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Quantum cosmology with a positive-definite action

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We argue that the indefiniteness of the Euclidean Einstein action is more serious in the cosmological context than in the asymptotically Euclidean context. To correct this, we consider a positive-definite action containing quadratic curvature terms. The physical states Ψ are now functions of both a three-metric q_{ab} and extrinsic curvature K_{ab}, and satisfy a differential equation analogous to the Wheeler-DeWitt equation. This equation has the form of a Schrödinger equation with q_{ab} playing the role of "time." By adopting Hartle and Hawking's boundary condition on the Euclidean function integral, we obtain a "preferred" solution to this equation. It is shown that in a simple minisuperspace model this wave function describes an inflationary universe.

I. INTRODUCTION

In a recent paper, Hartle and Hawking[1] have shown that there exists a "preferred" wave function in quantum gravity. This wave function has been further investigated by Hawking and Luttrell[2,3] who argue that it might reasonably represent the "quantum state of the universe." It is most conveniently defined in terms of the Euclidean functional-integral approach to quantum gravity.[4] In this approach, given a three-geometry $\hat{h}_{ab}$ one can formally define a wave function $\psi_{\hat{h}}[h_{ab}]$ by

$$\psi_{\hat{h}}[h_{ab}]=\int D[g_{ab}]e^{-I_{\rm GR}[g_{ab}]}\,, \tag{1.1}$$

where $I_{\rm GR}$ is the Euclidean Einstein action and the integral is over all Riemannian four-geometries which induce h_{ab} on one boundary and $\hat{h}_{ab}$ on the other. One can show that this wave function automatically satisfied the Wheeler-DeWitt equation, and hence represents a physical quantum state. Thus one obtains a quantum state for each choice of $\hat{h}_{ab}$. The preferred wave function mentioned above is obtained by letting $\hat{h}_{ab}$ degenerate to a point. To be more precise, if h_{ab} is a metric on a compact three-manifold Σ, then $\psi[h_{ab}]$ is defined by a Euclidean functional integral over all Riemannian four-metrics g_{ab} on manifolds M whose only boundary is Σ such that g_{ab} induces h_{ab} on Σ. If one thinks of $\psi_{\hat{h}}[h_{ab}]$ as the wave function "arising from $\hat{h}_{ab}$," then $\psi[h_{ab}]$ is the wave function "arising from nothing."[5] It is a possible candidate for the quantum state of the universe.

Unfortunately, there is a serious problem with this idea. The Euclidean form of the action for general relativity is not positive definite and hence the functional integral (1.1) is not well defined. To deal with this problem, it has been suggested that one must deform the contour of integration in the functional integral from an integral over real Euclidean metrics to an integral over complex metrics with $\text{Re}\, I_{\rm GR}[g_{ab}]\geq 0$. Unfortunately, a completely satisfactory contour has not yet been found. Since this is the main motivation for what follows, we now review the situation in a little more detail.

The Euclidean Einstein action is[6]

$$I_{\rm GR}[g_{ab}]=\frac{1}{16\pi}\int_M(-R+2\Lambda)dV-\frac{1}{8\pi}\int_{\partial M}K\,d\Sigma\,, \tag{1.2}$$

where R is the scalar curvature of g_{ab}, Λ is a constant, and K is the trace of the extrinsic curvature of the boundary. The fact that $I_{\rm GR}$ is not positive definite has mainly been discussed in the context of asymptotically Euclidean metrics (and $\Lambda=0$). In this context, the following specific prescription was proposed[7] for dealing with this problem. Write a general asymptotically Euclidean metric $\tilde{g}_{ab}$ as $\tilde{g}_{ab}=\Omega^2 g_{ab}$ where Ω is a positive function and g_{ab} is a metric with zero scalar curvature. The positive-action theorem[8] states that asymptotically Euclidean metrics with zero scalar curvature have positive action. Thus, for $\Omega=1$, $I_{\rm GR}$ is positive definite. For $\Omega\neq 1$, the action includes a volume integral of $-(\nabla\Omega)^2$. To make this positive, one now writes the conformal factor as $\Omega=1+iy$ and does the functional integration over all functions y that vanish asymptotically.

There are at least two difficulties with this prescription. First, it is simply not true that a general asymptotically

Euclidean metric can be conformally rescaled to one with zero scalar curvature. (A necessary condition is that the lowest eigenvalue of the conformally invariant Laplacian $-\nabla^2+\frac{1}{6}R$ should be positive.) Therefore this prescription completely omits a large class of possible configurations from the functional integral. A second difficulty concerns the presence of matter fields. Since the action for the matter fields involves the metric, it will no longer be positive when one rotates $\Omega=1+iy$. So the total action (matter plus gravity) can become negative.

The situation with regard to the indefiniteness of the Euclidean Einstein action is even worse in the cosmological context. To see this, fix a compact manifold M with boundary Σ and consider metrics $\tilde{g}_{ab}$ on M that induce a fixed metric h_{ab} on Σ. If we write $\tilde{g}_{ab}$ in the form

$$\tilde{g}_{ab}=\Omega^2 g_{ab} \ , \tag{1.3}$$

then from (1.2) we have

$$16\pi I_{\rm GR}(\tilde{g}_{ab})=\int_M[-\Omega^2 R-6(\nabla\Omega)^2+2\Lambda\Omega^4]dV \\ -2\int_{\partial M}\Omega^2 K\,d\Sigma \ . \tag{1.4}$$

Even if we choose $\Lambda>0$, this action can clearly be negative. Let us try to follow the procedure used in the asymptotically Euclidean case. Since extrema of the action satisfy $R_{ab}=\Lambda g_{ab}$, it seems natural to require that $R=4\Lambda$. It is tempting now to simply rotate $\Omega\to i\Omega$ so that the volume term in the action becomes positive definite. However, this violates the boundary condition that the induced metric on Σ must be h_{ab}.

