the boundary of a spacetime,[2–4] since by definition a spacetime is smooth and all irregular points have been excised. Unfortunately, no one definition of a singularity has been found which is applicable in all situations.[5,6] One of the more useful ways to attach a boundary to a singular spacetime is by a b (bundle)-boundary construction.[2,3] The b-boundary is the projection into a spacetime of a natural boundary attached to a higher-dimensional Riemannian manifold associated with the spacetime. In Schmidt's original[7] b-boundary construction, the Riemannian manifold is the bundle of frames over spacetime, with positive-definite metric induced by the affine connection. Boundary points of the frame bundle are determined by giving end points to all Cauchy sequences which do not converge in the frame bundle. The bundle boundary is then projected down to make a boundary for the spacetime. There are other (equivalent) formulations of the b-boundary construction,[8,9] but we will only need to use the one stated in order to describe the nature of singularities.

The major difficulty with the singularity theorems which prove the existence of singularities in large classes of spacetimes is that even though they predict incomplete geodesics in inextendible spacetimes, they do not describe what causes the incompleteness. In other words, they say nothing about the nature of the spacetimes or the type of singularity present. Early attempts at classifying singularities did not try to divide all singularities into different categories but rather were meant as meaningful descriptions of the types of singularities known in exact solutions.[2,10] Later classification schemes attempted to divide all singularities of a sufficiently pathological nature into one category or another. The first such comprehensive scheme was proposed by Ellis and King;[11] it was later refined by Ellis and Schmidt.[12]

The Ellis-Schmidt classification scheme describes the singularity structure of a spacetime (M,g) on which the Riemann tensor is k times continuously differentiable (i.e., is C^k); it uses a b-boundary construction to determine the location of singular points. If the b boundary is nonempty, there are two possibilities.

1. A point q in the b boundary is a C^r $(r \geq 0^-)$ regular boundary point if the spacetime (M,g) can be embedded in a larger spacetime (M',g') such that the Riemann tensor is C^r and q is an interior point in M'; or

2. a point q in the b boundary is a C^r $(r \geq 0^-)$ singular boundary point if the spacetime (M,g) is not extendible through q in a C^r way.

A singular boundary point q can then be classified as a scalar curvature singularity, a nonscalar curvature singularity, or a quasiregular singularity. We have the following definitions:[12]

1. A singular point q is a C^k (or C^{k-}) quasiregular singularity $(k \geq 0)$ if all components or derivatives of the Riemann tensor $R_{abcd;e_1 \cdots e_k}$ evaluated in an orthonormal (ON) frame parallel propagated (PP) along an incomplete geodesic ending at q are C^0 (or C^{0-}). In other words, the Riemann tensor components and derivatives tend to finite limits (or are bounded) in every PPON frame.

2. A singular point q is a C^k (or C^{k-}) curvature singularity $(k \geq 0)$ if this is not true, and it can be categorized as either (a) a C^k (or C^{k-}) nonscalar curvature singularity if all scalars in g_{ab}, η_{abcd}, and $R_{abcd;e_1 \cdots e_k}$ tend to a finite limit (or are bounded), i.e., tend to C^0 (or C^{0-}) functions, or (b) a C^k (or C^{k-}) scalar curvature singularity if some scalar does not tend to a C^0 (or a C^{0-}) function.

Scalar curvature singularities are the best known and most thoroughly investigated class of singularities. These "big bang" and "black hole" singularities correspond to one's usual concept of a real physical singularity, near which something "physical" diverges and near which all observers feel unbounded tidal forces. Less well understood are the nonscalar curvature singularities. No curvature scalars diverge in this case, yet some Riemann tensor components evaluated in a PPON frame along an incomplete curve do not tend to finite limits (or become unbounded). In fact, there is an ON basis in which all components of the Riemann tensor and its derivatives $R_{abcd;e_1 \cdots e_k}$ are well behaved; the Lorentz transformation which relates this basis to the PPON basis along an incomplete curve is, however, badly behaved. The physical effect is that all observers who fall into a nonscalar curvature singularity feel infinite tidal forces, but observers can move arbitrarily close to the singularity on other curves and feel no untoward effects.[13]

The least well-understood singularities are the quasiregular ones. No observers near a quasiregular singularity, including those who fall into the singularity itself, feel unbounded tidal forces. In all reasonable frames the Riemann tensor is completely finite. It is this unusual class of singularities which is our subject. In fact, in this paper a particularly interesting class of spacetimes with quasiregular singularities is identified and chosen for special consideration: the "Taub-NUT (Newman-Unti-Tamburino)-type" cosmologies.[14,15] These cosmologies are characterized by incomplete geodesics which spiral infinitely around a topologically closed spatial dimension. To our knowledge these are the only exact cosmological solutions to Einstein's equations which possess quasiregular singularities.

We study in particular three classes of exact solutions to Einstein's equations with Taub-NUT-type singularities: $R^3 \times S^1$ and $R^1 \times T^3$ flat Kasner universes, the two-parameter family of Taub-NUT universes, and an infinite-dimensional subclass of Einstein-Rosen-Gowdy spacetimes which we call Moncrief universes. Therefore our study encompasses the entire spectrum of cosmological solutions to Einstein's equations, including flat spacetimes, spatially homogeneous but anisotropic spacetimes, and inhomogeneous spacetimes.

The plan of this paper is as follows. In Sec. II, we discuss the general nature of spacetimes with quasiregular singularities. In Sec. III, we explore the Taub-NUT-type examples of spacetimes with quasiregular singularities by investigating their global structure, geodesic and test-field behavior, and the solution of a Cauchy problem. Finally, in Sec. IV, we examine the likelihood of the existence of generalized Taub-NUT-type cosmologies.

II. SPACETIMES WITH QUASIREGULAR SINGULARITIES

At a C^k (or C^{k-}) quasiregular singularity the Riemann tensor components $R_{abcd;e_1 \cdots e_k}$ in a PPON frame are at least C^0 (or C^{0-}). In this section, we will briefly review some of the well-known general properties of this mildest sort of true singularity, before specializing to those which are Taub-NUT-type.

A great deal is known about quasiregular singularities.[12] One of the most important properties is their local extendibility:[16,17]

If a point p is a C^k quasiregular singularity (k an integer >0), then every curve γ ending at p has a neighborhood U in the spacetime M such that U is isometric to a neighborhood U' in another spacetime M' with the image of γ being extendible in M'.

One must be careful, as Beem[18] warns, to be precise in one's definition of local extendibility (e.g., the definition in Hawking and Ellis[3] is incomplete), or one can find oneself claiming that Minkowski spacetime is locally extendible. Notice that the local extendibility property points out the global nature of quasiregular singularities.

All known examples[12,19] of spacetimes with quasiregular singularities are made by cutting and gluing together pieces of a regular spacetime (i.e., a nonsingular one). A simple example is the Riemannian cone in two dimensions: The vertex is a quasiregular singularity, since the cutting and pasting of a flat Riemannian space to make a cone causes two separate lines in the Riemannian space to be only a single line in the cone; therefore, there are not enough directions at the vertex for it to be a regular point. In all such cut and pasted quasiregular singularities, the singularity arises because there are too few or too many points near the singularity for it to be a regular point.[12]

Ellis and Schmidt[12] have classified such quasiregular singularities as either *elementary* or *complicated*. An elementary quasiregular singularity arises from identifications of a regular spacetime under a discrete group of isometries that leaves a set of points ("fixed points") invariant. Certain subclasses of elementary singularities may be defined by describing the nature of the set of fixed points. However, a complete classification scheme of these relatively well-behaved elementary singularities has not been found. Complicated quasiregular singularities are even more difficult to characterize since they are made by gluing together elementary quasiregular singularities.

Another useful division of quasiregular singularities exists (see, e.g., Ellis and Schmidt[12] and Tipler, Clarke, and Ellis[3]). Quasiregular singularities can be considered *specialized, holes,* or *primeval* depending on their properties:

1. A *specialized* quasiregular singularity q occurs if the spacetime is specialized on a curve γ ending at q, and points not Hausdorff separated from it, if the limiting values in a parallel-propagated frame on γ of the Weyl and Ricci tensors are invariant under some Lorentz transformation. Those quasiregular singularities corresponding to imprisoned incompleteness (e.g., Taub-NUT spacetime) are necessarily specialized. In addition, Clarke[20,21] has shown that any quasiregular singularity which can be reached from a globally hyperbolic region of spacetime is specialized:

If M is a globally hyperbolic, maximal C^{0-} spacetime, if p is the singular end point of an inextendible, incomplete causal curve, and if the Riemann tensor is not specialized at p, then p is a C^{0-} curvature singularity.

The condition that the spacetime is specialized at q is usually taken to indicate that the spacetime is Petrov-type D (although it could be Petrov-type O or N) and electrovac (although the physically "unrealistic" cases of negative pressure or negative density are possible).[20,12]

2. A *hole* is a quasiregular singularity which develops without reasonable cause from regular Cauchy data. A hole-free spacetime M is one where "if S is any acausal surface in M, then there is no spacetime M' with a 1:1 isometric mapping $\theta:D(S)\rightarrow M'$ such that $\theta(D(S))$ is properly contained in $D(\theta(S))$," where D signifies a domain of dependence.[20]

3. A *primeval* quasiregular singularity is one that has existed for all time. Clarke[21] has shown that if a strongly causal spacetime is nowhere specialized, hole-free, and contains only quasiregular singularities, then every singularity accessible on a future-directed causal curve is primeval.

A handful of exact solutions with quasiregular singularities are known. Here we will examine Taub-NUT, flat Kasner with $R^1\times T^3$ and $R^3\times S^1$ topology, and the Moncrief universes. In each, the quasiregular singularity is specialized. In fact, each of these singular spacetimes is Taub-NUT type: Each contains incomplete geodesics which spiral infinitely about a topologically closed spatial dimension.

III. TAUB-NUT-TYPE COSMOLOGIES

In this section we examine three classes of exact cosmological solutions to Einstein's equations with Taub-NUT-type singularities. We discuss in each case global properties of two inequivalent maximal Hausdorff extensions and a maximal non-Hausdorff extension and then describe the geodesic and test-field behaviors in one maximal Hausdorff extension of each cosmology.

A. Spacetimes

Flat Kasner spacetime. The Kasner universes[22] are a special class of Bianchi type-I spatially homogeneous but anisotropic solutions of the vacuum Einstein equations. A particular example is the flat Kasner universe, with metric

$$ds^2=-dt^2+t^2d\psi^2+d\theta^2+d\phi^2 \; . \tag{3.1}$$

On an R^4 manifold, a simple coordinate transformation converts this metric into the usual Minkowski metric; in that case, the points $t=0$ in Eq. (3.1) indicate only a coor-

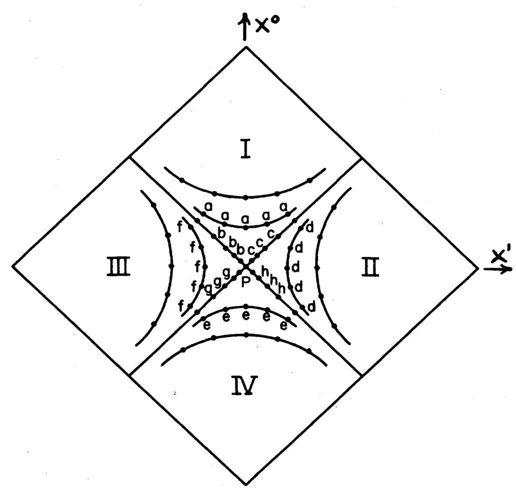

FIG. 1. Penrose diagram of the covering space of a (t,ψ) slice of the flat Kasner universes. R^4 flat Kasner is only a portion (Region I) of Minkowski spacetime (Regions I + II + III + IV). For $R^1\times T^3$ or $R^3\times S^1$ flat Kasner, the points a are identified, the points b are identified, etc., and the original spacetimes can be extended in two inequivalent Hausdorff ways across the null hypersurfaces shown: One extension is given by Regions I + II, and the other by Regions I + III. A maximally extended non-Hausdorff spacetime consists of regions I + II + III + IV with the point P removed.

dinate singularity. However, if the manifold structure has the topology $R^1 \times T^3$ or $R^3 \times S^1$, for example, in which one or more of the spatial dimensions are wrapped up, then the spacetime possesses a true singularity.

