Geometrical scaling in high-energy hadron collisions

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The concept of geometrical scaling for high-energy elastic hadron scattering is analyzed and its basic equations are solved in a consistent way. It is shown that they are applicable to a rather small interval of momentum transfers, e.g., maximally for $|t| \lesssim 0.15$ GeV^2 for pp scattering at CERN ISR energies ($\sqrt{s}=30$–63 GeV).

I. INTRODUCTION

The idea of geometrical scaling (GS) in high-energy hadron collisions was first phenomenologically introduced by Dias de Deus[1] in order to explain Koba-Nielsen-Olesen scaling in a simple way. Formally it states that the inelastic overlap function $G_{\rm in}(s,b^2)$ scales, s being the total energy squared and b the impact-parameter value. $G_{\rm in}(s,b^2)$ becomes a function of one variable $\beta=\pi b^2/\sigma_{\rm in}$ only when $s\to\infty$, i.e., $G_{\rm in}(\beta)$; $\sigma_{\rm in}(s)$ is the total inelastic cross section.

The unitarity condition in the impact-parameter space binds together this inelastic overlap function with the elastic one in the following manner:

$$\mathrm{Im}\,G_{\rm el}(s,b^2)=|G_{\rm el}(s,b^2)|^2+G_{\rm in}(s,b^2)\ . \qquad (1.1)$$

Generally, $G_{\rm el}(s,b^2)$ is complex. In order to derive its analogical scaling properties, Buras and Dias de Deus[2] were forced to accept the validity of the shadow-scattering limit. In this case the real part of the elastic overlap function (i.e., elastic amplitude) can be neglected at each impact parameter b. Using this assumption one can rewrite Eq. (1.1) in the form

$$G_{\rm el}(s,b^2)=\frac{i}{2}(1-[1-4G_{\rm in}(s,b^2)]^{1/2})\ . \qquad (1.2)$$

The geometrical scaling in the inelastic overlap function then implies the GS in the elastic one, i.e., $G_{\rm el}(s,b^2)$ similarly becomes the function of one scaling variable β only, i.e., $G_{\rm el}(\beta)$ when $s\to\infty$.

Using the Fourier-Bessel transform

$$F(s,t)\sim\int_0^\infty b\,db\,J_0(b\sqrt{-t}\,)G_{\rm el}(s,b^2)\ , \qquad (1.3)$$

where t is the four-momentum transfer squared and J_0 is the Bessel function of zero order, one can easily derive[2] (in the shadow-scattering limit and with spin effects neglected) that the function

$$\phi(s,\tau)=\frac{1}{\sigma_{\rm in}(s)}\frac{d\sigma_{\rm el}(s)}{d\tau}=\frac{1}{\sigma_{\rm in}{}^2(s)}\frac{d\sigma}{d|t|}\ , \qquad (1.4)$$

where $\tau=|t|\ \sigma_{\rm in}$ and $d\sigma/dt$ is the corresponding differential cross section, scales:

$$\phi(s,\tau)\underset{s\to\infty}{\longrightarrow}\phi(\tau)\ . \qquad (1.5)$$

By means of GS [see Eq. (1.5)], Buras and Dias de Deus[2] have tried to explain some of the experimental regularities in pp elastic scattering at the CERN ISR: the rise of the total cross section $\sigma_{\rm tot}$, the rise of the slope of the diffraction peak, and the decrease of the position of the diffraction dip as the energy increases within the ISR energy range. As the GS also predicts the constant ratio $\sigma_{\rm el}/\sigma_{\rm in}$ ($\sigma_{\rm el}$ being the total elastic cross section), one can slightly change the definition of the scaling variable:

$$\tau=|t|\,\sigma_{\rm tot}\ . \qquad (1.6)$$

Further progress is also due to Dias de Deus,[3] who has suggested a prescription on how to construct the Pomeron amplitude satisfying the GS requirements and describing high-energy hadron collisions.

