Triple hadronic-energy correlations in high-energy e^-e^+ annihilation

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Triple hadronic-energy correlations are suggested as a natural means for characterizing three-jet events and testing QCD in high-energy e^-e^+ annihilation with $\gamma+Z^0$ exchange. The general analysis of the triple correlation is given in terms of possible structure functions, and lowest-order QCD contributions are presented.

I. INTRODUCTION

A few years ago Brown and collaborators[1] suggested a hierarchy of increasingly finely grained, but still inclusive cross sections, which can be calculated in QCD. This hierarchy consists of energy-weighted cross sections. So far the most useful one has been the energy-energy correlation cross section (or energy-weighted angular correlation), extensively studied both theoretically[2,3] and experimentally[4] in e^-e^+ annihilation. What has been actually measured is the normalized, angle-integrated energy-energy correlation

$$\frac{1}{\sigma_{\rm tot}}\frac{d\Sigma}{d\cos\chi}$$

and the corresponding asymmetry, where χ is the angle between the two directions in which the hadronic energies are detected. (One has to sum up over all the directions, keeping χ fixed.)

The next member of the hierarchy is the normalized triple energy correlation

$$\frac{1}{\sigma_{\rm tot}}\frac{d^3\Sigma}{d\Omega_1 d\Omega_2 d\Omega_3}\,.$$

In this case the energies deposited into three directions (with unit vectors $\hat{r}_1,\hat{r}_2,\hat{r}_3$, respectively) should be measured. Actually, this is the first member of the hierarchy including threefold energy correlations of three particles belonging to three jets, while the double energy correlation takes into account such cases by counting the three particles in the corresponding pairs. Obviously, it is again useful to integrate over some angles in order to obtain quantities, which may be measured with better statistics. In the present paper we define and calculate the triple energy correlation for high-energy e^-e^+ annihilation, i.e., including both the virtual γ and Z^0 annihilation channels. We argue that the normalized, angle-integrated triple energy correlation is a natural way to study both three- and four-jet events.

Choosing the three calorimeters to lie in a plane, the triple energy correlation is measured as a twice-differential quantity. Experimentally one measures

$$\frac{1}{\sigma_{\rm tot}}\frac{d\Sigma_{\rm planar}}{d\cos\chi_1 d\cos\chi_2} = \frac{6}{\Delta\chi_1\sin\chi_1\Delta\chi_2\sin\chi_2}\frac{1}{N}\sum_{A=1}^{N}\sum\frac{E_{Aa}E_{Ab}E_{Ac}}{W^3}\,, \qquad (1)$$

where W means the total c.m. energy, A specifies the events ($A=1,\ldots,N$), while a,b,c specify the individual particles with energies E_{Aa},E_{Ab},E_{Ac}. The momenta of the particles a,b,c lie in a plane and the angle between the momenta of particles a and b (a and c) is χ_1 to $\chi_1+\Delta\chi_1$ (χ_2 to $\chi_2+\Delta\chi_2$). The second sum runs over all triplets of particles of the event A with the appropriate geometry. Each distinct triplet of particles is counted only once, while an individual particle may contribute in several triplets. Since the three calorimeters lie in a plane, it is clear that this quantity is sensitive to the three-jet events, and the multijets contribute little.

Choosing the three calorimeters in nonplanar positions, the triple energy correlation is measured as a threefold differential quantity. Experimentally it is measured as

$$\frac{1}{\sigma_{\rm tot}}\frac{d\Sigma}{d\cos\chi_1 d\cos\chi_2 d\cos\chi_3} = \frac{6}{\Delta\chi_1\sin\chi_1\Delta\chi_2\sin\chi_2\Delta\chi_3\sin\chi_3}\times\frac{1}{N}\sum_{A=1}^{N}\sum\frac{E_{Aa}E_{Ab}E_{Ac}}{W^3}\,, \qquad (2)$$

where the meaning of χ_1 and χ_2 is the same as above and the angle between the momenta of particles b and c lies between χ_3 and $\chi_3+\Delta\chi_3$. Since the kinematics is nonplanar, this quantity is sensitive to four-jet events; two and three jets do not contribute. Including all the self-correlations ($b=c$ at $\chi_3=0$, etc.) in Eq. (2) assures that the integral of Eq. (2) over the whole allowed χ_1,χ_2,χ_3 spaces is one.

Since triple-energy-correlation measurements involve the determination of at least a twice-differential quantity, i.e.,

$$\frac{1}{\sigma_{\rm tot}}\frac{d\Sigma_{\rm planar}}{d\cos\chi_1 d\cos\chi_2}\,,$$

the most hopeful situation for measuring it is around the Z^0 peak. This is why we have included in our calculation both γ and Z^0 exchanges. A measurement at the Z^0 peak is also advantageous from the point of view of fragmentation corrections and next-to-the-lowest-order QCD corrections, which are much smaller at the Z^0 peak than at presently accessible energies. The fragmentation correction is expected[5] to decrease as W^{-2}, i.e., we get almost a factor-of-$\frac{1}{10}$ suppression. Note that for the usual double energy correlation the fragmentation correction decreases only as $1/W$.

The organization of the paper is as follows. In Sec. II we discuss the triple energy correlation for general, nonplanar calorimeter positions. Section III deals with the planar positions and the corresponding $O(\alpha_s)$ QCD contributions. Section IV contains a discussion.