A second problem is that there is no analog of the positive-action theorem. For $\Omega=1$ and $R=4\Lambda$, the action becomes

$$8\pi I_{\rm GR}=-\Lambda V-\int K\,d\Sigma \ , \tag{1.5}$$

where V is the four-volume. Since the first term is negative definite and $K>0$ for spheres in a small neighborhood of a point this action can clearly be negative. In fact, instead of a positive-action theorem, one has the following:

Negative-action conjecture: Let M be a four-manifold with boundary Σ. Then for any metric on M with $R=4\Lambda$, the action $I_{\rm GR}$ defined by (1.5) is negative.

Intuitively, the idea is that in order for $I_{\rm GR}$ to be positive one needs $\int K<0$. But this means that one has to go past the "equator" of the geometry and by this time the volume is too big to be canceled. It seems unlikely that the action (1.5) is even bounded from below since one can construct "cylindrical" spaces with $R=4\Lambda$ and arbitrarily large volume.

In short, even if we ignore the fact that one cannot conformally rescale a general metric to one with $R=4\Lambda$,[9] and the fact that matter actions will no longer be positive, this procedure does not yield a positive-definite gravitational action.[10]

In this paper, the problem of an indefinite gravitational action is resolved by explicitly adding curvature-squared terms to the Einstein Lagrangian. The idea that the fundamental gravitational action is not the Einstein action but rather one involving quadratic terms in the curvature has received considerable attention recently for a variety of reasons.[11] Here, the main motivation is to improve the convergence of the Euclidean functional integral. We can then adopt exactly the same boundary condition that has been proposed for the Einstein Lagrangian to obtain a preferred "wave function of the universe."

In the past, theories described by curvature-squared Lagrangians were thought to suffer from instabilities and violation of unitarity. However, these conclusions were based entirely on a perturbation analysis off Minkowski spacetime. There is now growing evidence[11,12] that the perturbation theory is misleading, and that the full nonperturbative theory will be free of these difficulties.

The idea of computing the wave function of the universe in a theory described by a curvature-squared Lagrangian has also been discussed recently by Hawking and Luttrell.[3] However their approach is quite different from that adopted here. They add higher derivatives to mimic the effect of a massive scalar field in the gravitational Lagrangian. But their action is not positive definite.

What is the exact form of the action that we adopt? Perhaps the most attractive candidate for the action is simply to take the two independent curvature-squared Lagrangians

$$I_1=\int A(C_{abcd})^2+BR^2 \ , \tag{1.6}$$

where A and B are dimensionless coupling constants. This action was the starting point of the "induced gravity" program whose goal was to induce the scalar curvature term by quantum effects.[13] Unfortunately this program has recently run into difficulty because of the discovery of nonperturbative ambiguities in the value of the induced Newton's constant.[14] Therefore we must include the original Einstein action term:

$$I_2=\int A(C_{abcd})^2+BR^2-\frac{1}{16\pi}R \ . \tag{1.7}$$

This action, however, is *not* positive definite: Consider a metric whose curvature is small with R positive over a large volume. Then the curvature-squared terms will be negligible compared to the Einstein term which (for large enough volumes) can become arbitrarily negative. Thus we have to add a cosmological constant term Λ to make the action positive. So we are led to take the fundamental gravitational action to be

$$I=\tfrac{1}{4}\int A(C_{abcd})^2+B(R-4\Lambda)^2 \ , \tag{1.8}$$

where $32\pi\Lambda B=1$. Typically, one takes $B\simeq 1$ which implies that Λ is of order one in Planck units. This is, of course, much greater than the observational limits and is another manifestation of the well-known cosmological-constant problem.[15] If the Λ^2 term in (1.8) is somehow canceled in the effective action, then it is easy to show that any extrema of the resulting I whose typical radius of curvature is greater than the Planck length is an approximate solution to Einstein's equation. Thus one can recover general relativity in the "long wavelength" limit. It will be shown in the next section that one does *not* have to add a boundary term to this action.

The plan of this paper is the following. In Sec. II we begin by reviewing the canonical analysis of the Lorentzian theory described by the action I. This analysis shows that physical quantum states are described by wave functions ψ depending on a three-metric g_{ab} and extrinsic curvature K_{ab}, satisfying two differential equations coming from the classical constraints. The first equation implies that ψ is invariant under diffeomorphisms on the three-surface. The second has the form of a Schrödinger equation with q_{ab} playing the role of "time." We then consider the preferred wave function defined by the Euclidean functional integral. We will see that if one analytically continues the Euclidean extrinsic curvature to imaginary values (corresponding to real Lorentzian extrinsic curvature), then the wave function satisfies the constraint equations of the Lorentzian theory. In Sec. III this wave function is evaluated in a simple minisuperspace model. We find that ψ represents a superposition of classical de Sitter spacetimes. Hence it may be an appropriate description of an early inflationary phase in our universe.

II. COSMOLOGICAL WAVE FUNCTIONS

The wave function ψ is, of course, a function on configuration space. In order to identify the appropriate configuration space as well as to find the differential equations that ψ must satisfy, one must do a canonical analysis. We therefore begin this section with a discussion of the canonical theory. The second half of this section then examines the Euclidean functional integral which defines ψ and discusses its semiclassical evaluation.

A. Canonical analysis of the Lorentzian theory

The Lorentzian form of our fundamental gravitational action I is

$$\tilde{I}=-\tfrac{1}{4}\int AC_{abcd}C^{abcd}+B(R-4\Lambda)^2\ . \tag{2.1}$$

This action has been cast into canonical form by Boulware.[16] We now briefly review his results.