A differential equation for the invariant scaling function generating both the real and imaginary parts of the elastic amplitude has then been derived by Dias de Deus and Kroll.[4] They have supposed that this equation can be used in the whole region from $t=0$ to the dip and have derived some predictions concerning the dip behavior for the pp scattering at different energies. A similar approach has been applied in Ref. 5 to $p\bar{p}$ scattering (see also Refs. 6 and 7). These predictions have, however, been disqualified by the experimental data.[8] The question can arise whether the idea of geometrical scaling itself is disqualified by these experimental data or whether the application of the GS to the dip region is not justified. We will show in the following that the latter possibility is valid.

The rest of the paper is organized as follows: Section II deals briefly with a Pomeron amplitude satisfying the GS requirements; the basic equation for GS and its different modifications suitable for numerical calculations are given. Section III then contains a new solution of the basic equation of GS for the case of pp elastic scattering at various ISR energies. In Sec. IV the new shape of the scaling function is discussed in comparison with previous results.

II. BASIC EQUATIONS DESCRIBING GEOMETRICAL SCALING

In this section we briefly sketch a derivation of differential equations for a Pomeron amplitude satisfying the GS requirements. Following Dias de Deus,[3] we define as a Pomeron amplitude the dominant amplitude in the fixed-t, $s \to \infty$ region, corresponding to a singularity in a j plane at $j=1$ with a slope $\alpha(0)=1$ and being crossing symmetric under the $\nu \to -\nu$ transform, where $\nu \equiv (s-u)/2$. The simplest form of such an amplitude exhibiting the GS can be written as

$$F(\nu,t) = -\left[\nu \exp\left[-i\frac{\pi}{2}\right]\right] R^2\left[\nu \exp\left[-i\frac{\pi}{2}\right]\right] \times \varphi\left[tR^2\left[\nu\exp\left[-i\frac{\pi}{2}\right]\right]\right], \tag{2.1}$$

where

$$\sigma_{\mathrm{tot}}(\nu) \sim \frac{\mathrm{Im}F(\nu,0)}{\nu} = R^2(\nu)$$

and φ and R^2 are real analytic functions.

Expanding the function $R^2(\nu)$ into a Taylor series around a point $\xi = \ln\nu$ and keeping only a linear term, one obtains

$$R^2\left[\xi - i\frac{\pi}{2}\right] \simeq R^2(\xi) - i\frac{\pi}{2}\frac{d}{d\xi}R^2(\xi) . \tag{2.2}$$

Substituting (2.2) into (2.1), expanding

$$\varphi\left[\left[1 - i\frac{\pi}{2}\frac{d}{d\xi}\right]R^2(\xi)t\right]$$

around $R^2(\xi)t$ up to linear terms, collecting the real and imaginary parts separately and using instead of the ξ variable the s variable, one finally obtains[3]

$$\mathrm{Im}F(s,\tau) = \mathrm{Im}F(s,0)\varphi(\tau) \tag{2.3}$$

and

$$\mathrm{Re}F(s,\tau) = \mathrm{Re}F(s,0)\frac{d}{d\tau}[\tau\varphi(\tau)] . \tag{2.4}$$

Note that these two equations can be derived only if the following normalization conditions hold:

$$\varphi(0) = 1 \tag{2.5}$$

and

$$\left.\frac{d}{d\tau}[\tau\varphi(\tau)]\right|_{\tau=0} = 1 . \tag{2.6}$$

Clearly, condition (2.6) means that $\varphi'(0)$ must be finite. Both Eqs. (2.3) and (2.4) are generally regarded as basic equations for GS in elastic high-energy scattering. In the present form they are practically the same as the equations of Martin[9] which determine the scaling of the elastic amplitude at asymptotic energies. Their scaling properties are based on the asymptotic behavior of the scattering amplitude derived by Auberson, Kinoshita, and Martin.[10]

Equations (2.3) and (2.4) can be combined together in order to obtain the $d\sigma/d\tau$ distribution[4]

$$\frac{d\sigma}{d\tau}(s,\tau) = \frac{d\sigma}{d\tau}(s,0) \times \left[\varphi^2(\tau) + \rho^2\left[\frac{d}{d\tau}[\tau\varphi(\tau)]\right]^2\right] . \tag{2.7}$$

Comparing Eq. (2.7) with Eqs. (1.4) and (1.5) one can see that the second term in Eq. (2.7) violates the original concept of GS.