II. TRIPLE ENERGY CORRELATION IN THE NONPLANAR-DETECTOR-POSITION CASE

In terms of exclusive cross sections, the triple energy correlation cross section is defined[1] as

$$\frac{d^3\Sigma}{d\Omega_1 d\Omega_2 d\Omega_3}=\sum_{N=2}^{\infty}\int\prod_{d=1}^{N}E_d{}^{-1}(d^3p_d)\frac{d^N\sigma}{E_1{}^{-1}(d^3p_1)\cdots E_N{}^{-1}(d^3p_N)}S_N \times\left[\sum_{a,b,c=1}^{N}\frac{E_aE_bE_c}{W^3}\delta(\Omega_1-\Omega_a)\delta(\Omega_2-\Omega_b)\delta(\Omega_3-\Omega_c)\right], \quad (3)$$

where $d^N\sigma/E_1{}^{-1}(d^3p_1)\cdots E_N{}^{-1}(d^3p_N)$ denotes the exclusive N-particle cross section and S_N is a factor taking into account phase-space reduction for identical particles. The unit vectors pointing into the directions of the calorimeters will be denoted by $\hat{r}_1,\hat{r}_2,\hat{r}_3$, with the corresponding solid angles $\Omega_1,\Omega_2,\Omega_3$. Inclusion of the diagonal terms ($a=b, a=c$, etc.) assures that the integral of Eq. (3) with respect to any of the solid angles Ω_i reproduces the double energy correlation.

The calculation of the triple energy correlation goes parallel to that of the energy-energy correlation. We shall therefore follow Ref. 3. To lowest order in the electroweak interaction, the triple energy correlations are given by an energy-weighted phase-space integral of a squared amplitude $|T|^2$, with

$$|T|^2\propto\sum_f\left|\langle f_+|\mathcal{F}_{\gamma\mu}|0\rangle\frac{1}{W^2}\langle 0|j^\mu_\gamma|e^-e^+\rangle+\langle f_+|\mathcal{F}_{\text{weak}\,\mu}|0\rangle\frac{1}{W^2-M_Z^2+iM_Z\Gamma_Z}\langle 0|j^\mu_{\text{weak}}|e^-e^+\rangle\right|^2, \quad (4)$$

where $j^\mu_{\gamma(\text{weak})}$ ($\mathcal{F}^\mu_{\gamma(\text{weak})}$) is the lepton (hadron) electromagnetic (weak) current, and $|f_+\rangle$ is an arbitrary outgoing hadronic final state that can occur. We rewrite this as

$$|T|^2\propto\sum_f|a_1v^\mu\langle f_+|V_\mu|0\rangle+a_2v^\mu\langle f_+|A_\mu|0\rangle+a_3a^\mu\langle f_+|V_\mu|0\rangle+a_4a^\mu\langle f_+|A_\mu|0\rangle|^2, \quad (5)$$

where v^μ (a^μ) is the matrix element of the vector (axial-vector) leptonic current and V_μ (A_μ) means the hadronic vector (axial-vector) current. All the coupling constants as well as the γ and Z^0 propagators are included in $a_1,\ldots,a_4$, which are given in Ref. 3 in the standard model, for hadron production through a quark-antiquark pair of flavor f.

As recognized by Brown and Li,[2] the final state is effectively invariant under charge conjugation, therefore

$$\sum_f\langle 0|V^\mu|f_+\rangle\langle f_+|A^\nu|0\rangle=0. \quad (6)$$

Moreover, for massless quarks (i.e., much above any quark threshold) we also have for hadron production through one $q\bar{q}$ pair

$$\sum_f\langle 0|V^\mu|f_+\rangle\langle f_+|V^\nu|0\rangle=\sum_f\langle 0|A^\mu|f_+\rangle\langle f_+|A^\nu|0\rangle\equiv\bar{V}^{\mu\nu}. \quad (7)$$

Neglecting final state interactions, TCP invariance yields

$$\bar{V}^{\mu\nu}=\bar{V}^{\nu\mu}. \quad (8)$$

Now, we perform the necessary integrations and polarization sums over the final-state variables and multiply by the necessary energy factors. The final result is denoted by $V^{\mu\nu}$. This is again a symmetric tensor depending on W and the directions $\hat{r}_1,\hat{r}_2,\hat{r}_3$. (In the planar case only $\hat{r}_1,\hat{r}_2$ enter.) Thus, the triple energy correlation is proportional to

$$V_{\mu\nu}[v^\mu v^{\nu*}(|a_1|^2+|a_2|^2)+a^\mu a^{\nu*}(|a_3|^2+|a_4|^2)+v^\mu a^{\nu*}(a_1a_3^*+a_2a_4^*)+a^\mu v^{\nu*}(a_1^*a_3+a_2^*a_4)]. \quad (9)$$