Fix a compact three-manifold Σ. The canonical variables for the action $\tilde{I}$ can be taken to be the following: a three-metric q_{ab} on Σ and its conjugate momentum p^{ab} which is a symmetric second-rank tensor density that involves third time derivatives of the metric; and a tensor K_{ab} on Σ together with its conjugate momentum P^{ab} which is also a symmetric second-rank tensor density that involves second time derivatives of the metric. If we evolve $(q_{ab},K_{ab},P^{ab},p^{ab})$ to obtain a four-dimensional spacetime, then all these fields except p^{ab} acquire a simple geometric interpretation: q_{ab} is the induced metric on the surface Σ, K_{ab} is its extrinsic curvature, and P^{ab} is simply related to the four-dimensional curvature by

$$P^{ab}=-2q^{1/2}(2AE^{ab}+Bq^{ab}R)\ , \tag{2.2}$$

where $q=\det q_{ab}$, $E^{ab}=C^{ambn}t_m t_n$ is the electric part of the Weyl tensor with respect to the unit normal to Σ, and R is the four-dimensional scalar curvature. (Boulware in fact chooses $-P^{ab}$ as a "position variable" and K_{ab} as the conjugate momentum. The resulting formalism is of course completely equivalent. However, we will see in Sec. II B that, from the standpoint of quantization, the q_{ab}, K_{ab} representation is more convenient.)

The variables $(q_{ab},K_{ab},P^{ab},p^{ab})$ cannot be chosen independently. Just as in general relativity there are constraints arising from the invariance of the action under diffeomorphisms. These constraints are

$$\begin{aligned}C={}&K_{ab}p^{ab}-\frac{1}{8Aq^{1/2}}(P^T_{ab})^2-\frac{P^2}{144Bq^{1/2}}\\&+\tfrac{1}{2}P^{ab}(K_{ab}K+{}^3R_{ab})-\tfrac{1}{4}P({}^3R-K^{ab}K_{ab}+K^2)\\&+\tfrac{1}{2}D_aD_bP^{ab}-Aq^{1/2}(C_{abcd}t^d)^2\\&-2B\Lambda q^{1/2}({}^3R+K_{ab}K^{ab}-K^2)+4\Lambda^2Bq^{1/2}=0\end{aligned} \tag{2.3}$$

and

$$C_m=-\tfrac{1}{2}P^{ab}D_mK_{ab}+D_a(K_{bm}P^{ab})+D_aP^a_m=0\ , \tag{2.4}$$

where $(\ )^2$ means square the tensor with the metric q^{ab} [so $(C_{abcd}t^d)^2$ involves just spatial derivatives of K_{ab}] and P^T_{ab}, and P are the tracefree and trace parts of P_{ab}.

In the standard canonical approach to quantization the physical states of the system are represented by functions on configuration space $\psi(q_{ab},K_{ab})$ satisfying the operator version of the constraints:

$$\mathbf{C}\psi=0\ , \tag{2.5}$$

$$\mathbf{C}_m\psi=0\ , \tag{2.6}$$

where $\mathbf{C}$ and $\mathbf{C}_m$ are obtained by replacing p^{ab} by $(1/i)\delta/\delta q_{ab}$ and P^{ab} by $(1/i)\delta/\delta K_{ab}$ in (2.3) and (2.4). These constraint operators have similar factor-ordering and regularization problems as the analogous operators arising from the Einstein action.

The vector constraint (2.4) has exactly the same interpretation as the analogous constraint in general relativity. Classically, it generates diffeomorphisms on the three-surface. Quantum mechanically, it requires that the wave function depend only on the equivalence classes $\{(q_{ab},K_{ab})\}$ where two pairs (q_{ab},K_{ab}) and (q'_{ab},K'_{ab}) are said to be equivalent if they are related by a diffeomorphism on the three-surface Σ. This is easily seen as follows. Let N^m be any vector field on Σ. Then from (2.4) and a simple integration by parts we have

$$\begin{aligned}\int_\Sigma N^mC_m&=\int_\Sigma[-\tfrac{1}{2}P^{ab}N^mD_mK_{ab}\\&\qquad -P^{ab}K_{bm}D_aN^m-p^{ab}D_aN_b]\\&=-\tfrac{1}{2}\int_\Sigma[(\mathscr{L}_NK_{ab})P^{ab}+(\mathscr{L}_Nq_{ab})p^{ab}]\ .\end{aligned} \tag{2.7}$$

Therefore, replacing P^{ab} by $(1/i)\delta/\delta K_{ab}$ and P^{ab} by $(1/i)\delta/\delta q_{ab}$ we see that $\int_\Sigma N^m\mathbf{C}_m\psi=0$ says that ψ is unchanged if we simultaneously change q_{ab} and K_{ab} by the diffeomorphism generated by N^m.

The scaler constraint (2.5) is the analog of the Wheeler-DeWitt equation for the Einstein action. Interestingly enough, the form of this equation is quite different. The Wheeler-DeWitt equation resembles a wave equation on the configuration space. Equation (2.5) has the general structure of a Schrödinger equation with the three-metric q_{ab} playing the role of time. With the simplest choice of factor ordering, this equation takes the following form:

$$K_{ab}\, i\frac{\delta}{\delta q_{ab}} = G_{abcd}\frac{\delta}{\delta K_{ab}}\frac{\delta}{\delta K_{cd}} + A_{ab}\frac{\delta}{\delta K_{ab}} + V \ , \quad (2.8)$$

where G_{abcd} is a "metric" which depends only on q_{ab} and is positive definite. Although q_{ab} probably contains information about physical fields as well as the time, the fact that it naturally appears in a first-order form might have a significant effect in improving our understanding of time in quantum gravity. Notice that since q_{ab} and K_{ab} are independent variables the quadratic term is independent of the usual factor-ordering problems present in the analogous term in the Wheeler-DeWitt equation. However the linear-momentum term does have factor-ordering ambiguities since A_{ab} as well as V depend on both q_{ab} and K_{ab}. The precise expressions for G_{abcd}, A_{ab} and V can be read off from (2.3).[17]

The fact that wave functions in this theory depend on K_{ab} as well as q_{ab} has an important consequence. Penrose has argued[18] from considerations of entropy that the "big bang" should be very different from the "big crunch." In a theory where the quantum states are functions only of the three-metric, it is difficult to understand how this asymmetry could come about. However in this theory, the structure of the wave function for small three-volume and $K>0$ can be quite different from its structure for small three-volume and $K<0$. If the preferred wave function defined by the Euclidean functional integral has this property, then this might provide a possible "explanation" for the observed time asymmetry of our universe.