Equation (2.7) mutually binds together the following quantities: the experimentally determined ratio of the real and imaginary parts of the elastic amplitude in a forward direction

$$\rho = \frac{\mathrm{Re}F(s,0)}{\mathrm{Im}F(s,0)}$$

the experimentally measured distribution $d\sigma/d\tau$ and the scaling function $\varphi(\tau)$. Therefore if one postulates that this equation correctly describes the elastic-scattering data in a large interval of momentum transfers at present energies one can determine any of those quantities provided the two remaining ones are given. In principle, there are two ways it can be used: either one takes as the input variables the ρ and the $d\sigma/d\tau$ quantities and calculates the scaling function $\varphi(\tau)$, or one starts with the function $\varphi(\tau)$ and with the ρ value at some fixed energy and then predicts the $d\sigma/d\tau$ distribution for a given process as the scaling function $\varphi(\tau)$ is supposed to be a common function to all interacting hadrons. The ρ value can be determined independently, e.g., by means of a dispersion-relation technique.

If one further denotes

$$D(\tau) = (1+\rho^2)\frac{d\sigma(s,\tau)/d\tau}{d\sigma(s,0)/d\tau} , \tag{2.8}$$

then one derives for $d\varphi/d\tau$ an expression

$$\frac{d\varphi(\tau)}{d\tau} = \frac{1}{\rho\tau}\{-\rho\varphi(\tau) \pm [D(\tau)-\varphi^2(\tau)]^{1/2}\} ; \tag{2.9}$$

the upper sign must be taken in Eq. (2.9) if $\varphi'(0)$ is to be finite. According to Eqs. (2.3) or (2.4) the function $\varphi(\tau)$ must be real for real values of τ. Therefore, in the region of applicability of the GS it must hold

$$|\varphi(\tau)| \le [D(\tau)]^{1/2} . \tag{2.10}$$

As was stressed before, the scaling function $\varphi(\tau)$ can be determined by the numerical solution of Eq. (2.9), which represents a rather hard problem as the differential cross section or $D(\tau)$ changes very rapidly (practically in seven orders of magnitude from forward scattering to the dip in the case of pp collisions at the ISR); and this property must be shared somewhat by the φ function, too. This difficulty can be removed, if some other functions which may be energy-dependent are introduced:

(i) Making the substitution

$$\omega_s(\tau) = \frac{\varphi(\tau)}{[D(\tau)]^{1/2}} , \tag{2.11}$$

Eq. (2.9) can be transformed to

$$\frac{d\omega_s(\tau)}{d\tau}=-f(s,\tau)\omega_s(\tau)+\frac{1}{\rho}[1-\omega_s{}^2(\tau)]^{1/2}\ , \tag{2.12}$$

where

$$f(s,\tau)=\frac{1}{\tau}+\frac{dA(s,\tau)/d\tau}{A(s,\tau)} \tag{2.13}$$

and

$$A(s,\tau)=\left[\frac{d\sigma}{d\tau}(s,\tau)\right]^{1/2}\ . \tag{2.14}$$

The $\omega_s(\tau)$ function must be real as well as the scaling function $\varphi(\tau)$. Therefore,

$$|\omega_s(\tau)|\leq 1\ . \tag{2.15}$$

(ii) Another possibility consists in introducing the phase of the elastic amplitude

$$F(s,\tau)=i\,|F(s,\tau)|\,e^{-i\alpha_s(\tau)}\ ; \tag{2.16}$$

$\alpha_s(\tau)$ determines its real and imaginary parts:

$$\begin{aligned}\mathrm{Re}F(s,\tau)&=|F(s,\tau)|\sin\alpha_s(\tau)\ ,\\ \mathrm{Im}F(s,\tau)&=|F(s,\tau)|\cos\alpha_s(\tau)\ .\end{aligned} \tag{2.17}$$