For an $R^3 \times S^1$ flat Kasner universe, the points (t,ψ,θ,ϕ) and $(t,\psi+na,\theta,\phi)$ are identified, where n takes on all integer values from $-\infty$ to $+\infty$, a is a nonzero constant, and the coordinate ranges are $t \in (0,\infty)$, $\psi \in [0,a]$, $\theta \in (-\infty,\infty)$, and $\phi \in (-\infty,\infty)$. For an $R^1 \times T^3$ flat Kasner universe, the points (t,ψ,θ,ϕ) and $(t,\psi+na,\theta+mb,\phi+lc)$ are identified, where n,m,l take on all integer values from $-\infty$ to ∞, a,b,c are nonzero constants, and the coordinate ranges are $t \in (0,\infty)$, $\psi \in [0,a]$, $\theta \in [0,b]$, and $\phi \in [0,c]$. Both the $R^3 \times S^1$ and $R^1 \times T^3$ cosmologies can be extended through $t=0$ to one of two inequivalent maximal Hausdorff spacetimes with metric

$$ds^2 = \pm 2 d\psi' dt' + 2t'(d\psi')^2 + (d\theta')^2 + (d\phi')^2 , \qquad (3.2)$$

where $t' = t^2/2$, $\psi' = \psi \mp \ln t$, $\theta' = \theta$, and $\phi' = \phi$.

Figure 1 illustrates the covering space for a (t,ψ) slice of either $R^1 \times T^3$ or $R^3 \times S^1$ flat Kasner; it is simply a portion of an (x^0,x^1) slice of a two-dimensional Minkowski spacetime with $ds^2 = -(dx^0)^2 + (dx^1)^2$. Region I $(x^0 > |x^1|)$ is isometric to a slice of the original flat Kasner cosmology, since the coordinate transformation $x^0 = t\cosh(\psi)$ and $x^1 = t\sinh(\psi)$ yields a Minkowski metric with the point identification

$$(x^0,x^1) \leftrightarrow (x^0\cosh(na) + x^1\sinh(na) ,\; x^1\cosh(na) + x^0\sinh(na)) . \qquad (3.3)$$

The physics beyond the surfaces $x^0 = x^1$ $(x^1 > 0)$ and $x^0 = -x^1$ $(x^1 < 0)$ is not, however, determined from the evolution of Cauchy data given on a spacelike slice in the original flat Kasner spacetime. The boundaries $x^0 = |x^1|$ are Cauchy horizons; one must choose the type of extension made across these boundaries. An analytic extension across $x^0 = x^1$ $(x^1 > 0)$ into Region II or across $x^0 = x^1$ $(x^1 < 0)$ into Region III yields one of the maximal Hausdorff spacetimes given by Eq. (3.2); choice of the upper (lower) sign gives a metric analytic over Regions I + III (Regions I + II). If both extensions are performed, however, the resulting spacetime is non-Hausdorff. In fact, one can extend the original flat Kasner spacetime to a maximal non-Hausdorff spacetime which includes Region I + II + III + IV, modulo the action of the isometry group of this spacetime, but which excludes the point $P = (x^0,x^1) = (0,0)$. Since timelike and null geodesics hit P, even this maximal non-Hausdorff extension is incomplete. For a discussion of the extension process, see the Appendix.

The point P which must be omitted in the original flat Kasner spacetime and also in any extensions, including those which are non-Hausdorff, can be identified as the location of a singularity. This singularity is quasiregular according to the Ellis and Schmidt classification scheme,[12] since the Riemann tensor in any frame is convergent, in fact zero. It is the only singular point in the non-Hausdorff extensions; it exists in both Hausdorff extensions as well, so we term it an essential quasiregular singularity. In the Hausdorff extensions, the entire line $x^0 = x^1$ (or the line $x^0 = -x^1$) in the covering space forms the boundary of the manifold, and is a quasiregular singularity according to the classification scheme. We term the boundary line a nonessential quasiregular singularity (except for the point P) since the line is nonsingular in the non-Hausdorff extension or in the alternate Hausdorff extension. In the Hausdorff extensions the boundary line maps into $t' = 0$, $\psi' = \pm\infty$ in the metrics of Eq. (3.2), so the singularity is contained in the $t' = 0$ null hypersurface, but with infinite values of ψ'.

Taub-NUT spacetime. Taub-NUT spacetime[1,23–25] is an analytic extension of the original Taub universe, the spatially homogeneous, anisotropic, vacuum solution to Einstein's equations with topology $R^1 \times S^3$ and with Bianchi type-IX symmetries. The Taub metric is[23]

$$ds^2 = -U^{-1}dt^2 + (2l)^2 U(d\psi + \cos\theta\, d\phi)^2 + (t^2 + l^2)(\sin^2\theta\, d\phi^2 + d\theta^2) , \qquad (3.4)$$

where $U(t) = -1 + 2(mt + l^2)(t^2 + l^2)^{-1}$, m,l are positive constants, ψ,θ,ϕ are Euler angle coordinates on S^3 with ranges $0 \le \Psi \le 4\pi$, $0 \le \theta \le \pi$, and $0 \le \phi \le 2\pi$, and $t_- < t < t_+$, where $t_\pm = m \pm (m^2 + l^2)^{1/2}$. The constant-$t$ spatially homogeneous hypersurfaces of Taub spacetime turn null at t_+ and t_-; there are extensions across these lightlike hypersurfaces into regions isometric to the NUT spacetime of Newman, Tamburino, and Unti.[24] In the NUT universes, constant-t hypersurfaces are timelike, and the spacetimes are spatially inhomogeneous and acausal. The extensions of Taub spacetime are discussed by Misner and Taub,[1,25] Ryan and Shepley,[10] and Hawking and Ellis.[2]

There are two inequivalent analytic maximal Hausdorff extensions of Taub spacetime, given by

$$ds^2 = U(2l)^2(d\bar{\psi} + \cos\bar{\theta}\, d\bar{\phi})^2 \mp 2(2l)(d\bar{\psi} + \cos\bar{\theta}\, d\bar{\phi})d\bar{t} + (\bar{t}^2 + l^2)(d\bar{\theta}^2 + \sin^2\bar{\theta}\, d\bar{\phi}^2) , \qquad (3.5)$$

where

$$\bar{\psi} = \psi \pm t(2l)^{-1} \mp (2K_+)^{-1}\ln(t - t_+) \pm (2K_-)^{-1}\ln(t - t_-)$$

with $K_\pm = \pm l(t_+ - t_-)[(t_\pm)^2 + l^2]^{-1}$. In both extensions, the region $t_- < t < t_+$ is isometric to the original Taub spacetime, and both extensions have closed timelike lines in the NUT regions (i.e., for $t < t_-$ and $t > t_+$). There are still incomplete geodesics, although no larger Hausdorff extension is possible; any further extension is non-Hausdorff. A global picture of Taub-NUT spacetime is given in Fig. 2. The points P are quasiregular singularities,[12] and the lines $t = t_+$ and $t = t_-$ are Cauchy and Killing horizons in this maximal non-Hausdorff spacetime. For further discussion, see Hawking and Ellis[2] and the Appendix.

Moncrief spacetime. The Moncrief universes[26] are an infinite-dimensional class of Einstein-Rosen-Gowdy[27,28] spacetimes without curvature singularity. They have diagonal metrics

$$ds^2 = e^{2a}(-dt^2 + d\theta^2) + t^2 e^{2b} d\psi^2 + e^{-2b} d\phi^2 \qquad (3.6)$$

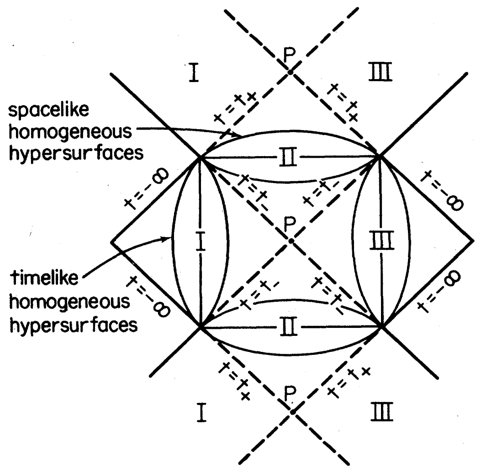

FIG. 2. Penrose diagram of the covering space of a (t,ψ) slice of the maximal non-Hausdorff extension of Taub-NUT spacetime. Each point represents a two-sphere. The points P which are omitted from the spacetime are quasiregular singularities, and the null hypersurfaces $t=t_{\pm}$ are Cauchy and Killing horizons.

defined on an $R^1\times T^3$ manifold with

$$b(t,\theta)=\sum_{n=-\infty}^{\infty} a_n J_0(nt)\sin(n\theta+\gamma_n) \tag{3.7}$$

and

$$a(t,\theta)=b(t,\theta)+\int_0^t ds\left[s\left[\frac{\partial b(s,\theta)}{\partial s}\right]^2+s\left[\frac{\partial b(s,\theta)}{\partial\theta}\right]^2\right], \tag{3.8}$$

where J_0 is the regular Bessel function of zeroth order. To avoid convergence of series questions, only a finite number of the coefficients a_n are allowed to be nonzero. The coordinate ranges are $t\in(0,\infty)$, $\psi\in[0,2\pi]$, $\theta\in[0,2\pi]$, and $\phi\in[0,2\pi]$.

Two inequivalent maximal Hausdorff extensions across the null hypersurface $t=0$ are given by[26]

$$ds^2=\mp\frac{1}{4t'}(e^{2a'}-e^{2b'})(dt')^2\pm\tfrac{1}{2}e^{2b'}d\psi'dt' +\frac{t'}{4}e^{2b'}(d\psi')^2+e^{2a'}(d\theta')^2+e^{-2b'}(d\phi')^2\,, \tag{3.9}$$

where

$$b'(t',\theta')=\sum_{n=0}^{\infty} a_n' J_0(nt'^{1/2})\sin(n\theta'+\gamma_n) \tag{3.10}$$

and

$$a'(t',\theta')=b'(t',\theta')+\tfrac{1}{2}\int_0^t ds'\left[4s'\left[\frac{\partial b(s',\theta')}{\partial s'}\right]^2+\left[\frac{\partial b(s',\theta')}{\partial\theta'}\right]^2\right]. \tag{3.11}$$

The region $t'>0$ is isometric to the original spacetime defined by Eq. (3.6); the original and extended coordinates are related by the transformation $t'=t^2/2$, $\psi'=2(\psi\mp lnt)$, $\theta'=\theta$, and $\phi'=\phi$, where we have imposed the appropriate coordinate identifications on ψ', θ', and ϕ'.

These extensions across the null hypersurface at $t'=0$ have been chosen to be analytic. We illustrate the analyticity explicitly in order to introduce notation which will be used in later sections of this paper. Clearly a' and b' are analytic. This implies immediately that all metric components except for $g_{t't'}=\mp(4t')^{-1}[\exp(2a')-\exp(2b')]$ are analytic. To examine $g_{t't'}$, one can expand a' and b' in power series away from t':

$$b'(t',\theta')=\sum_{j=0}^{\infty}\frac{(t')^j}{(j!)^2 4j}f^{(2j)}(\theta') \tag{3.12}$$

and

$$a'(t',\theta')=\sum_{j=0}^{\infty}\frac{(t')^j}{4j}\left[\frac{f^{(2j)}}{(j!)^2}+\sum_{k=0}^{j-1}\frac{2(j-k)}{j(k!)^2[(j-k)!]^2}[kf^{(2k)}f^{(2j-2k)}+(j-k)f^{(2k+1)}f^{(2j-2k-1)}]\right], \tag{3.13}$$

where

$$f(\theta')=\sum_{n=0} a_n\sin(n\theta'+\gamma_n) \tag{3.14}$$

and

$$f^{(n)}=\frac{d^n f(\theta')}{d\theta'^n}\,. \tag{3.15}$$

Thus, near $t'=0$ one finds

$$b'(t',\theta')=f(\theta')+\tfrac{1}{4}f^{(2)}(\theta')t'+O(t'^2)\,, \tag{3.16}$$

$$a'(t',\theta')=f(\theta')+\tfrac{1}{4}[f^{(2)}+2(f^{(1)})^2]t'+O(t'^2)\,, \tag{3.17}$$

$$e^{\pm 2b'}=e^{\pm 2f}[1\pm\tfrac{1}{2}f^{(2)}t'+O(t'^2)]\,, \tag{3.18}$$

$$e^{\pm 2a'}=e^{\pm 2f}\{1\pm\tfrac{1}{2}[f^{(2)}+2(f^{(1)})^2]t'+O(t'^2)\}\,. \tag{3.19}$$

Therefore, the $g_{t't'}$ component is

$$g_{t't'}=\mp\frac{1}{4t'}[e^{2a'}-e^{2b'}] \tag{3.20a}$$

$$=\pm\tfrac{1}{4}e^{2f}(f^{(1)})^2+O(t') \tag{3.20b}$$

which is clearly analytic (cf., Moncrief[26]). Therefore, both of the inequivalent Hausdorff extensions of Eq. (3.9) are analytic. As for the $R^1\times T^3$ and $R^3\times S^1$ flat Kasner and Taub-NUT spacetimes, each of these extensions is maximal, and one cannot make both extensions and still retain the Hausdorff nature of these spacetimes. This point is further discussed in the Appendix. The null hypersurface at $t=0$ in the maximal Hausdorff extensions is a Cauchy and Killing horizon which contains a quasiregular singularity, in the sense that the boundary points $t=0$, $\psi'=\infty$ are the end points of incomplete geodesics. A quasiregular singularity is still present in a maximal non-Hausdorff extension, and its location at P is clearly illustrated in Fig. 1.