B. Euclidean functional integral

The canonical analysis discussed above shows that the appropriate configuration space for the theory described by the action (2.1) consists of pairs (q_{ab}, K_{ab}) where q_{ab} is a metric and K_{ab} a symmetric tensor on a compact three-manifold Σ. We can now obtain a preferred Euclidean wave function ψ_E by adopting the same boundary condition on the functional integral as used for the Einstein Lagrangian. That is

$$\psi_E(q_{ab}, K_{ab}) = \int D[g_{ab}]e^{-I[g]} \ , \quad (2.9)$$

where the integral is over all Euclidean metrics g_{ab} on manifolds M whose boundary is Σ such that the induced metric and extrinsic curvature on Σ are q_{ab} and K_{ab}, respectively.

Before discussing the relation between ψ_E and the physical states of the theory, i.e., wave functions satisfying (2.5) and (2.6), we first consider the question of whether the action I given by (1.8) should be supplemented by boundary terms. For the Einstein action there are two separate arguments which require that a boundary term be added to the integral of the scalar curvature. In the first argument, the boundary term is obtained by demanding that solutions of the classical field equation be extrema of the action under all perturbations that vanish on the boundary, i.e., $\delta q_{ab}=0$. In order to satisfy this condition one has to "cancel" the second derivative terms in R by adding the integral of the extrinsic curvature. The analogous condition for the curvature-squared action I is that solutions be extrema of I under all perturbations satisfying $\delta q_{ab}=\delta K_{ab}=0$ on the boundary. This is because one expects to get a unique classical solution only if one fixes both q_{ab} and K_{ab} on the boundary. But, it is easy to verify that solutions of the classical field equation are indeed extrema of I (with no extra boundary terms added) under all perturbations satisfying these conditions.

The second argument, which leads to the same boundary term for the Einstein action, is the following. Consider a transition from an initial configuration (q_1, K_1) on a surface Σ_1 to a configuration (q_2, K_2) on a surface Σ_2, followed by a transition from (q_2, K_2) to a final configuration (q_3, K_3) on a surface Σ_3. One expects that the amplitude to go from the initial to the final configuration should be obtained by integrating over all configurations on the intermediate surface Σ_2. This requires

$$I[g_1+g_2]=I[g_1]+I[g_2] \ , \quad (2.10)$$

where g_1 is a metric that induces (q_1, K_1) on Σ_1 and (q_2, K_2) on Σ_2, g_2 is a metric that induces (q_2, K_2) on Σ_2 and (q_3, K_3) on Σ_3, and g_1+g_2 is the metric obtained by taking the union of the two. Since q_1 and q_2 induce the same metric *and intrinsic curvature* on Σ_2 the four-dimensional curvature of g_1+g_2 will not contain any δ functions (although it may be discontinuous). Hence *condition (2.10) is automatically satisfied by the action (1.8) without having to add any boundary terms.* Since the configuration space for the Einstein Lagrangian is smaller, the three metrics induced on Σ_2 by g_1 and g_2 will still agree; but the extrinsic curvatures need not. This results in a δ function in the Ricci curvature proportional to the difference of the extrinsic curvatures on Σ_2. Hence condition (2.10) will hold for I_{GR} only if one includes the surface term.

To summarize, in both cases the action I given by (1.8) satisfies the desired properties without having to add any surface terms.[19]

We now turn to the relation between ψ_E and the physical states of the theory, i.e., wave functions annihilated by the constant operators. Since ψ_E is a function of Euclidean extrinsic curvature, one would not expect it to satisfy (2.5) and (2.6). If we choose coordinates near Σ so that the Lorentzian extrinsic curvature is $K^L_{ab}=\partial q_{ab}/\partial t$, then the Euclidean action can be obtained from the Lorentzian action by formally setting $\tau = it$. The Euclidean extrinsic curvature is then

$$K^E_{ab}=\frac{\partial q_{ab}}{\partial \tau} = -i\frac{\partial q_{ab}}{\partial t} = -iK^L_{ab} \ . \quad (2.11)$$

Thus we expect that

$$\psi(q_{ab}, K^L_{ab}) \equiv \psi_E(q_{ab}, -iK^L_{ab}) \quad (2.12)$$

will satisfy the Lorentzian constraint equations. We now show that, at least formally, this is indeed the case.

Let M be a manifold containing a compact three-dimensional submanifold Σ (*not* on its boundary). Given q_{ab} and K^E_{ab} on Σ we define $\psi_E(q_{ab}, K^E_{ab})$ by the Euclidean functional integral (2.9). Since there is no external measure of time, ψ_E does not depend on where Σ is placed in M. If we took a different submanifold Σ' and put the same fields q_{ab}, K^E_{ab} on Σ', then the value of ψ_E is left unchanged. But by the "same fields" we mean that there exists a diffeomorphism $\phi:\Sigma \rightarrow \Sigma'$ which takes the fields on Σ to the fields on Σ'. Hence ψ_E must be invariant under diffeomorphisms. (One can think of a diffeomorphism as just corresponding to a "change of variables" in the functional integral.) Now consider an infinitesimal diffeomorphism generated by a vector field ξ^a. Taking the variation of both sides of (2.9) and assuming the measure is invariant we get

$$0=\int D[g]\left[\int_M C^{ab}\nabla_{(a}\xi_{b)}\right]e^{-I[g]} . \tag{2.13}$$

where $C^{ab}\equiv \delta I/\delta g_{ab}$ is the classical (Euclidean) field equation for the action I. Using the fact that $\nabla_a C^{ab}=0$, we can integrate by parts inside the large parentheses to obtain

$$0=\int D[g]\left[\int_\Sigma C^{ab}\xi_a d\Sigma_b\right]e^{-I[g]} . \tag{2.14}$$

The quantity in large parentheses is just the (Euclidean) constraint equations. Equation (2.14) shows that the operator versions of these constraints automatically annihilate the wave function.