As $|F(s,\tau)|\sim A(s,\tau)$, Eq. (2.9) can be transformed to

$$\left[\frac{d\alpha_s(\tau)}{d\tau}+\frac{1}{\rho\tau}\right]\sin\alpha_s(\tau)-f(s,\tau)\cos\alpha_s(\tau)=0\ . \tag{2.18}$$

Both these new equations (2.12) and (2.18) are nonlinear differential equations of the first order like Eqs. (2.7) and (2.9). Their solutions are uniquely determined by initial conditions

$$\omega_s(0)=\frac{1}{(1+\rho^2)^{1/2}} \tag{2.19}$$

and

$$\alpha_s(0)=\arctan\rho\ , \tag{2.20}$$

which are equivalent to initial condition (2.5).

The differential cross section can be parametrized at small values of t by

$$\frac{d}{dt}\sigma(s,t)=Be^{bt} \tag{2.21}$$

and the slope of the diffraction peak is in a simple relation to the first derivatives of the newly introduced functions at $\tau=0$. It holds

$$\varphi'(0)=-\frac{1+\rho^2}{1+2\rho^2}\frac{b}{2\sigma_{\rm tot}}\ , \tag{2.22}$$

$$\omega_s'(0)=\frac{\rho}{(1+2\rho^2)(1+\rho^2)^{1/2}}\frac{b}{2\sigma_{\rm tot}}\ , \tag{2.23}$$

$$\alpha_s'(0)=-\frac{\rho}{1+2\rho^2}\frac{b}{2\sigma_{\rm tot}}\ . \tag{2.24}$$

III. SCALING FUNCTION DETERMINED BY NUMERICAL SOLUTION OF MODIFIED DIFFERENTIAL EQUATIONS

As has already been mentioned, Eq. (2.9) together with the initial condition (2.5) defines the scaling function $\varphi(\tau)$ for any $\tau>0$. However, there are some troubles with the numerical solution of Eq. (2.9) due to a rapid change of the scaling function with increasing τ. In analyzing Eqs. (2.12) or (2.18) one is in a much better position.

We will start with the examination of Eq. (2.12). It is evident that the zeros in imaginary or real parts of elastic amplitude correspond to the zeros of $\omega_s(\tau)$ or $d\omega_s(\tau)/d\tau$, respectively (for the real part see Appendix). If one starts from the initial condition (2.19) one can see from Eq. (2.23) that the value of $\omega_s(\tau)$ increases slowly till it reaches its maximum value $\omega_s(\tau_0)=1$. If Eq. (2.12) has a real solution also for $\tau>\tau_0$ then an extremum must exist at this point and the first derivative must be equal to zero, which requires

$$f(s,\tau_0)=0\ . \tag{3.1}$$

It is, therefore, useful to examine the detailed behavior of the function $f(s,\tau)$ for $\tau>0$ for different collision processes. We have used the experimental data for *pp* elastic scattering at the energies $\sqrt{s}=30.7$, 44.7, 52.8, 62.5 GeV (taken from Ref. 11). The τ dependence of $f(s,\tau)$ at $\sqrt{s}=52.8$ GeV is given in Fig. 1; for other energies only unsubstantial deviations exist. There are three zeros in the region of interest; two of them lying between the dip and the origin. If $\omega_s(\tau)$ is to remain real in the whole interval, these zeros should coincide with the zeros in the real part of the elastic amplitude; and any other zeros in the considered amplitude should not exist.