B. Geodesic behavior

In this section we examine the behavior of geodesics in one maximal Hausdorff extension of each Taub-NUT-type cosmology considered in Sec. III A. In particular, we look at the extensions:

(1) $R^1\times T^3$ or $R^3\times S^1$ flat Kasner spacetime with

$$ds^2=2d\psi\,dt+2t\,d\psi^2+d\theta^2+d\phi^2\ . \tag{3.21}$$

(2) $R^1\times S^3$ Taub-NUT spacetime with

$$ds^2=2(2l)^2(d\psi+\cos\theta\,d\phi)^2+2(2l)(d\psi+\cos\theta\,d\phi)dt+(t^2+l^2)(d\theta^2+\sin^2\theta\,d\phi^2)\ . \tag{3.22}$$

(3) $R^1\times T^3$ Moncrief spacetime with

$$ds^2=\frac{1}{4t}[e^{2a}-e^{2b}]dt^2+\tfrac{1}{2}e^{2b}d\psi\,dt+\frac{t}{4}e^{2b}(d\psi)^2+e^{2a}(d\theta)^2+e^{-2b}(d\phi)^2\ , \tag{3.23}$$

where primes on the coordinates have been suppressed.

The equations governing the geodesic paths $x^\mu(\lambda)$ are

$$u^\mu_{;\sigma}u^\sigma=0\ , \tag{3.24}$$

where $u^\mu=\dot{x}^\mu=dx^\mu/d\lambda$ and where λ is an affine parameter. We scale λ so that $E=u^\mu u_\mu=0,-1$ for null and timelike paths, respectively. We will show that in the neighborhood of the null hypersurface which evolves in each cosmology, the geodesic paths behave similarly in all of the Taub-NUT-type universes.

Flat Kasner spacetime. Four first integrals of the motion in extended flat Kasner spacetime (with either $R^1\times T^3$ or $R^3\times S^1$ topology) are

$$\dot{t}+2t\dot{\psi}=C_1\ , \tag{3.25a}$$

$$\dot{\theta}=C_2\ , \tag{3.25b}$$

$$\dot{\phi}=C_3\ , \tag{3.25c}$$

$$2\dot{\psi}(\dot{t}+t\dot{\psi})+\alpha=0\ , \tag{3.25d}$$

where C_1,C_2,C_3 are arbitrary constants, and $\alpha=C_2{}^2+C_3{}^2-E$. The case $\alpha=0$ corresponds to null geodesics with no θ or ϕ motion. If also $\dot{\psi}=0$, then $\dot{t}=C_1$, so these geodesics are straight lines extending from $t=+\infty$ to $t=-\infty$, passing through $t=0$, and they are complete in the affine parameter λ. If $\dot{\psi}\neq0$, then $\dot{t}=-C_1$, so that if $C_1\neq0$, the null geodesics follow the paths $\psi=\psi_0-\ln|t|$, which start at $t=+\infty$ or $t=-\infty$ and then spiral in the ψ direction as they approach $t=0$. These geodesics are incomplete, approaching $t=0$ in a finite affine length in spite of spiraling an infinite number of times as $t\to0$. If $\dot{\psi}\neq0$, but $C_1=0$, the geodesic is confined to the null hypersurface $t=0$.

If $\alpha\neq0$, both null and timelike geodesics are permitted, and there can also be θ and ϕ motion. If $C_1=0$, geodesics obey $d\psi/dt=-(2t)^{-1}$, and so spiral infinitely as they approach $t=0$. If $C_1\neq0$, then the tangent vector is

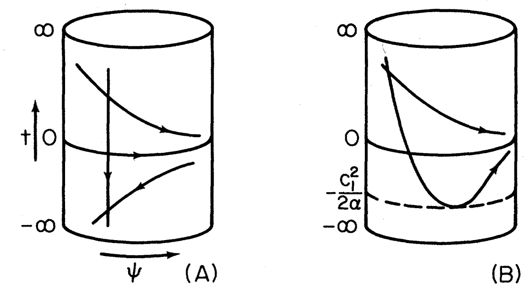

FIG. 3. Null and timelike geodesic orbits for an extended $R^1\times T^3$ or $R^3\times S^1$ flat Kasner universe: $A(t,\psi)$ slice is shown. (A) Null geodesic behavior when θ and ϕ are both constant ($\alpha=0$). (B) Null geodesic behavior when θ and ϕ are not both constant ($\alpha>0$), or timelike geodesic behavior ($\alpha\geq1$).

$$\begin{bmatrix}\dot{t}\\ \dot{\psi}\\ \dot{\theta}\\ \dot{\phi}\end{bmatrix}=\begin{bmatrix}(C_1{}^2+2\alpha t)^{1/2}\\ \frac{1}{2t}[C_1\mp(C_1{}^2\mp2\alpha t)^{1/2}]\\ C_2\\ C_3\end{bmatrix} \tag{3.26}$$

and so

$$\frac{d\psi}{dt}=-\frac{1}{2t}\frac{[(C_1{}^2+2\alpha t)^{1/2}\mp C_1]}{(C_1{}^2+2\alpha t)^{1/2}}\ . \tag{3.27}$$

These geodesics all begin at $t=+\infty$, and depending upon the choice of sign, either spiral toward $t=0$ from above, or else pass through the $t=0$ hypersurface to the negative t region, turn back at $t=-C_1{}^2/2\alpha$, and spiral toward $t=0$ from below. All $\alpha\neq0$ geodesics are incomplete; see Fig. 3.

Each geodesic on the extended flat Kasner spacetime corresponds to a straight-line geodesic in the I + III (i.e., $x^0>x^1$) region of the covering (Minkowski) spacetime, as pictured in Fig. 1. The slope and x^0 intercept of the straight lines may be identified as the extended flat Kasner geodesic constants $\pm(1+\alpha)^{1/2}$ and C_1, respectively. Thus the $C_1=0$ geodesics are those which strike the origin $x^0=x^1=0$ of Minkowski spacetime. A geodesic segment which crosses the line $x^1=-x^0$ (for $x^0\neq0$) is a pass-through segment in flat Kasner, passing through $t=0$ at a finite value of ψ. A geodesic segment which approaches a boundary point $x^1=x^0$ (including $x^0=0$) is a spiraling segment in flat Kasner, approaching $t=0$ as $\psi\to\infty$. The behavior of geodesics as illustrated in Fig. 3 is easily understood by considering the straight-line timelike geodesics in the $x^0>x^1$ region of the covering spacetime.

Taub-NUT spacetime. Geodesics in Taub-NUT spacetime have been investigated by Misner and Taub,[1] Shepley,[29] and Ryan and Shepley.[10] The spacetime has four Killing vectors:

$$\xi_1=-\sin\phi\frac{\partial}{\partial\theta}-\cos\phi\left[\cot\theta\frac{\partial}{\partial\phi}-\csc\theta\frac{\partial}{\partial\psi}\right]\ , \tag{3.28a}$$

$$\xi_2=\cos\phi\frac{\partial}{\partial\theta}-\sin\phi\left[\cot\theta\frac{\partial}{\partial\phi}-\csc\theta\frac{\partial}{\partial\psi}\right], \quad (3.28b)$$

$$\xi_3=\partial/\partial\phi\ , \quad (3.28c)$$

$$\eta=-\partial/\partial\psi\ , \quad (3.28d)$$

and four corresponding constants of the motion,

$$p_a=u^\mu\xi_{\mu a}\quad (a=1,2,3)\ , \quad (3.29)$$

$$p_{||}=-u^\mu\eta_\mu\ , \quad (3.30)$$

where u^μ is a solution of the geodesic equation (3.24). Define also $p^2=p_\perp{}^2-p_{||}{}^2$, where $p^2=p^ap_a$; then orientation of the coordinate axes to let $p_1=p_2=0$ and $p_3=p\geq 0$ gives $p_{||}=p\cos\theta$. The first integrals of motion are then

$$\dot t=\pm[(4l^2)^{-1}p_{||}{}^2+U(p_\perp{}^2(t^2+l^2)^{-1}-E)]^{1/2}\ , \quad (3.31a)$$

$$\dot\psi=(2lU)^{-1}(\dot t+p_{||}(2l)^{-1})-p_{||}(t^2+l^2)^{-1}\ , \quad (3.31b)$$

$$\dot\theta=0\ , \quad (3.31c)$$

$$\dot\phi=p(t^2+l^2)^{-1}\ . \quad (3.31d)$$

From these equations one can study the behavior of geodesics in the vicinity of the null hypersurfaces at $t_\pm=m\pm(m^2+l^2)^{1/2}$. For example, in the neighborhood of t_+ let $t=t_+(1+\delta)$ where $\delta\ll 1$. If $p_{||}=0$, then $d\psi/dt=\dot\psi/\dot t\sim-1/(2l\delta)$; these geodesics all spiral as the singularity at t_+ is approached. If $p_{||}\neq 0$, there are two cases corresponding to the upper and lower signs in the equation for $\dot t$. For the upper $(+)$ sign, $d\psi/dt\sim-1/(l\delta)$, and the geodesics spiral. For the lower $(-)$ sign, $d\psi/dt\sim$constant, and the geodesics pass through $t=t_+$ without obstruction.

Spiraling geodesics wrap up an infinite number of times in a finite affine length, since Eq. (3.31a) can be used to show that $\lambda\sim\delta^{1/2}$ for spiraling geodesics in a compact neighborhood of $t=t_+$: these geodesics are all incomplete. Some geodesics which pass through $t=t_+$ are complete; see, for example, Ryan and Shepley.[10] Geodesic behavior near t_+ (or t_-) in Taub-NUT is qualitatively the same as that near $t=0$ in the flat Kasner universes, and geodesic paths near $t_\pm$ resemble those near $t=0$ in Fig. 3.

Moncrief spacetime. Geodesic behavior in the Moncrief universes is complicated by the spatial inhomogeneity of these spacetimes. The lower degree of symmetry allows only two independent Killing vector fields and only three easily obtained first integrals of motion. The remaining second-order equation is nontrivial, so we investigate geodesic behavior only in the region of interest near $t=0$. Three first integrals are

$$\dot t+t\dot\psi=k'\ , \quad (3.32a)$$

$$\dot\phi=k\,e^{2b}\ , \quad (3.32b)$$

$$E=-\frac{1}{4t}(e^{2a}-e^{2b})\dot t^2-\frac{1}{2}e^{2b}\dot t\dot\psi+\frac{t}{4}e^{2b}\dot\psi^2$$
$$+e^{2a}\dot\theta^2+e^{-2b}\dot\phi^2\ , \quad (3.32c)$$

where k and k' are constants. If $k'=0$, then

$$\dot t=-t\dot\psi\ , \quad (3.33a)$$

$$\dot\psi=\pm 2t^{-1/2}[g(t,\theta,\dot\theta)]^{1/2}\ , \quad (3.33b)$$

where

$$g(t,\theta,\dot\theta)\equiv\dot\theta^2+k^2e^{2(b-a)}-E\,e^{-2a}\ . \quad (3.34)$$

These geodesics spiral infinitely in the ψ direction as $t\to 0$, since $d\psi/dt=-t^{-1}$. If $k'\neq 0$, then

$$\dot t=\mp k\,e^{-2b}F(t,\theta,\dot\theta)\ , \quad (3.35a)$$

$$\dot\psi=k't^{-1}e^{-2b}[1\pm F(t,\theta,\dot\theta)]\ , \quad (3.35b)$$

where

$$F(t,\theta,\dot\theta)=[e^{2(b-a)}+4t(k')^{-2}e^{4b}g(t,\theta,\dot\theta)]^{1/2}\ . \quad (3.36)$$

The quantity $e^{2(b-a)}\to 1$ as $t\to 0$, so upper signs in Eqs. (3.35) correspond to spiraling behavior, since then $d\psi/dt\sim-2t^{-1}$; lower signs correspond to pass-through behavior, since then $d\psi/dt$ is finite as $t\to 0$.