Unlike the case of the Einstein Lagrangian, the operator version of the Euclidean constraints is *not* the same as the operator version of the Lorentzian constraints. This is easily seen as follows. The Euclidean constraints are obtained from the Lorentzian constraints (2.3) and (2.4) by setting $K^L_{ab}=iK^E_{ab}$, $p^{ab}_L=ip^{ab}_E$, and leaving q_{ab} and P^{ab} unchanged. The Euclidean constraint operators are obtained from the constraints by replacing $P^{ab}\rightarrow -\delta/\delta K^E_{ab}$ and $p^{ab}_E \rightarrow -\delta/\delta q_{ab}$. The net effect is thus to send $q_{ab}\rightarrow q_{ab}$ $P^{ab}_L \rightarrow (1/i)\delta/\delta q_{ab}$ (which agrees with the Lorentzian operator equation) and $K^L_{ab}\rightarrow iK^E_{ab}$, $P^{ab}_L\rightarrow -\delta/\delta K^E_{ab}$ (which does not). Thus to obtain a solution of the Lorentzian operator constraints we must analytically continue the Euclidean wave function to imaginary values of the extrinsic curvature:

$$\psi[q_{ab},K^L_{ab}]=\psi_E[q_{ab},-iK^L_{ab}] . \tag{2.15}$$

This procedure will yield a solution to the constraints (2.5) and (2.6) for any wave function defined by the Euclidean functional integral. By adopting Hartle and Hawking's preferred boundary condition,[1] we obtain a unique physical state which might represent the quantum state of our universe.

One cannot, of course, calculate the functional integral (2.9) exactly. (In fact, the precise form of the measure is still an open question.) However, one can get some information about ψ from a semiclassical approximation. Since we are interested in ψ_E only for imaginary values of the extrinsic curvature, the semiclassical approximation is somewhat complicated. For example, it is not sufficient to minimize the Euclidean action with q_{ab} and K^E_{ab} fixed on the boundary and then analytically continue $\psi_E(q_{ab},K^E_{ab})=e^{-I_{\min}[g_{ab}]}$. This is because the analytic continuation may cross "Stokes lines" where the asymptotic form of the integral (for small $\hbar$) changes character. Instead, one must do a steepest-descents analysis in which one looks for complex extrema of the action with q_{ab} and $-iK^L_{ab}$ fixed on the boundary. Then, if one can deform the original contour of integration into a steepest-descents contour passing through this extrema, the semiclassical approximation to ψ is

$$\psi(q_{ab},K^L_{ab})\approx N e^{-I[g_{ab}]} , \tag{2.16}$$

where g_{ab} is the complex extremum and N includes contributions from quadratic fluctuations.

We now briefly explain what is meant by complex extrema of the action. Given a real manifold, one can always complexify the tangent space and introduce complex tensor fields. (Since we want to consider the boundary of our manifold $\partial M=\Sigma$ to be a three-dimensional submanifold, it is better not to view M as a complex two-manifold, even when this is possible.) A complex metric is simply an invertible, symmetric second-rank complex tensor field. There exists a unique, torsion-free covariant derivative compatible with each complex metric, and one can write down the field equations just as for real metrics. The only subtlety comes at the boundary. The unit normal to Σ can be defined by taking any function f such that $f=0$, $\nabla_a f\neq 0$ on Σ and setting

$$n_a=\frac{\nabla_a f}{(g^{nm}\nabla_m f\nabla_n f)^{1/2}} . \tag{2.17}$$

The problem is that the square root in the denominator is only defined up to an overall sign. In other words, one cannot distinguish the "outgoing" and "ingoing" normals for a general complex metric. This does not affect the definition of q_{ab} but it does affect the definition of K_{ab}. In our case this ambiguity is resolved as follows. Since we begin with an integral over real Euclidean metrics, there is a well-defined outgoing normal. We now require that as we deform the contour of integration, the unit normal changes continuously. This gives a well-defined definition of K_{ab}.

We conclude this section with a few remarks about another representation for the quantum theory based on the action (2.1). Instead of choosing (q_{ab},K_{ab}) to be "position variables," we could have chosen the configuration space to consist of pairs (q_{ab},P^{ab}). Although the resulting theory should be completely equivalent, the (q_{ab},P^{ab}) representation is less desirable for at least three reasons.

First, the scalar constraint equation (2.5) changes its character completely: Since the only quadratic momentum term in the (q_{ab},P^{ab}) representation is $K_{ab}p^{ab}$, the analog of the Wheeler-DeWitt equation would no longer resemble Schrödinger's equation but rather a second-order equation in a space whose metric has six positive and six negative eigenvalues at each point. Second, the condition for the composition of amplitudes in the (q_{ab},P^{ab}) representation is more complicated than $I[g_1+g_2]=I[g_1]+I[g_2]$. In fact, this latter condition can no longer be satisfied for *any* choice of boundary terms. This is because the extrinsic curvature need not be continuous across the boundary, so the four-dimensional curvature computed from g_1+g_2 may contain δ-function contributions. Since the action involves the square of the curvature, the left-hand side can diverge when the right-hand side is finite. Finally, it is difficult to obtain nontrivial information about $\psi(q_{ab},P^{ab})$ from a semiclassical approximation. One can show that the wave function defined by the Euclidean functional integral with q_{ab} and P^{ab} held fixed on the boundary automatically satisfies the Lorentzian constraint without any analytic continuation. Although this would normally be a significant advantage, in our case it is not, since the metric which minimizes the action holding q_{ab} and P^{ab} fixed on the boundary is in fact independent of P^{ab}. The curvature becomes discontinuous on the boundary.