As to Eq. (2.18) these zeros should coincide with $\sin\alpha_s(\tau)=0$ or

$$\alpha_s(\tau)=k\pi\ . \tag{3.2}$$

If $\alpha_s\neq k\pi$, Eq. (2.18) can be written in the form

$$\frac{d\alpha_s(\tau)}{d\tau}=-\frac{1}{\rho\tau}+f(s,\tau)\cot\alpha_s(\tau)\ . \tag{3.3}$$

If $\alpha_s(\tau)$ tends to $k\pi$ at other values of τ [where $f(s,\tau)$ is nonzero] one would obtain an irregular solution with a derivative growing to infinity. At such points the function $\omega_s(\tau)$ would become complex, and the same would hold for the scaling function $\varphi(\tau)$.

Consequently, an important question arises: Is the solution of Eqs. (2.12), (2.18), and (2.9) regular in the whole τ region considered in the previous papers (quoted in the Introduction) or at what τ does this solution cease to be regular and the consequences of GS become inapplicable to experimental data?

The only way to obtain an answer is to solve the corresponding differential equations with appropriate initial conditions. These equations must be solved in a numerical way. As the values of $d\sigma/dt$ are known in discrete points and burdened by experimental errors, the determination of the derivative needed in establishing the

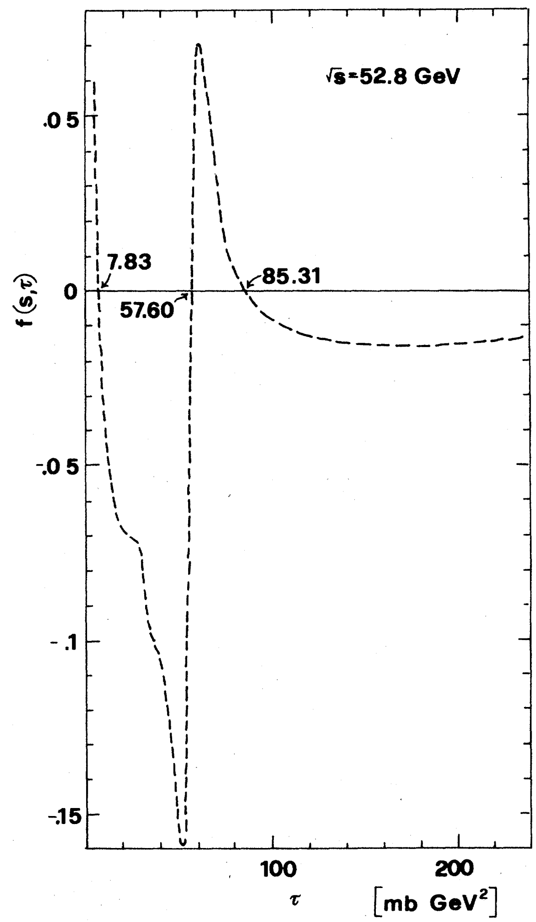

FIG. 1. Plot of function $f(s,\tau)$ vs τ for the pp scattering at the energy $\sqrt{s}=52.8$ GeV.

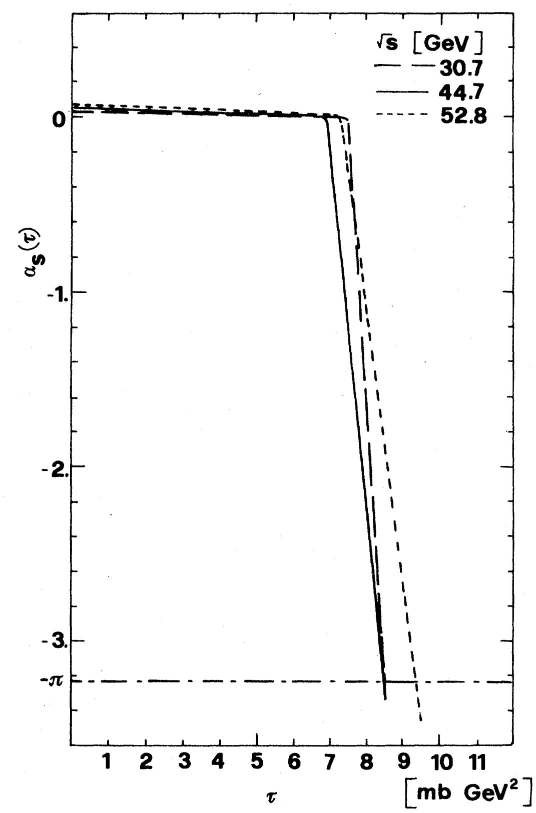

FIG. 2. Plot of phase $\alpha_s(\tau)$ vs scaling variable τ calculated by means of Eq. (2.18) for pp scattering at various ISR energies.