A complete description of the geodesics, including their completeness, requires the solution of the θ equation of motion

$$\ddot\theta+a'\dot\theta^2+2\dot a\dot t\dot\theta+\frac{1}{4t}(a'-b'e^{2(b-a)})\dot t^2-\frac{b'}{2}e^{2(b-a)}\dot t\dot\psi$$
$$-\frac{t}{4}e^{2(b-a)}b'\dot\psi^2+e^{-2(a+b)}b'\dot\phi^2=0\ , \quad (3.37)$$

where $\dot x^\mu=dx^\mu/d\lambda$, but where an overdot or prime on a or b means a partial derivative with respect to t or θ. Using the first integrals of Eqs. (3.32), this equation becomes

$$\ddot\theta+2a'\dot\theta^2+(a'+b')k^2e^{2(b-a)}-Ea'e^{-2a}$$
$$\mp 2\dot a\dot\theta G(t,\theta,\dot\theta)=0\ , \quad (3.38)$$

where

$$G(t,\theta,\dot\theta)=\begin{cases}2t^{1/2}[g(t,\theta,\dot\theta)]^{1/2} & (\text{if } k'=0)\ ,\\ k'e^{-2b}F(t,\theta,\dot\theta) & (\text{if } k'\neq 0)\ .\end{cases} \quad (3.39)$$

We wish to find $\dot t,\dot\psi,\dot\theta,\dot\phi$ as functions of t near $t=0$. We begin by assuming θ can be expanded as a power series in t for a particular geodesic:

$$\theta=\theta_0+\theta_1t+\theta_2t^2+\cdots\ . \quad (3.40)$$

Then the functions $a(t,\theta)$ and $b(t,\theta)$ can also be written as power series in t, and therefore $\dot t,\dot\psi,\dot\theta,\dot\phi$ can be found as functions of t and the coefficients θ_i using the known first integrals. Finally, constraints on the θ_i can be found using the second-order θ equation.

Using Taylor series, the function $f(\theta)$ defined by Eq. (3.14) is

$$f(\theta)=f(\theta_0+\theta_1t+\theta_2t^2+\cdots)$$
$$=f_0+\theta_1f_0^{(1)}t+(\theta_2f_0^{(1)}+\tfrac{1}{2}\theta_1{}^2f_0^{(2)})t^2+\cdots\ , \quad (3.41)$$

where

$$f^{(n)}\equiv\left.\frac{d^{(n)}f(\theta)}{d\theta^n}\right|_{\theta=\theta_0}\ . \quad (3.42)$$

Therefore, along the geodesic,

$$b(t,\theta(t))=f_0+(\theta_1 f_0^{(1)}+\tfrac{1}{4}f_0^{(2)})t+O(t^2)\ , \tag{3.43a}$$

$$a(t,\theta(t))=f_0+[\theta_1 f_0^{(1)}+\tfrac{1}{4}f_0^{(2)}+\tfrac{1}{2}(f_0^{(1)})^2]t+O(t^2)\ . \tag{3.43b}$$

Now $\dot\theta=d\theta/d\lambda=\dot t d\theta/dt$; either Eq. (3.33a) or (3.35a) can then be solved for $\dot t$ to give in each case

$$\dot t^2=[1-4t(d\theta/dt)^2]^{-1}e^{-2a}[k'^2e^{-2b}+4t(k^2e^{2b}-E)]\ . \tag{3.44}$$

All quantities on the right can be expanded in powers of t along the geodesic; the result is

$$\dot t=\mp 2t^{1/2}e^{-f_0}(k^2e^{f_0}-E)^{1/2}+O(t^{3/2}) \tag{3.45}$$

if $k'=0$, and

$$\dot t=\mp k'e^{-2f_0}\{1+[2\theta_1{}^2-2\theta_1 f_0^{(1)}-\tfrac{1}{2}f_0^{(2)}-\tfrac{1}{2}(f_0^{(1)})^2+2e^{2f_0}(k')^{-2}(k^2e^{2f_0}-E)]t\}+O(t^2) \tag{3.46}$$

if $k'\neq 0$.

Geodesic completeness can be described using these two expressions. Both Eq. (3.45) and Eq. (3.46) can be integrated to give the affine parameter λ in terms of t. If $k'=0$, then $\lambda\sim\pm t^{1/2}$ for small t, so all such geodesics spiral an infinite number of times in the ψ direction as $t\to 0$, and all are incomplete. If $k'\neq 0$, $\lambda\sim\mp t$, and there are two classes of geodesics: the upper sign corresponds to those which spiral infinitely in a finite affine length and are therefore incomplete; the lower sign corresponds to those which pass through $t=0$ without obstruction. However, if the geodesic is timelike, it is nevertheless incomplete, since it turns around at some $t<0$ and then spirals infinitely, in a finite proper time, just below $t=0$. Null geodesics with $k'=0$ may be complete or incomplete, however, as in flat Kasner or Taub-NUT. Figure 3 is again a good representation of timelike geodesic paths near $t=0$.

To complete the description of geodesics near the singularity, one can easily find expressions for $\dot\psi,\dot\theta,\dot\phi$ analogous to Eqs. (3.45) and (3.46) for $\dot t$, expanded in powers of t along a geodesic. Finally, the results may be substituted into the second-order equation (3.38), which in effect imposes constraints upon the coefficients θ_i. The result is that θ_0 and θ_1 are arbitrary constants, corresponding to the value of θ and the slope $d\theta/dt$ of the geodesic as $t\to 0$. The higher coefficients $\theta_2\theta_3,\ldots,$ are then determined by θ_1 and other known quantities.

C. Test-field behavior

In order to understand more fully the nature of the null hypersurface which forms a barrier between the globally hyperbolic and noncausal regions in each maximally extended Hausdorff Taub-NUT-type cosmology, we examine the behavior of scalar test fields in a neighborhood of the null hypersurface. We find that these fields behave quite similarly in each cosmology.

The scalar wave equation is[30]

$$(\Box-\xi R-M^2)=0\ , \tag{3.47}$$

where $\Box$ is the Laplace-Beltrami operator, ξ indicates the coupling ($\xi=0$ for minimal coupling and $\xi=\tfrac{1}{6}$ for conformal coupling), and M is the mass. Since each cosmology we consider is a vacuum solution of the field equations, the curvature scalar $R=0$, and so scalar wave behavior is independent of curvature coupling.

Flat Kasner spacetime. In the extended flat Kasner coordinates (t,ψ,θ,ϕ), Eq. (3.47) becomes

$$-2t\Phi_{,tt}-2\Phi_{,t}+2\Phi_{,t\psi}+\Phi_{,\theta\theta}+\Phi_{,\phi\phi}-M^2\Phi=0\ . \tag{3.48}$$

The mode solutions are

$$\Phi^\lambda_{\kappa\mu}=\tau^\lambda_{\kappa\mu}(t)\exp[i(\kappa\psi+\lambda\theta+\mu\phi)]\ , \tag{3.49}$$

where each κ,λ,μ is a positive or negative integer (or zero) if the corresponding coordinate is periodic, or a continuous parameter if the corresponding coordinate is unbounded. The function $\tau^\lambda_{\kappa\mu}(t)$ obeys

$$t\ddot\tau^\lambda_{\kappa\mu}+(1-i\kappa)\dot\tau^\lambda_{\kappa\mu}+\tfrac{1}{2}q\tau^\lambda_{\kappa\mu}=0\ , \tag{3.50}$$

where $q=\lambda^2+\mu^2+M^2$. The solutions may be expressed in terms of Hankel functions as has previously been shown in the electromagnetic field case;[31,32] however, we are interested primarily in the field near the $t=0$ singularity, so it is more convenient to express the solution as a power series. If $\kappa\neq 0$, then

$$\tau(t;\kappa\neq 0)=\sum_{n=0}^{\infty}a_n^{\kappa\lambda\mu}t^n+e^{i\kappa\ln|t|}\sum_{n=0}^{\infty}b_n^{\kappa\lambda\mu}t^n\ ; \tag{3.51}$$

if $\kappa=0$, then

$$\tau(t;\kappa=0)=(1+\ln|t|)\sum_{n=0}^{\infty}a_n^{0\lambda\mu}t^n+\sum_{n=0}^{\infty}b_n^{0\lambda\mu}t^n\ . \tag{3.52}$$

The $a_0^{\kappa\lambda\mu}$ and $b_0^{\kappa\lambda\mu}$ are arbitrary constants, and the recursion relations are

$$a_{n+1}^{\kappa\lambda\mu}=\frac{-qa_n^{\kappa\lambda\mu}}{2(n+1)(n+1-i\kappa)}\quad(\text{all }\kappa)\ , \tag{3.53a}$$

$$b_{n+1}^{\kappa\lambda\mu}=\frac{-qb_n^{\kappa\lambda\mu}}{2(n+1)(n+1+i\kappa)}\quad(\kappa\neq 0)\ , \tag{3.53b}$$

$$b_{n+1}^{0\lambda\mu}=\frac{-q}{2(n+1)^2}\left[b_n^{0\lambda\mu}-\frac{2}{n+1}a_n^{0\lambda\mu}\right]\ . \tag{3.53c}$$

For the $R^1\times T^3$ cosmology the general wave solution is therefore

$$\Phi=\Phi^{(0)}+\Phi^{(1)}+\Phi^{(2)}\ , \tag{3.54}$$

where

$$\Phi^{(0)}=\ln|t|\sum_{\lambda,\mu}\left[\sum_n a_n^{0\lambda\mu}t^n\right]e^{i(\lambda\theta+\mu\phi)}\ , \tag{3.54a}$$

$$\Phi^{(1)}=\sum_{\kappa,\lambda,\mu}\left[\sum_n b_n^{\kappa\lambda\mu}t^n\right]e^{i(\kappa\ln|t|+\kappa\psi+\lambda\theta+\mu\phi)}, \quad (3.54b)$$

$$\Phi^{(2)}=\sum_{\kappa,\lambda,\mu}\left[\sum_n a_n^{\kappa\lambda\mu}t^n\right]e^{i(\kappa\psi+\lambda\theta+\mu\phi)}. \quad (3.54c)$$

Here the sum $\sum_{\kappa,\lambda,\mu}$ is over all integer values of κ, λ, and μ from $-\infty$ to $+\infty$, including $\kappa=0$. For the $R^3\times S^1$ cosmology, where ψ is the periodic coordinate, the sums over λ and μ become integrals, and the coefficients $a_n^{\kappa\lambda\mu}$ and $b_n^{\kappa\lambda\mu}$ become continuous functions of λ and μ. Note that in both cosmologies, $\Phi^{(0)}$ is independent of ψ, and has a logarithmic amplitude divergence as $t\to 0$; $\Phi^{(1)}$ is finite in amplitude but has a logarithmic phase divergence; and $\Phi^{(2)}$ has no divergences.