The leptonic tensors $v^\mu v^{\nu*},\ldots$ are easily calculated and are given, e.g., in Ref. 3. The leptons are taken to be massless, therefore only the space-space part of V_{ik} enters in Eq. (9). V_{ik} is a symmetric tensor, hence we get the decomposition

$$V^{ik}=A_1\hat{r}_1^i\hat{r}_1^k+A_2\tfrac{1}{2}(\hat{r}_1^i\hat{r}_2^k+\hat{r}_2^i\hat{r}_1^k)+A_3\tfrac{1}{2}(\hat{r}_1^i\hat{r}_3^k+\hat{r}_3^i\hat{r}_1^k)+A_4\hat{r}_2^i\hat{r}_2^k+A_5\tfrac{1}{2}(\hat{r}_2^i\hat{r}_3^k+\hat{r}_3^i\hat{r}_2^k)+A_6\hat{r}_3^i\hat{r}_3^k, \quad (10)$$

so that in general the six structure functions A_i are sufficient for describing V^{ik}. The A_i's depend on W and the angles χ_1,χ_2,χ_3 (where $\cos\chi_1=\hat{r}_1\cdot\hat{r}_2$, $\cos\chi_2=\hat{r}_1\cdot\hat{r}_3$, $\cos\chi_3=\hat{r}_2\cdot\hat{r}_3$). Since the three detectors are interchangeable, we obtain relations among the A_i's:

$$\begin{aligned}&A_4(\chi_1,\chi_2,\chi_3)=A_1(\chi_1,\chi_3,\chi_2),\quad A_6(\chi_1,\chi_2,\chi_3)=A_1(\chi_3,\chi_2,\chi_1),\\&A_3(\chi_1,\chi_2,\chi_3)=A_2(\chi_2,\chi_1,\chi_3),\quad A_5(\chi_1,\chi_2,\chi_3)=A_2(\chi_3,\chi_2,\chi_1),\end{aligned} \quad (11)$$

and

$$A_1(\chi_1,\chi_2,\chi_3)=A_1(\chi_2,\chi_1,\chi_3),\quad A_2(\chi_1,\chi_2,\chi_3)=A_2(\chi_1,\chi_3,\chi_2)\;. \tag{12}$$

Putting in all the details we have

$$\frac{d^3\Sigma}{d\Omega_1 d\Omega_2 d\Omega_3}=W^2V^{ik}\sum_f [\,v_iv_k^*(|a_{1f}|^2+|a_{2f}|^2)+a_ia_k^*(|a_{3f}|^2+|a_{4f}|^2)+v_ia_k^*(a_{1f}a_{3f}^*+a_{2f}a_{4f}^*)$$
$$+a_iv_k^*(a_{1f}^*a_{3f}+a_{2f}^*a_{4f})]\;, \tag{13}$$

where the sum goes over quark flavors.

This expression is rather complicated. However, our purpose is to integrate out over some of the angles, eventually we want to keep only the χ_1,χ_2,χ_3 dependences. First, we transform

$$d\Omega_1 d\Omega_2 d\Omega_3=d\cos\theta_1 d\phi_1 d\cos\theta_2 d\phi_2 d\cos\theta_3 d\phi_3$$

to the set of new variables

$$d\cos\chi_1 d\cos\chi_2 d\cos\chi_3 d\cos\theta_1 d\cos\theta_2 d\phi_1\;.$$

(θ_i and ϕ_i are polar and azimuthal angles measured in a coordinate system where the e^- moves along the z axis.) The result of this variable transformation is not simple, therefore we present only the result integrated over ϕ_1 and normalized to the total cross section calculated to lowest order in α_s through γ and Z^0 exchanges:

$$\frac{1}{\sigma_{\text{tot}}}\frac{d\Sigma}{d\cos\chi_1 d\cos\chi_2 d\cos\chi_3 d\cos\theta_1 d\cos\theta_2}$$
$$=8\pi^2\sum_{\theta_3=\theta_{3-},\theta_{3+}}[A_1\sin^2\theta_1+2A_2(\cos\chi_1-\cos\theta_1\cos\theta_2)+2A_3(\cos\chi_2-\cos\theta_1\cos\theta_3)+A_4\sin^2\theta_2$$
$$+2A_5(\cos\chi_3-\cos\theta_1\cos\theta_3)+A_6\sin^2\theta_3]\frac{1}{\Delta(\theta_1,\theta_2,\chi_1)}$$
$$\times\frac{\sin^2\theta_1}{|(\cos\chi_1\cos\theta_1-\cos\theta_2)\Delta(\theta_1,\theta_3,\chi_2)-el_{\pm}\Delta(\theta_1,\theta_2,\chi_1)(\cos\chi_2\cos\theta_1-\cos\theta_3)|}\;, \tag{14}$$

where

$$\cos\theta_{3+(-)}=\frac{-\cos\chi_3(\cos\chi_1\cos\theta_1-\cos\theta_2)-\cos\chi_2(\cos\chi_1\cos\theta_2-\cos\theta_1)\pm\Delta(\chi_1,\chi_2,\chi_3)\Delta(\chi_1,\theta_1,\theta_2)}{\sin^2\chi_1}\;,$$
$$el_{+(-)}=\text{sgn}[\pm\Delta(\chi_1,\chi_2,\chi_3)(\cos\chi_1\cos\theta_1-\cos\theta_2)+\Delta(\chi_1,\theta_1,\theta_2)(\cos\chi_3-\cos\chi_1\cos\chi_2)]\;,$$
$$\Delta(\alpha,\beta,\gamma)=(1+2\cos\alpha\cos\beta\cos\gamma-\cos^2\alpha-\cos^2\beta-\cos^2\gamma)^{1/2}\;, \tag{15}$$
$$\Delta(\alpha,\beta,\gamma)\text{ is real if }\cos\gamma\in[\cos(\alpha+\beta),\cos(\alpha-\beta)]\;.$$

This is the condition which ensures that α,β,γ $(0\le\alpha,\beta,\gamma\le180^\circ)$ are the sides of a convex spherical triangle. The reality conditions imposed on the various Δ's entering Eqs. (14) and (15) restrict the possible values of the various angles. Carrying out the sum over θ_3 in Eq. (14) one has to use el_+ for the θ_{3+} term and el_- for the θ_{3-} term.