values of $f(s,\tau)$ could bring some discrepancies during the solution of differential equations. Therefore, we have "smoothed" the experimental data beforehand by fitting the differential cross section with the help of the formula

$$\frac{d\sigma}{d\tau}=(a_0+a_1\tau)\exp(b_1\tau+b_2\tau^2+b_3\tau^3) + (c_0+c_1\tau)\exp(d_1\tau+d_2\tau^2+d_3\tau^3)\ . \tag{3.4}$$

It would be possible to take a greater number of free parameters, but the formula (3.4) seems to be quite sufficient for our purposes.

Let us now denote the individual roots of $f(s,\tau)$ seen in Fig. 1 by τ_0, τ_1, and τ_2, respectively. Their values are only weakly energy dependent. The solution of Eqs. (2.18) and (3.3), respectively, is then given in Fig. 2. (The undefined value of $\alpha_s'(\tau)$ at τ_0 [see Eq. (2.18)] can be determined by analytical continuation.) One can immediately see that the first value of τ fulfilling the condition (3.2) agrees fully with τ_0. The $|\alpha_s(\tau)|$ grows quickly reaching great values of many multiples of π at the values of τ not corresponding to any root of $f(s,\tau)$. Therefore, the solution of the differential equation becomes irregular at $\tau_i \simeq 9$ mb GeV^2.

This behavior is in full agreement with the numerical solution of Eq. (2.12). The $\omega_s(\tau)$ reaches its maximum value $\omega_s=1$ at $\tau=\tau_0$ where the first derivative is equal to zero. Then $\omega_s(\tau)$ tends to zero reaching this value at the same τ where $\alpha_s=-\pi/2$; that means the imaginary part of the amplitude equals zero at this point. The value $\omega_s=-1$ is then reached at τ_i showing again that ω_s ceases to be real for $\tau>\tau_i$. It may happen that around τ_1 and τ_2 the ω_s will again turn real but this no longer has any serious physical meaning.

We are forced to conclude that the equations of the GS derived in Ref. 3 have their physical meaning in a rather narrow interval $\tau\epsilon\langle 0,\ \tau_u\rangle$ only, where $\tau_u=8.5-9.5$ mb GeV^2 (which corresponds to $-t\in[0,0.2]$ GeV^2) when the energy increases from $\sqrt{s}=30.7$ to 62.5 GeV in the case of pp elastic scattering. They cannot be applied to larger momentum transfers as the scaling function $\varphi(\tau)$ ceases to be real, which contradicts one of the basic assumptions of GS. For smaller values of τ the $\varphi(\tau)$ is energy-independent in agreement with the GS requirements.

The calculated normalized real parts corresponding to the phases from Fig. 2 are shown in Fig. 3. The curves cross the zero line at $\tau=6.8-7.2$ mb GeV^2 (it corresponds to $t\cong-0.16$ GeV^2), then rapidly decrease in order to in-