Taub-NUT spacetime. In the extended Taub-NUT coordinates (t,ψ,θ,ϕ), the wave equation (3.22) becomes

$$(t^2-2mt-l^2)\Phi_{,tt}+2(t-m)\Phi_{,t}-(t^2+l^2)l^{-1}\Phi_{,t\psi}-tl^{-1}\Phi_{,\psi}+\cot^2\theta\Phi_{,\psi\psi}-2\cot\theta(\sin\theta)^{-1}\Phi_{,\psi\phi}$$
$$+\Phi_{,\theta\theta}+\cot\theta\Phi_{,\theta}+(\sin\theta)^{-2}\Phi_{,\phi\phi}-(t^2+l^2)M^2\Phi=0 . \quad (3.55)$$

The mode solutions are

$$\Phi_{\mu\kappa}^{\lambda}(t,\psi,\theta,\phi)=\tau_\kappa^\lambda(t)e^{i\kappa\psi}d_{\mu\kappa}^{\lambda}(\theta)e^{i\mu\phi}, \quad (3.56)$$

where μ is an integer, because $0\le\phi\le 2\pi$ with 0 and 2π identified, and κ is an integer or half-integer, because $0\le\psi\le 4\pi$ with 0 and 4π identified. The function $d_{\mu\kappa}^{\lambda}(\theta)$ obeys

$$\left[\frac{-d}{d\theta^2}-\cot\theta\frac{d}{d\theta}+(\sin\theta)^{-2}(\mu^2+\kappa^2-2\mu\kappa\cos\theta)\right]d_{\mu\kappa}^{\lambda}=\lambda(\lambda+1)d_{\mu\kappa}^{\lambda} \quad (3.57)$$

which is an equation encountered in the quantum mechanical problem of the symmetric top, where ψ,θ,ϕ are Euler angles of the top.[33] The $d_{\mu\kappa}^{\lambda}(\theta)$ are therefore the well-known Wigner functions,[34] except that κ and (as we will show) λ can take on half-integer as well as integer values.

If we let

$$d_{\mu\kappa}^{\lambda}=(1-\xi)^{|\mu+\kappa|/2}\xi^{|\mu-\kappa|/2}F(\xi), \quad (3.58)$$

where $\xi=\sin^2(\theta/2)$, then $F(\xi)$ satisfies the hypergeometric equation

$$\xi(1-\xi)F''+[c-(a+b+1)\xi]F'-abF=0 \quad (3.59)$$

with

$$a=\lambda+1+\tfrac{1}{2}(|\mu+\kappa|+|\mu-\kappa|), \quad (3.60a)$$

$$b=-\lambda+\tfrac{1}{2}(|\mu+\kappa|+|\mu-\kappa|), \quad (3.60b)$$

$$c=1+|\mu-\kappa| . \quad (3.60c)$$

The series

$$F(\xi)=\sum_{n=0}^{\infty}\beta_n\xi^n \quad (3.61)$$

with

$$\beta_{n+1}=\frac{(n+a)(n+b)}{(n+1)(n+c)}\beta_n \quad (3.62)$$

truncates at some n if b is zero or a negative integer; this it must do to keep $d_{\mu\kappa}^{\lambda}(\pi)$ finite. The truncation condition may be stated as follows: Let $x=|\mu+\kappa|+|\mu-\kappa|$; then if λ is an integer $(\lambda=0,1,2,\ldots)$, x must be an even number (or zero), with $x\le 2\lambda$. If λ is a half-integer $(\lambda=\frac{1}{2},\frac{3}{2},\ldots)$, x must be an odd number, with $x\le 2\lambda$. If λ is a half-integer, κ must also be a half-integer; but if κ is a half-integer, λ may be an integer or a half-integer. In terms of μ and κ, the rules regarding allowed values are as follows.

(1) If λ is an integer $(\lambda=0,1,2,\ldots)$, μ and κ can each take on all integer values from $-\lambda$ to $+\lambda$.

(2) If λ is an integer $(\lambda=1,2,3,\ldots)$, κ can also take on half-integer values from $-|\mu|+\frac{1}{2}$ up to $|\mu|-\frac{1}{2}$ for given μ, where μ can take on all integer values from $-\lambda$ to $+\lambda$, excluding zero.

(3) If λ is a half-integer $(\lambda=\frac{1}{2},\frac{3}{2},\frac{5}{2},\ldots)$, κ can take on all half-integer values from $-\lambda$ to $+\lambda$, and for given κ,μ can take on all integer values from $-|\kappa|+\frac{1}{2}$ to $|\kappa|-\frac{1}{2}$.

The equation for $\tau_\kappa^\lambda(t)$ is

$$(t^2-2mt-l^2)\ddot{\tau}_\kappa^\lambda+\left[2(t-m)-\frac{i\kappa}{l}(t^2+l^2)\right]\dot{\tau}_\kappa^\lambda$$
$$-\left[\lambda(\lambda+1)-\kappa^2+\frac{i\kappa t}{l}+(t^2+l^2)M^2\right]t_\kappa^\lambda=0 . \quad (3.63)$$

It can be solved by power series about either of the $t=t_\pm$ hypersurfaces. Let $t=t_\pm+T$, where $t_\pm$ means either t_+ or t_-; then if $\kappa\neq 0$,

$$\tau_\kappa^\lambda(T)=\sum_{n=0}^{\infty}a_n^{\kappa\lambda}T^n+\exp\left[\frac{i\kappa t_\pm}{l}\ln|T|\right]\sum_{n=0}^{\infty}b_n^{\kappa\lambda}T^n , \quad (3.64)$$

where the $a_0^{\kappa\lambda}$ and $b_0^{\kappa\lambda}$ are arbitrary. The recursion relations for the other $a_n^{\kappa\lambda}$ and $b_n^{\kappa\lambda}$ are easily obtained. If $\kappa=0$,

$$\tau_0^\lambda(T)=\left[1+\ln\left|\frac{T}{t_\pm}\right|\right]\sum_{n=0}^{\infty}a_n^{0\lambda}T^n+\sum_{n=0}^{\infty}b_n^{0\lambda}T^n , \quad (3.65)$$

where the $a_0^{0\lambda}$ and $b_0^{0\lambda}$ are arbitrary and the recursion relations for subsequent $a_n^{0\lambda}$ and $b_n^{0\lambda}$ can easily be obtained. The complete solution of the wave equation in the vicinity of t_+ or t_- is therefore

$$\Phi(t,\psi,\theta,\phi)=\Phi^{(0)}+\Phi^{(1)}+\Phi^{(2)} , \quad (3.66)$$

where

$$\Phi^{(0)}=\ln\left|\frac{T}{t_\pm}\right|\sum_{\lambda=0}^{\infty}\sum_{\mu=-\lambda}^{\lambda}\left[\sum_{n=0}^{\infty}a_n^{0\lambda\mu}T^n\right]d_{\mu 0}^{\lambda}(\theta)e^{i\mu\phi} , \quad (3.66a)$$

$$\Phi^{(1)}=\sum_{\kappa,\lambda,\mu}\left[\sum_{n=0}^{\infty}b_n^{\kappa\lambda\mu}T^n\right]e^{i\kappa[\psi+(t_\pm/l)\ln|T|]}d_{\mu\kappa}^{\lambda}(\theta)e^{i\mu\phi} , \quad (3.66b)$$

$$\Phi^{(2)}=\sum_{\kappa,\lambda,\mu}\left[\sum_{n=0}^{\infty}a_n^{\kappa\lambda\mu}T^n\right]e^{i\kappa\psi}d_{\mu\kappa}^{\lambda}(\theta)e^{i\mu\phi} , \quad (3.66c)$$

where

$$\sum_{\kappa,\lambda,\mu}\equiv\sum_{\lambda=0}^{\infty}\sum_{\mu=-\lambda}^{\lambda}\sum_{\kappa=-\lambda}^{\lambda}+\sum_{\lambda=1}^{\infty}\sum_{\substack{\mu=-\lambda\\(\mu\neq 0)}}^{\lambda}\sum_{\kappa=-|\mu|+1/2}^{|\mu|-1/2}+\sum_{\lambda=1/2}^{\infty}\sum_{\kappa=-\lambda}^{\lambda}\sum_{\mu=-|\kappa|+1/2}^{|\kappa|-1/2} . \quad (3.67)$$

Here each sum is from the lower limit to the upper limit noted for the quantity κ, λ, or μ, increasing by integer steps. Note that except for the form of the θ dependence and the selection rules for the indices, scalar waves on the Taub-NUT universe are very similar to those on the flat Kasner universe.

Moncrief spacetime. In the extended Moncrief coordinates (t,ψ,θ,ϕ) of Eq. (3.23), the wave equation (3.47) becomes

$$-t\Phi_{,tt}-\Phi_{,t}+2\Phi_{,t\psi}+\tfrac{1}{4}\Phi_{,\theta\theta}+\frac{1}{t}(e^{2(a-b)}-1)\Phi_{,\psi\psi}+\tfrac{1}{4}e^{2(a+b)}\Phi_{,\phi\phi}-\tfrac{1}{4}M^2e^{2a}\Phi=0 . \quad (3.68)$$

Mode solutions have the form $\Phi=F^{\kappa\mu}(t,\theta)e^{i\kappa\psi}e^{i\mu\phi}$ where κ and μ are integers, and $F^{\kappa\mu}$ obeys

$$-t\ddot{F}^{\kappa\mu}-(1-2i\kappa)\dot{F}^{\kappa\mu}+\tfrac{1}{4}(F^{\kappa\mu})''-g^{\kappa\mu}(t,\theta)F^{\kappa\mu}=0 , \quad (3.69)$$

where overdots and primes denote partial derivatives with respect to t and θ, respectively, and where

$$g^{\kappa\mu}(t,\theta)=\frac{\kappa^2}{t}(e^{2(a-b)}-1)+\frac{\mu^2}{4}e^{2(a+b)}+\frac{M^2}{4}e^{2a} . \quad (3.70)$$

Equation (3.68) cannot be separated in general, but it can be solved by power series. Using Eqs. (3.16) and (3.17) relating $a(t,\theta)$ and $b(t,\theta)$ to

$$f(\theta)=\sum_j a_j\sin(j\theta+\gamma_j) ,$$

we can write

$$g^{\kappa\mu}(t,\theta)=\sum_{m=0}^{\infty}g_m^{\kappa\mu}(\theta)t^m , \quad (3.71)$$

where the $g_m^{\kappa\mu}$ can be expressed in terms of $f(\theta)$: for example,

$$g_0^{\kappa\mu}(\theta)=\kappa^2(f'(\theta))^2+\frac{\mu^2}{4}e^{4f(\theta)}+\frac{M^2}{4}e^{2f(\theta)} . \quad (3.72)$$

Guided by the wave solutions in the flat Kasner universe, let

$$F^{\kappa\mu}(t,\theta)=\sum_n A_n^{\kappa\mu}(\theta)t^n+e^{i\kappa c\ln|t|}\sum_n B_n^{\kappa\mu}(\theta)t^n \quad (3.73)$$

for $\kappa\neq 0$, and

$$F^{0\mu}(t,\theta)=(1+\ln|t|)\sum_n A_n^{0\mu}(\theta)t^n+\sum_n B_n^{0\mu}(\theta)t^n \quad (3.74)$$

for $\kappa=0$, where the $A_n^{\kappa\mu}$ and $B_n^{\kappa\mu}$ are periodic functions of θ, and c is a number to be determined.