A simple example will illustrate this last point. Let φ be a function on the unit ball B in E^4 and set

$$S[\varphi]=\int_B(\nabla^2\varphi)^2 \; . \tag{2.18}$$

We wish to minimize S subject to the condition that on the boundary of B, $\varphi=0$ and $\nabla^2\varphi=K$ where K is some constant. Consider a sequence of functions φ_{r_0} which are constant for $r<r_0<1$ and then "bend" to satisfy $\nabla^2\varphi_{r_0}=K$ at $r=1$. One possibility is

$$\varphi_{r_0}(r)=\begin{cases}-K(1-r_0)^2/(8-6r_0), & 0\le r\le r_0 \; ,\\ K(r-1)(r+1-2r_0)/(8-6r_0), & r_0\le r\le 1 \; .\end{cases} \tag{2.19}$$

It is easy to see that both φ_{r_0} and φ'_{r_0} are continuous at $r=r_0$ and φ_{r_0} satisfies the boundary conditions for all r_0. The action $S[\varphi_{r_0}]$ is clearly minimized by taking the limit $r_0\rightarrow 1$. In this limit, the action approaches zero. In particular it is independent of the value of K. Since the action I does not contain any terms involving derivatives of the curvature, a similar thing can happen here. Thus, nontrivial information about $\psi(q_{ab},P^{ab})$ can only be obtained by computing the quadratic fluctuations.

III. MINISUPERSPACE MODEL

In this section we apply the general formalism discussed in Sec. II to a simple minisuperspace model. The model is obtained by restricting consideration only to metrics with isotropic three-surfaces. Since metrics of this type are all conformally flat, the Weyl squared term in the action I (1.8) will play no role. Furthermore, we set $\Lambda=0$. Although a physically realistic theory probably needs $\Lambda>0$, to include a positive Newton's constant, the choice $\Lambda=0$ is made for the following reason. It simplifies the semiclassical evaluation of the wave function considerably, and yields some information about the $\Lambda>0$ case. This is because Λ is negligible for certain regions of the configuration space. A complete analysis of the case $\Lambda>0$ will appear in a forthcoming paper.[20] For now we consider the simpler theory, and show that with $\Lambda=0$, the resulting wave function describes an inflationary universe.

We begin this section with a canonical analysis of the Lorentzian theory. Then we evaluate the functional integral semiclassically and interpret the resulting wave function.

A. Canonical analysis

We write the spacetime metric in the form

$$ds^2=e^{2\alpha(\eta)}(-d\eta^2+d\Omega_3{}^2) \; , \tag{3.1}$$

where $d\Omega_3{}^2$ is the metric on a unit three-sphere. The four-dimensional scalar curvature is then given by

$$R=6\,e^{-2\alpha}(1+\alpha'^2+\alpha'') \; , \tag{3.2}$$

where a prime denotes $d/d\eta$. The Lorentzian action is (2.1)

$$\begin{aligned}\tilde{I}&=\frac{-B}{4}\int R^2dV\\ &=-18B\pi^2\int(1+\alpha'^2+\alpha'')^2d\eta \; .\end{aligned} \tag{3.3}$$

To cast this fourth-order action into canonical form we must first vary $\tilde{I}$ with respect to the highest derivatives and set the results equal to a new variable Q:

$$Q\equiv-\frac{\delta\tilde{I}}{\delta\alpha''}=36B\pi^2(1+\alpha'^2+\alpha'') \; . \tag{3.4}$$

Solving for α'' in terms of Q we next write the action in the form

$$\tilde{I}=\int(\alpha'Q'-\bar{H})d\eta \; , \tag{3.5}$$

where

$$\bar{H}(\alpha,\alpha',Q)=(1+\alpha'^2)Q-Q^2/m \tag{3.6}$$

and we have set $m\equiv(72\pi^2B)$. The momenta conjugate to α and Q are now defined in the usual manner:

$$P_Q=\frac{\delta\tilde{I}}{\delta Q'}=\alpha' \; , \tag{3.7a}$$

$$P_\alpha=\frac{\delta\tilde{I}}{\delta\alpha'}=Q'-2\alpha'Q \; . \tag{3.7b}$$

The Hamiltonian is then

$$\begin{aligned}H&=P_\alpha Q'+P_\alpha\alpha'-L\\ &=P_\alpha P_Q+(1+P_Q{}^2)Q-Q^2/m \; .\end{aligned} \tag{3.8}$$

Classically, this Hamiltonian is constrained to vanish. In the quantum theory we require that the Hamiltonian operator must annihilate the wave function. As discussed in the last section we want to quantize in a representation where α and $P_Q=\alpha'$ are the configuration variables.[21] Therefore, we replace P_Q with x and Q with $-P_x$ to obtain

$$H=xP_\alpha-(1+x^2)P_x-P_x{}^2/m \; . \tag{3.9}$$

To obtain the Hamiltonian operator we replace P_α with $(1/i)\partial/\partial\alpha$ and P_x with $(1/i)\partial/\partial x$. Thus physical states of the theory must satisfy

$$xi\frac{\partial\psi}{\partial t}=\frac{1}{m}\frac{\partial^2\psi}{\partial x^2}+i\frac{\partial\psi}{\partial x}+ix^{2-p}\frac{\partial(x^p\psi)}{\partial x} \; , \tag{3.10}$$

where we have replaced α with the more suggestive variable t and written the last term with an arbitrary parameter p reflecting ambiguities in factor ordering. *This equation is just the ordinary Schrödinger equation for a one-dimensional system governed by the time-independent Hamiltonian*

$$H_0(x,p)=\frac{-1}{x}\left[\frac{p^2}{m}+(1+x^2)p\right] \; . \tag{3.11}$$

The relation between H_0 and H is the following. Since H is independent of α, P_α is constant. H_0 is simply $-P_\alpha$. The equations of motion generated by H_0 are completely equivalent to those generated by H. The only difference is that for H_0, the solutions are parametrized by t whereas for H, they are parametrized by η. Note that H_0 is not bounded from below.

Given any function $\psi(t_0,x)$ one can in principle evolve (3.10) to obtain an allowed quantum state of the system. In the next section we find which quantum state corresponds to placing Hartle and Hawking's boundary condition on the Euclidean functional integral.