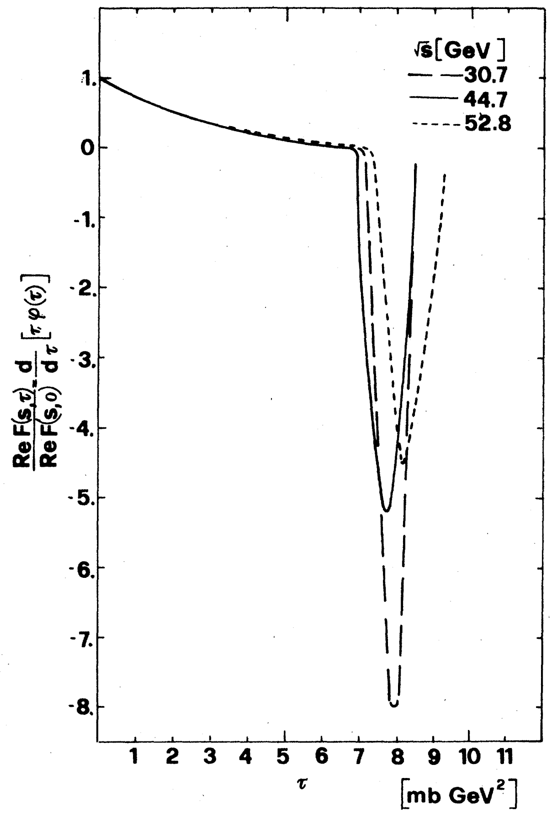

FIG. 3. Plot of normalized real parts of elastic amplitudes vs τ for pp scattering at different ISR energies and corresponding to the phases $\alpha_s(\tau)$ given in Fig. 2.

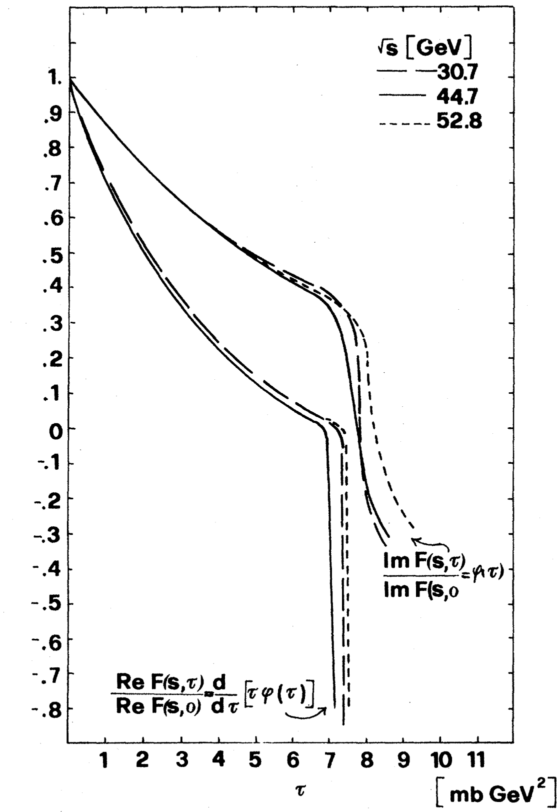

FIG. 4. Plot of the normalized imaginary parts of elastic amplitudes vs τ for pp scattering at different ISR energies and corresponding to the phases $\alpha_s(\tau)$ given in Fig. 2. For comparison this figure also contains the graphs of normalized real parts of elastic amplitudes shown in Fig. 3.

crease again, leaving their minimum values roughly at $t=-0.18$ GeV^2.

Similarly, the calculated normalized imaginary parts are exhibited in Fig. 4. The corresponding curves slowly decrease from their initial values and cross the zero line at the point $\tau=7.8$–8.3 mb GeV^2. This figure also shows the real parts from the previous figure in order to make a mutual comparison of both the real and imaginary parts. One can observe from Fig. 4 that the scaling function $\varphi(\tau)$ is energy independent for $|t| \lesssim 0.15$ GeV^2, which is the region of maximal applicability of the GS in pp scattering at the ISR.

Because of a gap in the data[11] at the energy $\sqrt{s}=62.5$ GeV at $-t \sim 0.1$ GeV^2, Figs. 2–4 do not contain the corresponding curves at this energy.