Equation (14) still looks quite complicated, however, its structure is simple. It is important, that initial-state polarizations and weak-interaction parameters (i.e., Z^0 mass and width and coupling constants) do not appear in Eq. (14). Similar results are valid for the normalized energy-energy correlations integrated over at least one azimuthal angle.[3] Integrating over the remaining θ angles leads to

$$\frac{1}{\sigma_{\text{tot}}}\frac{d\Sigma}{d\cos\chi_1 d\cos\chi_2 d\cos\chi_3}=\frac{64\pi^3}{3\Delta(\chi_1,\chi_2,\chi_3)}(A_1+2A_2\cos\chi_1+2A_3\cos\chi_2+A_4+2A_5\cos\chi_3+A_6)\;, \tag{16}$$

where $0\le\chi_i\le180^\circ$ with the condition $\Delta(\chi_1,\chi_2,\chi_3)=\text{real}$. The right-hand side of Eq. (16) is invariant under interchanges of the calorimeters, as may be checked using Eqs. (11) and (12).

The structure functions A_i may be calculated in lowest-order QCD from the known[6] $e^-e^+\to q\bar{q}q\bar{q},q\bar{q}gg$ cross sections.[7] Here we are concerned with the lowest-order QCD calculation for the planar case, when three-jet final states contribute.

III. TRIPLE ENERGY CORRELATION IN THE PLANAR-DETECTOR-POSITION CASE

The analysis of the triple energy correlation in this case goes parallel to that of the nonplanar case. As a consequence of the planarity the most differential quantity is

$$\frac{d\Sigma_{\text{planar}}}{d\chi_1 d\chi_2 d\cos\theta_1 d\cos\theta_2 d\phi_1} .$$

Still we may start the calculation from $d^3\Sigma_{\text{planar}}/d\Omega_1 d\Omega_2 d\Omega_3$, keeping in mind that this quantity contains a δ function, ensuring the planarity. The invariant decomposition of V_{ik} can be written as

$$V^{ik}=[S_1\hat{r}_1^i\hat{r}_1^k+S_2\hat{r}_2^i\hat{r}_2^k+S_3\tfrac{1}{2}(\hat{r}_1^i\hat{r}_2^k+\hat{r}_2^i\hat{r}_1^k)+S_4\delta^{ik}]\frac{1}{|a|}$$
$$\times\delta\left[\frac{1}{a}[\sin\theta_3\sin\theta_1\cos\theta_2\sin(\phi_1-\phi_3)+\sin\theta_3\sin\theta_2\cos\theta_1\sin(\phi_3-\phi_2)+\sin\theta_1\sin\theta_2\cos\theta_3\sin(\phi_2-\phi_1)]\right],$$
$$a=(\cos\theta_1-\cos\theta_2)\sin\theta_3\cos\phi_3+(\cos\theta_3-\cos\theta_1)\sin\theta_2\cos\phi_2+(\cos\theta_2-\cos\theta_3)\sin\theta_1\cos\phi_1\,, \quad (17)$$

where the S_i's depend only on χ_1,χ_2 and W. In a more suggestive form the δ function can be rewritten as $\delta(\hat{r}_3\cdot(\hat{r}_1 x\hat{r}_2)/a)$, where $\hat{r}_1 x\hat{r}_2$ denotes the vector product. The symmetry properties of the S_i's are

$$S_1(\chi_1,\chi_2)=\frac{\sin^2(\chi_1+\chi_2)}{\sin^2\chi_1}S_1(2\pi-\chi_1-\chi_2,\chi_2),\quad S_2(\chi_1,\chi_2)=S_1(\chi_1,2\pi-\chi_1-\chi_2)\,,$$
$$S_3(\chi_1,\chi_2)=\frac{\sin\chi_2}{\sin(\chi_1+\chi_2)}[S_1(\chi_2,\chi_1)-S_1(\chi_1,\chi_2)]-\frac{\sin(\chi_1+\chi_2)}{\sin\chi_2}S_1(\chi_1,2\pi-\chi_1-\chi_2)\,, \quad (18)$$
$$S_4(\chi_1,\chi_2)=S_4(\chi_1,2\pi-\chi_1-\chi_2)=S_4(2\pi-\chi_1-\chi_2,\chi_2)=S_4(\chi_2,\chi_1)\,.$$