Substitutions of $F^{\kappa\mu}$ (for $\kappa\neq 0$) into Eq. (3.68) yields an infinite set of constraint equations in the coefficients, one for each time dependence. The equation of order t^{-1} is $-i\kappa c(i\kappa c-2i\kappa)B_0^{\kappa\mu}=0$, so that for nonzero κ, c, and $B_0^{\kappa\mu}$ we must choose $c=2$. The subsequent equations are then

$$O(t^0):\ -(1-2i\kappa)A_1+\tfrac{1}{4}A_0''-A_0g_0=0 , \quad (3.75a)$$

$$-(1+2i\kappa)B_1+\tfrac{1}{4}B_0''-B_0g_0=0 , \quad (3.75b)$$

$$O(t^1):\ -2(2-2i\kappa)A_2+\tfrac{1}{4}A_1''-(A_0g_1+A_1g_0)=0 , \quad (3.76a)$$

$$-2(2+2i\kappa)B_2+\tfrac{1}{4}B_1''-(B_0g_1+B_1g_0)=0 , \quad (3.76b)$$

etc. Thus each $A_0^{\kappa\mu}(\theta)$ and $B_0^{\kappa\mu}(\theta)$ is an arbitrary periodic function, and the subsequent $A_n^{\kappa\mu}(\theta)$ and $B_n^{\kappa\mu}(\theta)$ are determined by the relations

$$A_n(\theta)=\left[\tfrac{1}{4}A_{n-1}''(\theta)-\sum_{j=0}^{n-1}A_j(\theta)g_{n-j-1}(\theta)\right]\Big/ n(n-2i\kappa) , \quad (3.77)$$

$$B_n(\theta)=\left[\tfrac{1}{4}B_{n-1}''(\theta)-\sum_{j=0}^{n-1}B_j(\theta)g_{n-j-1}(\theta)\right]\Big/ n(n+2i\kappa) , \quad (3.78)$$

for each $\kappa(\neq 0)$ and μ. For the $\kappa=0$ modes, substitution of $F^{0\mu}$ into Eq. (3.68) yields a similar set of equations, which can be satisfied if $A_0^{0\mu}(\theta)$ and $B_0^{0\mu}(\theta)$ are arbitrary periodic functions, and the subsequent $A_n^{0\mu}$ and $B_n^{0\mu}$ are found from

$$A_n(\theta)=\left[\tfrac{1}{4}A_{n-1}''(\theta)-\sum_{j=0}^{n-1}A_j(\theta)g_{n-j-1}(\theta)\right]\Big/ n^2 , \quad (3.79)$$

$$B_n(\theta)=\left[\tfrac{1}{4}B''_{n-1}(\theta)-\sum_{j=0}^{n-1}B_j(\theta)g_{n-j-1}(\theta)-2nA_n(\theta)\right]\Big/n^2\,, \tag{3.80}$$

for $\kappa=0$ and any μ. Therefore all of these functions can be determined from the A_0's and B_0's.

Now because each $A_n^{\kappa\mu}$ and $B_n^{\kappa\mu}$ is periodic, one can write

$$A_n^{\kappa\mu}(\theta)=\sum_{\lambda=-\infty}^{\infty} a_n^{\kappa\lambda\mu}e^{i\lambda\theta}\,, \tag{3.81}$$

$$B_n^{\kappa\mu}(\theta)=\sum_{\lambda=-\infty}^{\infty} b_n^{\kappa\lambda\mu}e^{i\lambda\theta}\,, \tag{3.82}$$

so that the most general solution of the wave equation can be written

$$\Phi=\Phi^{(0)}+\Phi^{(1)}+\Phi^{(2)}\,, \tag{3.83}$$

where

$$\Phi^{(0)}=\ln|t|\sum_{\lambda,\mu}\left[\sum_{n=0}^{\infty}a_n^{0\lambda\mu}t^n\right]e^{i(\lambda\theta+\mu\phi)}\,, \tag{3.83a}$$

$$\Phi^{(1)}=\sum_{\kappa,\lambda,\mu}\left[\sum_{n=0}^{\infty}b_n^{\kappa\lambda\mu}t^n\right]e^{i\kappa(\psi+2\ln|t|)}e^{i(\lambda\theta+\mu\phi)}\,, \tag{3.83b}$$

$$\Phi^{(2)}=\sum_{\kappa,\lambda,\mu}\left[\sum_{n=0}^{\infty}a_n^{\kappa\lambda\mu}t^n\right]e^{i(\kappa\psi+\lambda\theta+\mu\phi)}\,. \tag{3.83c}$$

Here each parameter κ,λ,μ extends from $-\infty$ to $+\infty$, including zero. The a_0's and b_0's are arbitrary constants, and the subsequent a_n and b_n coefficients can in principle be found from them. The solution is almost identical in form to that in the flat Kasner case. However, in the Moncrief case, terms of fixed λ are not solutions, because of the nonseparability of the wave equation. That is, individual modes in the Φ given above are not generally solutions of the wave equation. In the special case that κ, μ, and M are all zero the function $g^{\kappa\mu}(t,\theta)=0$, and so Eq. (3.68) is separable. The wave equation in t and θ is then essentially the same as that in the flat Kasner spacetime, and the individual modes (each with a different λ) are solutions of the wave equation.

Summary of test-field behavior. The scalar wave solutions are similar on all of the Taub-NUT-type spacetimes. To display the similarities, let T be the time away from a singularity ($T=t$ for the flat Kasner and Moncrief cosmologies, and $T=t-t_\pm$ for the Taub-NUT cosmology). Then the general wave solution is

$$\Phi=\Phi^{(0)}+\Phi^{(1)}+\Phi^{(2)}\,, \tag{3.84}$$

where

$$\Phi^{(0)}=\ln|T|\sum_{\lambda,\mu}\left[\sum_{n}a_n^{0\lambda\mu}T^n\right]f_{0\mu}^{\lambda}(\theta,\phi)\,, \tag{3.84a}$$

$$\Phi^{(1)}=\sum_{\kappa,\lambda,\mu}\left[\sum_{n}b_n^{\kappa\lambda\mu}T^n\right]e^{i\kappa\alpha\ln|T|}f_{\kappa\mu}^{\lambda}(\psi,\theta,\phi)\,, \tag{3.84b}$$

$$\Phi^{(2)}=\sum_{\kappa,\lambda,\mu}\left[\sum_{n}a_n^{\kappa\lambda\mu}T^n\right]f_{\kappa\mu}^{\lambda}(\psi,\theta,\phi)\,. \tag{3.84c}$$

Here

$$\alpha=\begin{cases}1 & \text{(flat Kasner)},\\ 2 & \text{(Moncrief)},\\ t_\pm/l & \text{(Taub-NUT)}\end{cases} \tag{3.85}$$

and

$$f_{\kappa\mu}^{\lambda}(\psi,\theta,\phi)=\begin{cases}e^{i(\kappa\psi+\lambda\theta+\mu\phi)} & \text{(flat Kasner or Moncrief)},\\ e^{i\kappa\psi}d_{\kappa\mu}^{\lambda}(\theta)e^{i\mu\phi} & \text{(Taub-NUT)}\,.\end{cases} \tag{3.86}$$

The coefficients are different in each case; simple recursion relations for the a_n's and b_n's exist for the flat Kasner and Taub-NUT cases, since the wave equation is separable. For the Moncrief waves the coefficients may be found by the method outlined previously. The summation signs in Eq. (3.84) denote sums or integrals as the spacetime topology requires.

D. The Cauchy problem

The Cauchy problem on a Taub-NUT-type spacetime consists in finding a solution of the wave equation (3.47) subject to the conditions

$$\Phi|_{t_0}=A(\psi,\theta,\phi)\,, \tag{3.87a}$$

$$\left.\frac{\partial\Phi}{\partial t}\right|_{t_0}=B(\psi,\theta,\phi)\,, \tag{3.87b}$$

where A and B are bounded functions of the spatial coordinates with periodicities as required by the spacetime topology. The hypersurface $t=t_0$ must be spacelike, so $t_0>0$ for the flat Kasner and Moncrief universes, and $t_-<t_0<t_+$ for the Taub-NUT universe. We will show that finite data on t_0 generally evolves into divergent fields as one approaches the $T=0$ null hypersurface.

From the mode solutions given by Eqs. (3.84), clearly Φ converges as $T\to0$ if and only if the coefficients $a_0^{0\lambda\mu}$, which enter into the $\ln|T|$ terms, vanish for all λ,μ; it is possible to find Cauchy data at t_0 which is consistent with this convergence condition. The mode coefficients $a_0^{\kappa\lambda\mu}$ and $b_0^{\kappa\lambda\mu}$ can all be evaluated in terms of A and B in the usual way, by multiplying the expressions for $\Phi|_{t_0}$ and $\partial\Phi/\partial t|_{t_0}$ by wave modes and integrating over the coordinates ψ,θ,ϕ.

We now specialize to the $R^1\times T^3$ flat Kasner spacetime for simplicity; other spacetimes can be treated analogously. If one expresses A as the Fourier series

$$A(\psi,\theta,\phi)=\sum_{\kappa,\lambda,\mu}A_{\kappa\lambda\mu}\exp[i(\kappa\psi+\lambda\theta+\mu\phi)] \tag{3.88}$$

with a similar expression for B, one finds that the coefficients $a_0^{0\lambda\mu}=0$ if and only if the Fourier coefficients $A_{0\lambda\mu}$ and $B_{0\lambda\mu}$ are related by

$$B_{0\lambda\mu}=A_{0\lambda\mu}\sum_{n=1}^{\infty}\alpha_n n t_0{}^{n-1}\Big/\sum_{n=0}^{\infty}\alpha_n t_0{}^n\ , \tag{3.89}$$

where

$$\alpha_n=\frac{(-1)^n q^n}{2^n(n!)^2} \tag{3.90}$$

with $q=\lambda^2+\mu^2+M^2$. The coefficients $b_0^{0\lambda\mu}$ are then given by

$$b_0^{0\lambda\mu}=A_{0\lambda\mu}\Big/\sum_{n=0}^{\infty}\alpha_n t_0{}^n\ . \tag{3.91}$$

The convergence of Φ therefore restricts the ψ-independent (i.e., $\kappa=0$) Fourier terms of the Cauchy data. If neither A nor B has $\kappa=0$ modes, then Φ converges and also Φ has no $\kappa=0$ modes, because all coefficients $a_0^{0\lambda\mu}=b_0^{0\lambda\mu}=0$. More generally, if the ψ-independent modes of A and B are related by Eq. (3.89), then Φ still converges, but it may have the ψ-independent modes

$$\Phi(\theta,\phi)=\sum_{\lambda,\mu}\left[\sum_n b_n^{0\lambda\mu}t^n\right]e^{i(\lambda\theta+\mu\phi)} \tag{3.92}$$

along with ψ-dependent modes which are in no way constrained by the convergence condition. Thus generic Cauchy data at t_0 yield a divergent field at $T=0$, but Cauchy data may be chosen which lead to a convergent field.

In the case of massless scalar fields without θ or ϕ dependence, an exact expression for the field in terms of the Cauchy data is

$$\Phi(t,\psi)=A(\psi)+\tfrac{1}{2}f(t_0)\int_{\psi}^{\bar\psi}d\psi' B(\psi')\ , \tag{3.93}$$

where

$$\bar\psi=\psi+2\int_{t_0}^{t}dt'/f(t') \tag{3.94}$$

and $f(t)=2t$ for flat Kasner, $f(t)=t$ for Moncrief, and $f(t)=-2lU(t)$ for Taub-NUT. Expanding $B(\psi)$ in Fourier series

$$B(\psi)=\sum_{\kappa}B_{\kappa 00}\exp(i\kappa\psi) \tag{3.95}$$

and evaluating the integral in Eq. (3.93), one sees that $\Phi(t,\psi)$ converges as $T\to 0$ if and only if $B_{000}=0$. This result is consistent with the general $R^1\times T^3$ flat Kasner case, because if $\lambda=\mu=M=0$, then $q=0$, so $\alpha_0=1$ and $\alpha_n=0$ for $n\neq 0$, according to Eq. (3.90). Then convergence of Φ allows arbitrary $A_{\kappa 00}$ and $B_{\kappa 00}$, except that $B_{000}=0$, according to Eq. (3.89).[35]

IV. GENERALIZED TAUB-NUT-TYPE COSMOLOGIES

The exact solutions of Einstein's equations studied in Sec. III all had quasiregular singularities with very similar characteristics; in this section we will discuss the generic occurrence of this Taub-NUT-type behavior in spatially homogeneous and in spatially inhomogeneous cosmologies.

A. Spatially homogeneous cosmologies

The occurrence of a quasiregular singularity in the Bianchi type-I model, flat Kasner, and the Bianchi type-IX model, Taub-NUT, leads one to wonder whether such a singularity can evolve in other spatially homogeneous cosmologies. It is well known that all spatially homogeneous and isotropic cosmologies, i.e., the Friedmann-Robertson-Walker universes, have scalar curvature singularities.[2] Only when the spacetime has less symmetry (i.e., the isometry group is three- or four-dimensional rather than six-dimensional) are other types of singularities possible.[3,12] Therefore only in the Kantowski-Sachs universes[36,37] or the anisotropic Bianchi models could Taub-NUT-type behavior occur.