IV. DISCUSSION

Some previous applications of GS[4–7] led to some interesting conclusions concerning the relations between differential cross sections at different energies for larger momentum transfers. The fact that the application of the Taylor series expansion of the scaling function at the dip region gives a very good agreement with experimental data if the appropriate values of ρ are used could be impressive and could suggest that some deeper relation might exist. We have, therefore, performed a similar analysis with different values of ρ. It is possible to show that the fits with the same χ^2/DF values can be obtained practically for any value of ρ ranging from 0.02 to 0.10 (at the dip region). Moreover, we have also used this method in the other regions of the τ variable, especially for small values of τ's. We have obtained the fits with the same value of χ^2/DF for completely different values of the ρ quantity (e.g., differing in two orders of magnitude). Therefore one must conclude that the method of the Taylor series expansion is not reliable.

Another problem concerns the τ dependence of the scaling function $\varphi(\tau)$ suggested in Ref. 4. The given behavior was obtained under some additional assumptions added to the basic assumptions of GS. It was supposed that the $\varphi(\tau)$ exhibits a monotone decrease from the forward scattering to the dip reaching the zero value just at the dip. It follows from our analysis that these additional assumptions are in contradiction to the used GS basic assumptions as the solution of differential equations of the first order is fully determined by one initial condition, i.e., by $\varphi(0)=1$. A false behavior for higher values of τ could be easily obtained when Eqs. (2.7) or (2.9) was made use of as a starting point for the numerical solution of the given problem.

An identical crossing-symmetric formula for elastic amplitude valid for asymptotic energies but for infinitesimally small values of $|t|$ has been derived also by Martin[9] from rather general assumptions concerning the amplitudes of collision processes. Our results show that the idea of GS can be applied to elastic *pp* scattering for small momentum transfers already at the ISR energies.

At the present time it is not possible to give a definite answer as to whether the consequences of GS can be applied to larger momentum transfers or not. Our limitation concerns Eq. (2.7) only. However, this equation is a result of some approximations applied to the general Pomeron amplitude (2.1). It follows from the approach leading to Eqs. (2.3) and (2.4) that the energy dependence appearing for $-t>0.15$ GeV^2 can be in principle compensated by the s dependence of $\sigma_{\rm tot}$ and that also the range of applicability of GS can be fundamentally broadened. However, if the given effect were taken into account the simplicity of the GS idea would be lost.

V. CONCLUSION

By introducing some other variables in addition to the scaling function $\varphi(\tau)$, it has been possible to solve the basic equation of GS numerically in a consistent way. The scaling function determined in such a way then differs significantly from the previous suggestions (see Ref. 4). It also follows that the applicability of the basic GS equation (2.7) is limited to a rather small interval of $|t|\lesssim 0.15$ GeV^2. The applicability range could be probably broadened if the more general crossing-symmetric amplitude (2.1) represented the actual starting point of the GS analysis.

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APPENDIX

Let us suppose that τ_0 is a root of the real part of elastic amplitude. Then using Eqs. (2.4) and (2.9) one obtains

$$\omega_s(\tau_0)=\pm 1\ . \tag{A1}$$

Using Eqs. (2.13), (2.14), and (2.8), one can derive that the following equality must hold

$$f(s,\tau_0)=\frac{1}{\tau_0}+\left.\frac{d[D(\tau)]^{1/2}/d\tau}{[D(\tau)]^{1/2}}\right|_{\tau_0}. \tag{A2}$$

If one again uses Eqs. (2.4) and (A2), then

$$\tau_0[D(\tau_0)]^{1/2}f(s,\tau_0)\omega_s(\tau_0)+\tau_0[D(\tau_0)]^{1/2}\frac{d}{d\tau}\omega_s(\tau_0)=0\ . \tag{A3}$$

But as $\tau_0[D(\tau_0)]^{1/2}\neq 0$ for $\tau_0\neq 0$, because of (A1) one obtains that

$$f(s,\tau_0)=\pm\omega_s'(\tau_0)\ . \tag{A4}$$

With the help of Eqs. (2.17) and (2.18) one immediately observes that the real part of the elastic amplitude has a root τ_0 if and only if the $\omega_s'(\tau)$ function has a root at the same τ_0.

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