In terms of V^{ik}, $d^3\Sigma_{\text{planar}}/d\Omega_1 d\Omega_2 d\Omega_3$ is expressed as in Eq. (13). Now, we transform to the variables $d\chi_1 d\chi_2 d\cos\theta_1 d\cos\theta_2 d\phi_1 d\theta_3$ and integrate over the variables ϕ_1,θ_3. The normalized result is again simple, independent of initial-state polarizations and weak-interaction parameters. It reads as

$$\frac{1}{\sigma_{\text{tot}}}\frac{d\Sigma_{\text{planar}}}{d\chi_1 d\chi_2 d\cos\theta_1 d\cos\theta_2}=\frac{16\pi^2}{\Delta(\chi_1,\theta_1,\theta_2)}[S_1\sin^2\theta_1+S_2\sin^2\theta_2+S_3(\cos\chi_1-\cos\theta_1\cos\theta_2)+2S_4]\,. \quad (19)$$

Integrating over the remaining angles, we get the following general form of the normalized planar triple energy correlation:

$$\frac{1}{\sigma_{\text{tot}}}\frac{d\Sigma_{\text{planar}}}{d\chi_1 d\chi_2}=\frac{64\pi^3}{3}(S_1+S_2+S_3\cos\chi_1+3S_4),\quad (0\le\chi_1,\chi_2\le 180^\circ,\ 180^\circ\le\chi_1+\chi_2\le 360^\circ)\,. \quad (20)$$

Using Eq. (18) it is easy to see that

$$\frac{1}{\sigma_{\text{tot}}}\frac{d\Sigma_{\text{planar}}}{d\chi_1 d\chi_2}$$

is symmetric under exchange of any two of the angles $\chi_1,\chi_2,\chi_3=360^\circ-\chi_1-\chi_2$. On the other hand, $d\Sigma_{\text{planar}}/d\cos\chi_1 d\cos\chi_2$ is not a completely symmetric function. Equation (20) is again valid for arbitrary initial-state polarizations and does not depend on weak-interaction parameters. The structure functions S_1,S_2,S_3 may be easily calculated from the known $e^-e^+\to q\bar{q}g$ cross sections in lowest-order QCD with the result

$$S_1(\chi_1,\chi_2)=-\frac{\alpha_s(W)}{2\pi^4}\frac{E_1{}^2E_2{}^2E_3{}^2}{W^6}E_1{}^2\left[\frac{1}{(W-2E_1)(W-2E_2)}+\frac{2}{(W-2E_1)(W-2E_3)}+\frac{1}{(W-2E_2)(W-2E_3)}\right],$$
$$S_4(\chi_1,\chi_2)=-(S_1+S_2+S_3\cos\chi_1)\,, \quad (21)$$

where $\alpha_s(W)$ is the strong coupling constant,

$$E_1=W\frac{-\sin(\chi_1+\chi_2)}{\sin\chi_1+\sin\chi_2-\sin(\chi_1+\chi_2)},\quad E_2=E_1\frac{\sin\chi_2}{-\sin(\chi_1+\chi_2)},\quad E_3=E_2\frac{\sin\chi_1}{\sin\chi_2}\,. \quad (22)$$

E_1,E_2,E_3 are the energies detected in the calorimeters 1,2,3 lying in the directions $\hat{r}_1,\hat{r}_2,\hat{r}_3$, respectively. S_2 and S_3 may be obtained using Eq. (18). These expressions of S_i determine

$$\frac{d\Sigma_{\text{planar}}}{d\chi_1 d\chi_2 d\cos\theta_1 d\cos\theta_2 d\phi_1}$$

even in the fully polarized case. The normalized and integrated energy correlation is obtained by combining the S_i's as in Eq. (20). Finally

$$\frac{1}{\sigma_{\text{tot}}}\frac{d\Sigma_{\text{planar}}}{d\chi_1 d\chi_2}=\frac{64\alpha_s(W)}{3\pi}E_1{}^2E_2{}^2E_3{}^2\frac{1}{W^6}\left[\frac{E_1{}^2+E_2{}^2}{(W-2E_1)(W-2E_2)}+\frac{E_1{}^2+E_3{}^2}{(W-2E_1)(W-2E_3)}+\frac{E_2{}^2+E_3{}^2}{(W-2E_2)(W-2E_3)}\right]. \tag{23}$$

A simple check of this result is that the sum rule

$$\int_{-1}^{-\cos\chi_1}\frac{d\Sigma_{\text{planar}}}{d\cos\chi_1 d\cos\chi_2}\frac{W}{E_3}d\cos\chi_2=\frac{d\Sigma}{d\cos\chi_1} \tag{24}$$

is valid, where $d\Sigma/d\cos\chi_1$ is the well known energy-energy correlation at lowest-order QCD. The $O(\alpha_s)$ result, Eq. (23), is expected to be a good approximation (provided fragmentation corrections are also added), when the three angles ($\chi_1,\chi_2,\chi_3=360°-\chi_1-\chi_2$) among the detectors are away from 0° and 180°. We may assume $\delta<\chi_i<180°-\delta$, with $\delta\approx 20°$—30°. Since we have conditions for all the three angles, this region is quite restricted. Figure 1 shows

$$\frac{1}{\sigma_{\text{tot}}}\frac{d\Sigma_{\text{planar}}}{d\chi_1 d\chi_2}$$

as a two-variable function for $\delta=30°$. The function is quite smooth, the smallest value is 0.0073, the largest value is 0.0175. If we choose a smaller value for δ, the function will be more sharply peaked at the edges. Obviously for $\chi_i=0°$ or 180° the $O(\alpha_s)$ result diverges.