B. Semiclassical evaluation of functional integral

We now consider positive-definite metrics of the form:

$$ds^2=e^{2\alpha(\eta)}(d\eta^2+d\Omega_3{}^2) . \tag{3.12}$$

The scalar curvature is

$$R=6\,e^{-2\alpha}(1-\alpha'^2-\alpha'') \tag{3.13}$$

[cf. (3.2)]. Recall that to obtain the "wave function of the universe" one must first compute

$$\psi_E(\alpha_0,\alpha_0')=\int D[\alpha]e^{-I[\alpha]} , \tag{3.14}$$

where the Euclidean action is

$$I[\alpha]=\frac{B}{4}\int R^2 dV \tag{3.15}$$

and the integral is over all metrics which are regular at the origin and induce α_0 and α_0' on the boundary. To see what regularity at the origin corresponds to for (3.12), we briefly consider an alternative form of the metric

$$ds^2=a^2(r)(dr^2+r^2 d\Omega_3{}^2) . \tag{3.16}$$

This metric is clearly regular at the origin if $a(r)=a(0)+O(r^2)$. Now comparing (3.12) and (3.16) we obtain

$$\frac{dr}{r}=d\eta,\quad 1+\frac{r}{a}\frac{da}{dr}=\alpha' . \tag{3.17}$$

Therefore the origin corresponds to $\eta=-\infty$ and regularity at the origin requires that for $\eta\ll 0$, $\alpha'\approx 1+O(e^{2\eta})$. After computing $\psi_E(\alpha_0,\alpha_0')$ one must then analytically continue α_0' to $-ix_0$ to obtain a solution to (3.10).

As we discussed earlier, in the semiclassical approximation, it is not sufficient to minimize the Euclidean action, and then analytically continue $\psi_E(\alpha_0,\alpha_0')=e^{-I_{\min}(\alpha_0,\alpha_0')}$ from real α_0' to imaginary values. Instead, one must do a steepest-descents analysis. This involves finding complex extrema of the action which satisfy the boundary conditions (denoting the boundary by $\eta=0$):

$$\begin{aligned}&\alpha'(\eta)=1+O(e^{2\eta})\quad \text{for } \eta\ll 0 ,\\ &\alpha(0)=\alpha_0 ,\\ &\alpha'(0)=-ix_0 .\end{aligned} \tag{3.18}$$

If the original contour of integration can be deformed into a steepest-descents contour passing through the extrema $\alpha(\eta)$, then the semiclassical approximation to ψ is

$$\psi(\alpha_0,x_0)\approx N\,e^{-I[\alpha]} , \tag{3.19}$$

where N includes contributions from quadratic fluctuations. In the following we will concentrate on just the exponent. We now find the complex extrema of the action I satisfying the boundary conditions (3.18).

Setting the variation of I with respect to α equal to zero we obtain the field equation

$$[(1-\alpha'^2-\alpha'')'-2\alpha'(1-\alpha'^2-\alpha'')]'=0 . \tag{3.20}$$

Using (3.13) this simplifies to

$$e^{2\alpha}R'=\text{constant} . \tag{3.21}$$

Since the scalar curvature must be finite at the origin where $\alpha\to-\infty$, this implies that $R=$constant. Hence, *all (isotropic) extrema of I have constant scalar curvature.* This result clearly holds for complex α as well as real α.

Since we have reduced the fourth-order equation (3.20) to the second-order equation $R=$constant, it is not clear that there will exist solutions satisfying our boundary conditions. Indeed, in the case of real α, one can show that all metrics with a regular origin and $R=$constant are in fact spaces of constant curvature. Since $\alpha'>-1$ for all spaces of constant curvature, there do not exist solutions with $\alpha_0'<-1$ on the boundary. (If one tries to minimize the action with $\alpha_0'<-1$ then the volume becomes unbounded, i.e., "the solution goes off to infinity.") However, we will now show that there *do* exist complex solutions satisfying (3.18) for all α_0,x_0.

We start with the metric for a real four-sphere:

$$e^{-\alpha}=M\cosh(\eta+c) , \tag{3.22}$$

where $M>0$ and c are constants. The scalar curvature is related to M by

$$R=12M^2 . \tag{3.23}$$

Taking the derivative of both sides of (3.22) we obtain

$$\alpha'=-\tanh(\eta+c) \tag{3.24}$$

which shows that the boundary condition as $\eta\to-\infty$ is satisfied for all M and c. For real c, $|\alpha'|<1$ for all η. However, we now let $c=i\theta$ where $-\pi/2<\theta<\pi/2$. Then

$$e^{-\alpha}=M\cosh(\eta+i\theta) \tag{3.25}$$

has $R=$constant and satisfies the boundary conditions (3.18) provided

$$e^{-\alpha_0}=M\cos\theta ,\quad x_0=\tan\theta . \tag{3.26}$$

We thus obtain complex extrema of I satisfying (3.18) for all α_0, x_0.

We could have obtained exactly the same solutions by starting with a real space of constant negative curvature, as we now show. The metric for a space of constant negative curvature is

$$e^{-\alpha}=-N\sinh(\eta-d) , \tag{3.27}$$

where $N>0$ and d are constants. (The point $\eta=d$ corresponds to infinity.) The scalar curvature is now $R=-12N^2$ and

$$\alpha' = -\coth(\eta - d) . \tag{3.28}$$

Setting $d = i\varphi$ $(0 < \varphi < \pi)$ we obtain a solution satisfying the boundary conditions (3.18) provided

$$e^{-\alpha_0} = iN \sin\varphi, \quad x_0 = \cot\varphi . \tag{3.29}$$

But the first condition says that N must be imaginary. If we set $N = -iM$ and $\varphi = \theta + \pi/2$, then this solution becomes identical to the one given by (3.25) and (3.26).

We now compute the action for these solutions:

$$I = 18\pi^2 B \int_{-\infty}^{0} (1 - \alpha'^2 - \alpha'')^2 d\eta . \tag{3.30}$$

From (3.24) we have

$$\alpha'' = (\alpha'^2 - 1) . \tag{3.31}$$

So

$$\begin{aligned} I &= 72\pi^2 B \int_{-\infty}^{0} (\alpha'^2 - 1)^2 d\eta \\ &= m \int_{1}^{-ix_0} (\alpha'^2 - 1) d\alpha' \\ &= m[i(x_0 + x_0^3/3) + \tfrac{2}{3}] , \end{aligned} \tag{3.32}$$

where we have again set $m = 72\pi^2 B$.