In each maximal Hausdorff Taub-NUT-type cosmology, the evolution of a null hypersurface is a key feature: It forms the barrier between a globally hyperbolic region of spacetime and a noncausal region which contains closed timelike lines. The null hypersurface is both a Cauchy horizon and a Killing horizon, and it contains a quasiregular singularity. We saw in Sec. III A and in the Appendix, that the singularity is a topological defect in each of these spacetimes, which remains even when we relax the Hausdorff requirement on the spacetime and consider the maximal non-Hausdorff extensions of these manifolds.

Here we consider a basis (a lightlike evolution basis) which may be used in the neighborhood of a null hypersurface in any Bianchi spatially homogeneous cosmology. This allows us to consider in detail the occurrence of Taub-NUT-type singularities in the general Bianchi type-I and -IX cosmologies and to discuss the possibility of Taub-NUT-type singularities in other spatially homogeneous cosmologies. The existence of such a lightlike evolution basis can be described by the following theorem, which was stated by Ryan and Shepley.[10]

Theorem I. If $H(t)$ is a hypersurface invariant under a three-dimensional simply transitive group G, and $H(\bar t)$ has lightlike geometry, then in a neighborhood of a point in $H(\bar t)$ there exists a basis of one-forms $\{\sigma^\mu\}$ such that (1) $\sigma^0=dt$ where t parametrizes the homogeneous hypersurfaces H and t is not unique, (2) $\sigma^i=b^i_j\omega^j$ where b^i_j are components of the t-dependent matrix $B(t)$ which is nonsingular at each t, (3) the one-forms ω^i obey $d\omega^i=\frac{1}{2}C^i_{jk}\omega^j\Lambda\omega^k$ where the C^i_{jk} are the structure constants of the group G, and (4) in the σ^μ system, the metric takes the block-diagonal form

$$ds^2=2(\sigma^0)(\sigma^1)+f(t)(\sigma^1)^2+(\sigma^2)^2+(\sigma^3)^2\ , \tag{4.1}$$

where $f(\bar t)=0$.

The proof of this theorem in the SO(3,R) case (i.e., Bianchi type-IX case) was first given by Shepley;[29] the proof for a general spatially homogeneous cosmology was obtained by Konkowski.[38] We will now use the theorem to investigate Bianchi type-I, Bianchi type-IX, and general homogeneous vacuum cosmologies.

Bianchi type-I. A Bianchi type-I model is characterized by structure constants $C^i_{st}=0$. The invariant basis $\{x_i\}$ is $\{\partial_1,\partial_2,\partial_3\}$ with dual one-forms $\omega^1=dx^1$, $\omega^2=dx^2$, and $\omega^3=dx^3$. Therefore, by Theorem I, if a null hypersurface evolves at $t=\bar t$, in a lightlike evolution basis the metric takes the form of Eq. (4.1),

$$ds^2=2\sigma^0\sigma^1+f(t)(\sigma^1)^2+(\sigma^2)^2+(\sigma^3)^2\ ,$$

where $f(\bar{t})=0$. Since $C^i_{st}=0$, the matrix $B(t)$ is diagonal at $t=\bar{t}$ and $\sigma^0=dt$, $\sigma^1=dx^1$, $\sigma^2=b_2dx^2$, and $\sigma^3=b_3dx^3$; therefore,

$$ds^2=2dt\,dx^1+f(t)(dx^1)^2+b_2{}^2(dx^2)^2+b_3{}^2(dx^3)^2 \ . \tag{4.2}$$

The functions $f(t)$, $b_2(t)$, and $b_3(t)$ are constrained by the field equations to obey the relations

$$\frac{-2\ddot{b}_3}{b_3}-\frac{2\ddot{b}_2}{b_2}-\ddot{f}=0 \ , \tag{4.3a}$$

$$\frac{2\dot{f}\dot{b}_3}{b_3}+\frac{\dot{f}\dot{b}_2}{b_2}+\ddot{f}=0 \ , \tag{4.3b}$$

$$\frac{2\dot{b}_2\dot{f}}{b_2}=0 \ , \tag{4.3c}$$

$$\frac{2\dot{b}_3\dot{f}}{b_3}=0 \ . \tag{4.3d}$$

Since $f(t)$ is assumed not to be a constant, Eqs. (4.3c) and (4.3d) imply that b_2 and b_3 are constants, and Eq. (4.3a) or (4.3b) implies that $f=at+b$, where a and b are constants. Therefore, at $t=\bar{t}$, the Riemann tensor $R^{\mu}_{\nu\sigma\tau}=0$, so that any Bianchi type-I cosmology that evolves a null hypersurface is flat at the null hypersurface. Furthermore, if we translate coordinates to relocate the zero of t, we can write the metric of Eq. (4.2) in the flat Kasner form of Eq. (3.21),

$$ds^2=2d\psi\,dt+2t(d\psi)^2+(d\theta)^2+(d\phi)^2 \ ,$$

at $t=0$. By continuity, it must take this form in a neighborhood of the null hypersurface. Therefore, whether a vacuum Bianchi type-I spacetime which evolves a null hypersurface develops in addition a quasiregular singularity depends entirely upon the topology of its manifold. In Sec. III A we showed that a flat Kasner metric defined on R^4 produces an extendible spacetime, but that the same metric on an $R^3\times S^1$ or $R^1\times T^3$ manifold leads to the development of a quasiregular singularity. Here, the spacetime metric takes the flat Kasner form near the null hypersurface, so exactly the same results are predicted.

We are considering general vacuum Bianchi type-I models; one would therefore like to know whether the homogeneous hypersurfaces can have a topology other than R^3, $R^2\times S^1$, and T^3. Ellis[39] has discussed this question for orientable hypersurfaces while drawing heavily on the work of Wolf.[40] Ellis shows that the only compact hypersurfaces invariant under the Bianchi type-I group have the topologies T^3, T^3 with one coordinate twisted by π, and T^3 with all three coordinates twisted by π. The noncompact possibilities are R^3, $R^2\times S^1$, $R^2\times S^1$ with the S^1 coordinate twisted by π, $R^1\times T^2$, and $R^1\times T^2$ with the T^2 coordinates twisted by π. Because all of these possible topologies for the homogeneous hypersurface involve identifications (except for the case of R^3), one would expect geodesics to "wrap up" with a finite affine length in all except R^3. Arguments against large Hausdorff extensions for the four-dimensional spacetimes would be the same as before, and in every case (except R^3), a Taub-NUT-type quasiregular singularity would form.

Bianchi type-IX. A similar analysis applies to Bianchi type-IX cosmologies.[10] We will state only the results.

Any vacuum Bianchi type-IX spacetime which contains a null hypersurface is described by a metric which can be transformed into the Taub-NUT form

$$ds^2=2\,dt\,\omega^1+f(t)(\omega^1)^2+b^2(t)[(\omega^2)^2+(\omega^3)^2] \tag{4.4}$$

in a neighborhood of the hypersurface. Here $d\omega^i=\frac{1}{2}\epsilon_{ijk}\omega^j\Lambda\omega^k$, $b(t)=[at+(4a)^{-1}]^{1/2}$ where a is a constant, and $f(t)$ equals zero at the null hypersurface.

Again, the topology of the homogeneous hypersurfaces is important. All such hypersurfaces can be covered by S^3, and are thus necessarily compact. As Ellis[39] argues, possible topologies are S^3, P^3 (real projective space), S^3/Z_n ($n>2$, and Z is the cyclic group), S^3/D_m ($m>2$, and D is the binary dihedral group), S^3/T, S^3/O, and S^3/I (where T, O, and I are the binary symmetry groups of the regular tetrahedron, the regular octahedron, and the regular isosahedron, respectively). The only other possibilities involve these manifolds with extra identifications corresponding to reflections or twists by π. The compactness of these three manifolds leads one to expect that if a null hypersurface exists, geodesics will wind up in finite proper time as in the Taub-NUT case, the spacetime will be inextendible to a larger Hausdorff manifold, and a quasiregular singularity will occur.

The general vacuum case. The possibility of Taub-NUT-type Bianchi type-I and -IX vacuum cosmologies has been considered in detail; we now focus on the occurrence of Taub-NUT-type singularities in general spatially homogeneous vacuum cosmologies.

In all Taub-NUT-type cosmologies the quasiregular singularity has been located in a null hypersurface. For any Bianchi cosmology, Theorem I gives the metric of Eq. (4.1) at a null hypersurface; one must then find the Riemann tensor and require that Einstein's equations are satisfied at the null hypersurface. The analysis is tedious, since one must consider each Bianchi type and all physically realistic stress-energy tensors; then one must test each resulting spacetime for the incomplete geodesic behavior which characterizes a maximal Hausdorff Taub-NUT-type cosmology (see, e.g., Clarke[41]). In order to find exact solutions of Einstein's equations with Bianchi symmetries and Taub-NUT-type singularities, we know of no substitution for this long, tedious procedure. There is, however, a way to show that one expects Taub-NUT-type cosmologies of each and every vacuum Bianchi type, and furthermore, that one would expect Taub-NUT-type Kantowski-Sachs universes.

In the vacuum Bianchi type-I and the Bianchi type-IX cases, a Taub-NUT-type singularity was associated both with the evolution of a null hypersurface and with a spacetime manifold whose homogeneous hypersurfaces were identified in some sense. The identifications caused classes of null and timelike geodesics to spiral infinitely with finite length as they approached the null hypersurface. In the general vacuum case, a Taub-NUT-type singularity is also expected to occur whenever a spacelike hypersurface turns null in a spacetime with identified homogeneous hypersurfaces, since one again expects irremovable incomplete paths.

Ellis and Schmidt[12] introduce the idea of an elementary quasiregular singularity. An elementary quasiregular singularity is defined as a singularity that arises from identifications of a regular spacetime under a discrete group of isometries that leaves a set of points invariant. The set of invariant points, the fixed points, must be deleted from the spacetime, since the spacetime can no longer be regular if they are left in after the identifications have been made. Since any geodesic which ends at an excised fixed point is incomplete, the spacetime would be singular, and since the spacetime is regular, the singularity would be quasiregular.

Clearly, the singularities in vacuum spatially homogeneous Taub-NUT-type cosmologies are elementary quasiregular singularities: The deleted fixed points are points in the null hypersurface; the manifold identifications are due to discrete subgroups of the isometry group of the spacetime. Such discrete subgroups are possible in all spatially homogeneous cosmologies (except some homogeneous ones), including both the Bianchi cosmologies and the Kantowski-Sachs cosmologies.[12] One might at first think it would be easy to list all the discrete subgroups of the isometry group in each case and therefore enumerate all vacuum spatially homogeneous Taub-NUT-type cosmologies. Unfortunately, as Ellis and Schmidt point out,[12] one cannot currently do this. The problem is equivalent to finding all discrete subgroups of the Lorentz group which give allowed identifications (e.g., not the reflection $t \rightarrow -t$) and generate fixed points. This is an unsolved problem (see, e.g., Schwarz[42]). Therefore, at present, all one can say is that Taub-NUT-type cosmologies are possible in each Bianchi class and in the Kantowski-Sachs models. Furthermore, one can note that a Taub-NUT-type singularity will occur only when an otherwise regular spacetime is identified under a discrete subgroup of the isometry group.

B. Spatially inhomogeneous cosmologies

The Moncrief universes are, in some sense, an inhomogeneous generalization of the $R^1 \times T^3$ flat Kasner universe. One might, therefore, reasonably expect similar inhomogeneous generalizations of the other Tabu-NUT-type spatially homogeneous cosmologies: $R^3 \times S^1$ flat Kasner, Taub-NUT, and the spacetimes discussed in the previous section. In this section, we more fully discuss these expectations.

Generalized Moncrief universes. In addition to Moncrief's infinite-parameter families of inhomogeneous vacuum Taub-NUT-type solutions we have discussed,[26] the existence of other classes of inhomogeneous cosmologies with Taub-NUT-type properties has been established as well. In particular Moncrief[43] has described the existence of an infinite dimensional family of Taub-NUT-type spacetimes defined on an $R^1 \times T^3$ manifold which satisfy the vacuum Einstein equations on some neighborhood of their Cauchy horizons. In subsequent work[44] he has extended the construction of generalized Taub-NUT-type inhomogeneous cosmologies to those on $R^1 \times S^3$ manifolds. In both cases, he assumes the existence of a single Killing vector field and shows that these cosmologies are exact solutions to Einstein's equations in a neighborhood of a compact null hypersurface. This compact null hypersurface is a Cauchy horizon and a Killing horizon, and it contains a quasiregular singularity in the sense that the boundary points $t=0, \psi'=\infty$ are the end points of incomplete geodesics.