Following Ref. 8, for comparison we have calculated the planar triple energy correlation in a theory with vector gluons and scalar quarks. The result is

$$\frac{1}{\sigma_{\text{tot}}}\frac{d\Sigma_{\text{planar}}}{d\chi_1 d\chi_2}=\frac{32\alpha_s(W)}{\pi}\frac{E_1{}^2E_2{}^2E_3{}^2}{W^6}\left[3-\frac{4}{3}\frac{E_1E_2\cos\chi_1}{(W-2E_1)(W-2E_2)}-\frac{4}{3}\frac{E_1E_3\cos\chi_2}{(W-2E_1)(W-2E_3)}-\frac{4}{3}\frac{E_2E_3\cos(\chi_1+\chi_2)}{(W-2E_2)(W-2E_3)}\right], \tag{25}$$

where σ_{tot} is the total cross section for scalar-quark production. Although this function looks at first sight quite different from the spin-half-quark result, numerically the two functions are very close to each other. For $\delta=30°$ they agree within 10%. Thus it would be difficult to distinguish experimentally QCD from the scalar-quark theory from a measurement of

$$\frac{1}{\sigma_{\text{tot}}}\frac{d\Sigma_{\text{planar}}}{d\chi_1 d\chi_2}.$$

Similar statements have been made for several other angle-integrated quantities in Ref. 8.

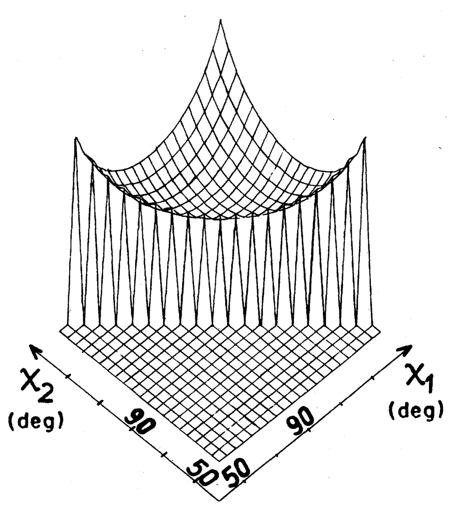

FIG. 1. Axonometric view of the normalized, planar triple-energy-correlation function for $\delta=30°$. The largest value of the function is at $\chi_1=60°$, $\chi_2=150°$; $\chi_1=150°$, $\chi_2=60°$; $\chi_1=\chi_2=150°$; while the smallest value is at $\chi_1=\chi_2=120°$. For $\chi_1+\chi_2<180°+\delta$ (a region where the QCD result is not applicable) the function has been arbitrarily set equal to zero.

IV. DISCUSSION

In summary, we have worked out triple energy correlations for high-energy e^-e^+ annihilation. There are two kinematical configurations in which triple energy correlations may be studied. The planar case is a measure of three-jet production, while in the nonplanar case the four-jet final states contribute.

It is a new feature of the triple energy correlation that it provides a possibility to study both the three- and four-jet events. A similar treatment of the usual energy-energy correlations (i.e., double energy correlations) leads to a trivial result. In that case the two possible kinematical configurations are back-to-back directions ($\hat{r}_1=-\hat{r}_2$) and $\hat{r}_1\neq-\hat{r}_2$. The former is sensitive to two-jet events, while the latter is sensitive to three-jet events. However, in this case the analog of $d\Sigma_{\text{planar}}/d\chi_1 d\chi_2$ is the fully integrated energy-energy correlation, i.e., the total cross section.

The calculation of triple energy correlations has been carried out for arbitrary initial-state polarizations. It is important that after normalization with the total cross section and integrating over at least one azimuthal angle, initial-state polarization and weak-interaction parameters (i.e., Z^0 mass and width and coupling constants) drop out. This is similar to what happens with the double energy correlations.[3]

There is no doubt that measurements of

$$\frac{1}{\sigma_{\rm tot}}\frac{d\Sigma_{\rm planar}}{d\chi_1 d\chi_2}$$

will be easier than those of

$$\frac{1}{\sigma_{\rm tot}}\frac{d\Sigma}{d\chi_1 d\chi_2 d\chi_3} .$$

The QCD result for the planar case is given by Eqs. (22) and (23). We emphasize that for measuring triple energy correlations a detailed event-by-event analysis is not required. This is a common advantage of the whole hierarchy of energy-weighted cross sections. We believe that

$$\frac{1}{\sigma_{\rm tot}}\frac{d\Sigma_{\rm planar}}{d\chi_1 d\chi_2}$$

probes the three-jet final state in a more natural way than the double energy correlation. In this case all the three jets figure in the same way, while in the double energy correlation one of the jets is always integrated over. The Z^0-peak energy region will be a particularly good possibility for measuring these quantities. At this energy the total cross section is large, and fragmentation and higher-order QCD corrections are expected to be small.

APPENDIX

To determine the size of the fragmentation correction the best method is to perform a complete Monte Carlo calculation making use of one (or more) of the existing fragmentation models. While we postpone this to a forthcoming publication, we present here a simple treatment, which already determines the energy dependence of the fragmentation correction. The line of reasoning strictly follows Ref. 9.