Now fix α_0 and x_0. It seems likely that the solution given by (3.25) and (3.26) is the only exact solution satisfying the prescribed boundary conditions. Although a detailed steepest-descents analysis is difficult for the infinite-dimensional path integral (3.14), it is reasonable to expect that the original contour of integration can be deformed to pass over this extrema. Thus, in the semiclassical approximation,

$$\psi(\alpha_0, x_0) \approx e^{-I[\alpha]} , \tag{3.33}$$

where I is given by (3.32). Replacing α_0 with t and x_0 with x we obtain our final result:

$$\psi(x,t) = e^{-im(x + x^3/3)} , \tag{3.34}$$

where we have dropped the irrelevant constant $2m/3$ from the action. Notice that the wave function is time independent. This is a direct result of the fact that the action is scale invariant, and hence is unchanged if a constant is added to α.

We now ask if ψ satisfies the Hamiltonian constraint of the canonical theory (3.10). Notice that

$$\frac{1}{i}\frac{\partial\psi}{\partial x} = -m(1+x^2)\psi . \tag{3.35}$$

If we choose the factor-ordering parameter $p = 2$ in (3.10) then the constraint equation becomes

$$-xi\frac{\partial\psi}{\partial t} = \frac{1}{i}\frac{\partial}{\partial x}\left[\frac{1}{im}\frac{\partial}{\partial x} + (1+x^2)\right]\psi . \tag{3.36}$$

Therefore ψ is an exact solution of the canonical constraint equation. It is perhaps surprising that the semiclassical approximation to the wave function satisfies the exact constraint equation. This is probably a result of the high degree of symmetry imposed on the metric together with scale invariance of the action. The fact that a certain choice of factor ordering is picked out is *not* surprising. It is (presumably) just the factor ordering which corresponds to choosing a scale-invariant measure in the path integral.[22] Notice that even though the one-dimensional Hamiltonian H_0 is unbounded from below, the wave function obtained by imposing Hartle and Hawking's boundary condition on the path integral corresponds to a zero-energy eigenstate.

The wave function ψ has a simple physical interpretation. It satisfies the operator equivalent of [see (3.35)]

$$p = -m(1+x^2) . \tag{3.37}$$

Recall from (3.4) that $p = -Q = -mRe^{2\alpha}/12$ and $x = \alpha'$. Therefore, classically, (3.37) becomes

$$R = 12\, e^{-2\alpha}(1 + \alpha'^2) . \tag{3.38}$$

Taking the derivative of both sides of this equation and using (3.2) we find $R' = 0$. In other words, Eq. (3.38) describes a classical spacetime of constant positive curvature, i.e., de Sitter spacetime. Notice that the value of the curvature R is not fixed by this equation. For a given α, any value of $R > 12e^{-2\alpha}$ can satisfy $p = -m(1+x^2)$. Since $\alpha \equiv t$, we arrive at the following physical interpretation of ψ: *The "wave function of the universe" $\psi(x,t)$ corresponds to a superposition of classical de Sitter spacetimes with $R > 12e^{-2t}$.* In this sense the wave function ψ can be said to describe an inflationary universe.

Since there is no length scale in this model, one could not expect to see a transition from the inflationary phase to a Robertson-Walker phase. In the future we will investigate more complicated models which include $\Lambda > 0$ and anisotropy. We hope to show that the wave function defined by imposing compact boundary conditions on the Euclidean functional integral with a positive-definite action gives a reasonable description of the observed universe.

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[10]If the negative-action conjecture is true, one might be tempted to simply change the overall sign of the action. After all, the sign of the Euclidean action was determined by "rotating" the Lorentzian action in the same direction that makes the matter action positive. But we have already seen that there are difficulties keeping both the matter and gravitational actions positive. However $-I_{\rm GR}$ would still be negative when conformal fluctuations were taken into account.

[11]See articles by D. Boulware, A. Strominger, and E. T. Tomboulis, in *Quantum Theory of Gravity,* edited by S. Christensen (Adam Hilger, Bristol, 1984); see also H. W. Hamber and R. M. Williams, Nucl. Phys. **B248**, 145 (1984).

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[15]Interestingly enough, one can obtain an arbitrarily small cosmological constant by taking $\Lambda=O(\epsilon)$ and $B=O(\epsilon^{-1})$. However, it now appears to be more difficult to recover general relativity in an appropriate limit.

[16]Boulware (Ref. 11).

[17]The $D_aD_bP^{ab}$ term in (2.3) can be viewed as part of the $A_{ab}\delta/\delta K_{ab}$ term in (2.8), by multiplying the constraint by an arbitrary function and integrating by parts twice.

[18]R. Penrose, in *General Relativity: An Einstein Centenary Survey* (Ref. 4).

[19]This argument shows that no boundary terms are needed, but it does not prove that they cannot exist. Indeed, in the asymptotically flat context ($\Lambda=0$) one expects that the Hamiltonian will contain a surface term which is just the usual Arnowitt-Deser-Misner (ADM) energy of general relativity. This suggests that the action I should contain the same boundary term as $I_{\rm GR}$. (I thank A. Ashtekar for pointing this out.) However in the cosmological context considered here, the surface term in the Hamiltonian vanishes, and we can consistently ignore surface terms in the action. For another discussion of surface terms in curvature-squared theories see N. Barth, Ph.D. thesis, University of North Carolina, 1983.

[20]G. T. Horowitz (in preparation).

[21]α' is related to the trace of the extrinsic curvature by $K=3\alpha' e^{-\alpha}$. In Friedmann models of our universe, α' is just the Hubble "constant."

[22]For a discussion of the relation between factor ordering and path-integral measures, see K. Kuchar, J. Math. Phys. **24**, 2122 (1983), and references cited therein.