In both the $R^1 \times T^3$ and $R^1 \times S^3$ generalizations described by Moncrief, the compact null hypersurfaces which evolve are ruled by closed null generators. However, analytic vacuum (or electrovacuum) spacetimes with compact null surfaces ruled by closed null orbits necessarily have a Killing symmetry.[45] Therefore, since the assumption of a Killing symmetry is essential to the construction of generalized Moncrief universes, more general examples (i.e., those without such a Killing symmetry) are not possible. Finally, Moncrief[46] has shown that the $R^1 \times T^3$ generalized Moncrief universes are unstable under generic linear perturbations. We expect $R^1 \times S^3$ generalized Moncrief universes to behave similarly. Both the linear perturbation analysis by Moncrief[46] and the studies by Moncrief and Isenberg[45] indicate the instability of these spacetimes and a probable change in the nature of the singularities in these models if a complete back-reaction calculation is carried out.

The general case. The discovery by Moncrief and Isenberg[45] that a Killing symmetry is necessary to obtain Taub-NUT-type spacetimes with compact null hypersurfaces shows that large classes of such spacetimes are not to be expected. However, one need not have compact hypersurfaces to obtain Taub-NUT-type spatially homogeneous cosmologies: Elementary quasiregular singularities occur whenever identifications are made of a regular spacetime in such a way that points have to be deleted from the manifold. Therefore, there may be large classes of inhomogeneous spacetimes which evolve identified but noncompact null hypersurfaces.

Again, one sees that the presence of a Taub-NUT-type quasiregular singularity is a topological effect, a defect in the topology of the spacetime itself. Furthermore, in an inhomogeneous spacetime, one expects that in addition to elementary quasiregular singularities, complicated quasiregular singularities might occur: These could be elementary quasiregular singularities pasted together, or other, possibly more physical, types of quasiregular singularities.

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APPENDIX: MAXIMAL NON-HAUSDORFF EXTENSIONS

A fiber-bundle approach best exhibits the impossibility of making both Hausdorff extensions simultaneously for

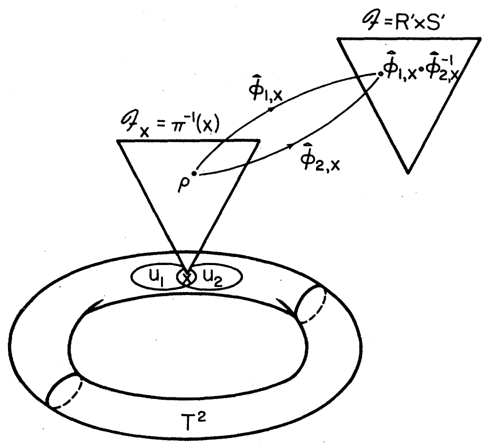

FIG. 4. $R^1\times T^3$ flat Kasner as a fiber bundle (M,B,π,G). The bundle (M,B,π) consists of the bundle space $M=R^1\times T^3$, the base space $B=T^2$, and the projection π: $R^1\times T^3\rightarrow T^2$. The typical fiber is $\mathscr{F}=R^1\times S^1$, and the structure group G is the isometry group of the flat Kasner spacetime. The family of open sets $\{u_j\}$ covers T^2, and $\phi_{j,x}$ is a homeomorphism of $\mathscr{F}_x$ onto $\mathscr{F}$.

the spatially homogeneous Taub-NUT-type cosmologies, $R^1\times T^3$ and $R^3\times S^1$ flat Kasner and $R^1\times S^3$ Taub-NUT, proving that each must be maximal. Furthermore, this process produces non-Hausdorff extensions, including one which may be termed the maximal non-Hausdorff extensions for each spacetime. (Hawking and Ellis[2] first used this method for Taub-NUT spacetime.) Extensions of the spatially inhomogeneous $R^1\times T^3$ Moncrief universes are more complicated and will be discussed separately at the end of this appendix.

Usually a non-Hausdorff spacetime is physically undesirable since spacetime points are not separable and geodesics may bifurcate.[2,47] In these universes, however, no geodesics bifurcate, and the non-Hausdorff property is a mere technicality. For further discussion of non-Hausdorff spacetimes, see Hajicek.[48,49]

Here, for definiteness, consider an unextended flat Kasner universe with a metric g given by Eq. (3.1) and a manifold $M=R^1\times T^3$. (This analysis is applicable also to $R^3\times S^1$ flat Kasner and to $R^1\times S^3$ Taub-NUT[2] when appropriate changes are made.) M is considered to be a fiber bundle over the two-torus T^2 with fiber $\mathscr{F}=R^1\times S^1$ (see Fig. 4); the bundle projection π: $M\rightarrow T^2$ is defined by $(t,\psi,\theta,\phi)\rightarrow(\theta,\phi)$. The group of translations T_3 maps fibers into fibers so that the pairs $(\mathscr{F},\tilde{g})$ are all isometric, where $\tilde{g}$ is the metric induced on the fiber by g on the bundle M. In particular, the fiber $\mathscr{F}$ is the (t,ψ) plane and $\tilde{g}$ is given by $ds^2=-dt^2+t^2(d\psi)^2$.

The tangent space T at any point in M can be decomposed into a vertical subspace V and a horizontal subspace H. V is spanned by the vectors $\partial/\partial t$ and $\partial/\partial\psi$; H is spanned by the vectors $\partial/\partial\theta$ and $\partial/\partial\phi$. The metric g on T can be decomposed therefore into two parts: g_V on V (where $g_V=\tilde{g}$) and g_H on H [where g_H is given by $ds^2=(d\theta)^2+(d\phi)^2$]. The interesting part of g is contained in g_V. We therefore consider analytic extensions of $(\mathscr{F},g_V)$ which will give, in turn, analytic extensions of (M,g).

Call the Hausdorff extension corresponding to Eq. (3.2), upper sign, $(\bar{\mathscr{F}},\bar{g}_V)$, where $\bar{g}_V$ is given by $ds^2=2d\bar{\psi}\,d\bar{t}+2\bar{t}(d\bar{\psi})^2$. Call the Hausdorff extension corresponding to Eq. (3.2), lower sign, $(\bar{\bar{\mathscr{F}}},\bar{\bar{g}})$ where obviously $\bar{\bar{g}}_V$ is given by $ds^2=-2d\bar{\bar{\psi}}\,d\bar{\bar{t}}+\bar{\bar{t}}(d\bar{\bar{\psi}})^2$. Both extensions are analytic. To see the relationship between the two extensions, we go to the covering space shown in Fig. 1. The covering space $(\tilde{\mathscr{F}}_0,\tilde{g}_V)$ of $(\mathscr{F},g_V)$ can be described in double-null coordinates (u,v). The metric $\tilde{g}_V$ is given by

$$ds^2=t^2du\,dv\ , \tag{A1}$$

where

$$u=\bar{\psi}=\psi+\ln|t| \tag{A2}$$

and

$$v=\bar{\bar{\psi}}=\psi-\ln|t|\ . \tag{A3}$$

This metric is analytic on $\tilde{\mathscr{F}}_0$, i.e., in Region I of Fig. 1. Actually, g_V is analytic on the entire covering space $\tilde{\mathscr{F}}$ which is Regions I + II + III + IV. The space $(\tilde{\mathscr{F}},\tilde{g}_V)$ itself has a one-dimensional group of isometries which is the Lorentz group of a two-dimensional Minkowski spacetime. Let G be the discrete subgroup of the Lorentz group generated by a nontrivial element of the isometry group. Then the action of G on $\bar{\mathscr{F}}$ (i.e., Regions I + III) or on $\bar{\bar{\mathscr{F}}}$ (i.e., Regions I + II) is properly discontinuous, so $(\bar{\mathscr{F}},\bar{g}_V)/G$ and $(\bar{\bar{\mathscr{F}}},\bar{\bar{g}}_V)/G$ are Hausdorff spacetimes. However, the action of G on the combination $\bar{\mathscr{F}}$ and $\bar{\bar{\mathscr{F}}}$ (i.e., Regions I + II + III) is not properly discontinuous, and it is easy to show[2] that the quotient space (I + II + III, $\tilde{g}_V)/G$ is a non-Hausdorff manifold, Finally, one finds that even though $(\tilde{\mathscr{F}},\tilde{g}_V)/G$ (where $\tilde{\mathscr{F}}$ = I + II + III + IV) is not a manifold, $(\tilde{\mathscr{F}}-\{P\},\tilde{g}_V)/G$ [where $P=(0,0)$] is a manifold.

By combining these extensions of the (t,ψ) plane with the (θ,ϕ) coordinates, corresponding extensions of the entire manifold (M,g) are possible. Such an analysis illustrates that the two inequivalent Hausdorff extensions of Eq. (3.2) really are unextendible analytic, maximal Hausdorff extensions, and it shows that $(\tilde{M}-\{P=(0,0,\theta,\phi)\},\tilde{g}_V)/G$ is the maximal non-Hausdorff extension.

This maximal extension shows the global properties of the spacetime most clearly. The omitted point P is a singularity (in fact it is a quasiregular singularity, a topological defect in the spacetime), the surfaces $u=0$ or $v=0$ are Cauchy horizons, and Regions II and III are static spacetimes which contain closed timelike lines. In fact, these static regions are isometric to the Rindler wedge[50] with metric, $ds^2=-(x^1)^2dt^2+(dx^1)^2+(dx^2)^2+(dx^3)^2$, which has the same form as the unextended flat Kasner metric of Eq. (3.21) but with x and t interchanged.[51]

By analogy with other Taub-NUT-type spacetimes, one expects a Moncrief universe to admit a non-Hausdorff extension without bifurcating geodesics. Such an analysis is complicated, however, by the spatial inhomogeneity of a Moncrief universe which manifests itself in metric coefficients with spatial, in addition to time, dependence. In the simpler Taub-NUT-type spacetimes, we carried out non-Hausdorff extensions by making both maximal Hausdorff extensions simultaneously. To attempt an analogous extension here, define two "asymptotically null" coordinates:

$$u = 2(\psi + \ln |t|) \tag{A4}$$

and

$$v = 2(\psi - \ln |t|) . \tag{A5}$$

Then the metric (3.23) becomes

$$ds^2 = \frac{e^{(u-v)/2}}{8}(e^{2a}+e^{2b})du\,dv + \frac{e^{(u-v)/2}}{16}(e^{2b}-e^{2a})(du^2+dv^2) + e^{2a}d\theta^2 + e^{-2b}d\phi^2 . \tag{A6}$$

As the null hypersurface at $t=0$ is approached, the second term in Eq. (A3) vanishes since

$$\lim_{t\to 0}(e^{2b}-e^{2a})=0 , \tag{A7}$$

and the (u,v)-coordinates turn null. It is for this reason we term these coordinates "asymptotically null." Furthermore, in the limit that Eq. (3.8) becomes the unextended flat Kasner metric, Eq. (A6) takes the double-null form used in the flat Kasner maximal, non-Hausdorff extension.

It is interesting to examine the metric (A6) in a neighborhood of the null hypersurface at $t=0$. In that limit the metric approximates the flat Kasner metric:

$$ds^2 \approx \frac{e^{(u-v)/2}}{4} du\,dv + d\theta^2 + d\phi^2 , \tag{A8}$$

and one can draw a (u,v) slice which locally resembles the covering space of maximal non-Hausdorff flat Kasner (see Fig. 1). As in the flat Kasner case, identifications due to the topology of the spacetime indicate that the point $P=(u=0,v=0)$, must be omitted from the spacetime. Clarke[41] has proved that the Moncrief universes contain an essential singularity, one that cannot be removed by a global extension, and this analysis has shown the location of that singularity.

Of course, this analysis has not proved that the non-Hausdorff extension which appears valid in a neighborhood of $t=0$ is actually a valid global extension. One might possibly be able to use a fiber-bundle analysis to make such a global extension even though spatial inhomogeneity would greatly complicate it. What this analysis has done is to provide a good picture of the singularity structure of these universes. It shows the location of the topological defect in these spacetimes, the point P, and it illustrates that the remaining portion of the surfaces at $t=0$ (i.e., $u=0$, $v\neq 0$ and $u\neq 0, v=0$) are Cauchy horizons.

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