The average number of hadrons, hadron pairs, hadron triplets produced by a quark moving with three momentum $\vec{p}$ is given in terms of the functions $f_1(\vec{h}_1;\vec{p})$, $f_2(\vec{h}_1,\vec{h}_2;\vec{p})$, and $f_3(\vec{h}_1,\vec{h}_2,\vec{h}_3;\vec{p})$, respectively, as

$$\begin{aligned} dn &= \frac{d^3h_1}{h_1^0} f_1(\vec{h}_1;\vec{p}) , \\ d^2n &= \frac{d^3h_1}{h_1^0}\frac{d^3h_2}{h_2^0} f_2(\vec{h}_1,\vec{h}_2;\vec{p}) , \\ d^3n &= \frac{d^3h_1}{h_1^0}\frac{d^3h_2}{h_2^0}\frac{d^3h_3}{h_3^0} f_3(\vec{h}_1,\vec{h}_2,\vec{h}_3;\vec{p}) , \end{aligned} \tag{26}$$

where $\vec{h}_1,\vec{h}_2,\vec{h}_3$ denote the momenta of the produced hadrons. By energy-momentum conservation we have the sum rules[10]

$$\begin{aligned} &\int \frac{d^3h_1}{h_1^0} h_1^\mu f_1(\vec{h}_1;\vec{p}) = p^\mu , \\ &\int \frac{d^3h_2}{h_2^0} h_2^\mu f_2(\vec{h}_1,\vec{h}_2;\vec{p}) = (p^\mu - h_1^\mu) f_1(\vec{h}_1;\vec{p}) , \\ &\int \frac{d^3h_3}{h_3^0} h_3^\mu f_3(\vec{h}_1,\vec{h}_2,\vec{h}_3;\vec{p}) = (p^\mu - h_1^\mu - h_2^\mu) f_2(\vec{h}_1,\vec{h}_2;\vec{p}) . \end{aligned} \tag{27}$$

In terms of the functions f_i we express the contribution of the $q\bar{q}$ final state to the triple energy correlation as

$$\begin{aligned} \frac{d^3\Sigma^{q\bar{q}}}{d\Omega_1 d\Omega_2 d\Omega_3} = \int d\Omega_p \frac{d\sigma}{d\Omega_p} \int \frac{h_1{}^2dh_1}{h_1^0}\frac{h_2{}^2dh_2}{h_2^0}\frac{h_3{}^2dh_3}{h_3^0} \\ \times \frac{h_1^0h_2^0h_3^0}{W^3}\{[f_1(\vec{h}_1;\vec{p})f_2(\vec{h}_2,\vec{h}_3;-\vec{p})+f_1(\vec{h}_2;\vec{p})f_2(\vec{h}_1,\vec{h}_3;-\vec{p}) \\ +f_1(\vec{h}_3;\vec{p})f_2(\vec{h}_1,\vec{h}_2;-\vec{p})+f_3(\vec{h}_1,\vec{h}_2,\vec{h}_3;\vec{p}) \\ +h_1^0\delta(\vec{h}_1-\vec{h}_2)f_1(\vec{h}_1;\vec{p})f_1(\vec{h}_3;-\vec{p})+h_1^0\delta(\vec{h}_1-\vec{h}_2)f_2(\vec{h}_1,\vec{h}_3;\vec{p}) \\ +h_2^0\delta(\vec{h}_2-\vec{h}_3)f_1(\vec{h}_3;\vec{p})f_1(\vec{h}_1;-\vec{p})+h_2^0\delta(\vec{h}_2-\vec{h}_3)f_2(\vec{h}_3,\vec{h}_1;\vec{p}) \\ +h_3^0\delta(\vec{h}_1-\vec{h}_3)f_1(\vec{h}_3;\vec{p})f_1(\vec{h}_2;-\vec{p})+h_3^0\delta(\vec{h}_1-\vec{h}_3)f_2(\vec{h}_3,\vec{h}_2;\vec{p}) \\ +h_1^0\delta(\vec{h}_1-\vec{h}_2)h_2^0\delta(\vec{h}_2-\vec{h}_3)f_1(\vec{h}_1;\vec{p})]+[\vec{p}\leftrightarrow-\vec{p}]\} , \end{aligned} \tag{28}$$

where $d\sigma/d\Omega_p$ is the cross section of $q\bar{q}$ production. The inclusion of self-correlations in Eq. (28) ensures the normalization condition

$$\int d\Omega_3 \frac{d^3\Sigma^{q\bar{q}}}{d\Omega_1 d\Omega_2 d\Omega_3} = \frac{d^2\Sigma^{q\bar{q}}}{d\Omega_1 d\Omega_2} , \tag{29}$$

where the right-hand side is the $q\bar{q}$ contribution to the usual double energy correlation. Introduce now[9] the new functions F_1,F_2,F_3 as

$$F_1(\eta_1)=\frac{2}{W}\int h_1{}^2dh_1f_1(\vec{h}_1;\vec{p}) ,$$

$$F_2(\eta_1,\eta_2,\chi_1)=\left[\frac{2}{W}\right]^2\int h_1{}^2dh_1h_2{}^2dh_2[f_2(\vec{h}_1,\vec{h}_2;\vec{p})+h_1^0\delta(\vec{h}_1-\vec{h}_2)f_1(\vec{h}_1;\vec{p})] , \tag{30}$$

$$\begin{aligned}F_3(\eta_1,\eta_2,\eta_3,\chi_1,\chi_2,\chi_3)=\left[\frac{2}{W}\right]^3\int h_1{}^2dh_1h_2{}^2dh_2h_3{}^2dh_3&\\ \times[f_3(\vec{h}_1,\vec{h}_2,\vec{h}_3;\vec{p})+h_1^0\delta(\vec{h}_1-\vec{h}_2)f_2(\vec{h}_1,\vec{h}_3;\vec{p})&\\ +h_2^0\delta(\vec{h}_2-\vec{h}_3)f_2(\vec{h}_3,\vec{h}_1;\vec{p})+h_3^0\delta(\vec{h}_1-\vec{h}_3)f_2(\vec{h}_3,\vec{h}_2;\vec{p})&\\ +(h_1^0)^2\delta(\vec{h}_1-\vec{h}_2)\delta(\vec{h}_2-\vec{h}_3)f_1(\vec{h}_1;\vec{p})] ,&\end{aligned}$$

where η_i is the angle between $\vec{h}_i$ and $\vec{p}$; and the meaning of χ_i is as before. The normalization conditions are

$$\int d\Omega_1F_1(\eta_1)=1,\quad \int d\Omega_2F_2(\eta_1,\eta_2,\chi_1)=F_1(\eta_1),\quad \int d\Omega_3F_3(\eta_1,\eta_2,\eta_3,\chi_1,\chi_2,\chi_3)=F_2(\eta_1,\eta_2,\chi_1) . \tag{31}$$

In terms of these functions we have

$$\begin{aligned}\frac{d^3\Sigma^{q\bar{q}}}{d\Omega_1d\Omega_2d\Omega_2}=\tfrac{1}{8}\int d\Omega_p\frac{d\sigma}{d\Omega_p}[&F_1(\eta_1)F_2(\pi-\eta_2,\pi-\eta_3,\chi_3)+F_1(\pi-\eta_1)F_2(\eta_2,\eta_3,\chi_3)\\ &+F_1(\eta_2)F_2(\pi-\eta_1,\pi-\eta_3,\chi_2)+F_1(\pi-\eta_2)F_2(\eta_1,\eta_3,\chi_2)\\ &+F_1(\eta_3)F_2(\pi-\eta_1,\pi-\eta_2,\chi_1)+F_1(\pi-\eta_3)F_2(\eta_1,\eta_2,\chi_1)\\ &+F_3(\eta_1,\eta_2,\eta_3,\chi_1,\chi_2,\chi_3)+F_3(\pi-\eta_1,\pi-\eta_2,\pi-\eta_3,\chi_1,\chi_2,\chi_3)] .\end{aligned} \tag{32}$$

The properties of the functions F_1,F_2 have been determined in Ref. 9 under the assumption of scaling, limited transverse momentum, and no backward production. The properties of F_3 are determined in a similar way. The results are as follows. $F_1(\eta)$ has a peak at $\eta\approx 0$ with a height $\propto W^2$ and width $\propto 1/W$; while for $\eta \gg \langle h_\perp\rangle/W$

$$F_1(\eta)\approx C\langle h_\perp\rangle/2\pi W\sin^3\eta ,$$

with $C\langle h_\perp\rangle$ constant. When both η_1 and η_2 are large compared to $1/W$, $F_2(\eta_1,\eta_2,\chi_1)\propto 1/W^2$, and it is strongly peaked when either η_1 or η_2 becomes very small. Similarly,

$$F_3(\eta_1,\eta_2,\eta_3,\chi_1,\chi_2,\chi_3)\propto 1/W^3$$

when all η_i are large compared to $1/W$, and it is strongly peaked whenever one η_i becomes very small. The existence of the peaks for small η_i follows from the normalization conditions (31).

Using the above properties we observe that the integral in Eq. (32) yields nonleading contributions as $W\to\infty$ except for small angular regions about directions which are aligned either collinearly or anticollinearly with the detection directions Ω_1, Ω_2, or Ω_3. Note that we assume $\chi_i\neq 0$, $\chi_i\neq\pi$. Consider, e.g., the region when $\eta_1\approx 0$, i.e., $\Omega_p\approx\Omega_1$, $\chi_1\approx\eta_2$, $\chi_2\approx\eta_3$. We get in this case

$$\begin{aligned}\frac{1}{8}\frac{d\sigma}{d\Omega_1}[&F_2(\pi-\chi_1,\pi-\chi_2,\chi_3)\int d\Omega_1F_1(\eta_1)+F_1(\pi-\chi_1)\int d\Omega_1F_2(\eta_1,\eta_3,\chi_2)+F_1(\pi-\chi_2)\int d\Omega_1F_2(\eta_1,\eta_2,\chi_1)\\ &+\int d\Omega_1F_3(\eta_1,\eta_2,\eta_3,\chi_1,\chi_2,\chi_3)]=\frac{1}{8}\frac{d\sigma}{d\Omega_1}[F_2(\pi-\chi_1,\pi-\chi_2,\chi_3)+F_1(\pi-\chi_1)F_1(\chi_2)+F_1(\pi-\chi_2)F_1(\chi_1)\\ &+F_2(\chi_1,\chi_2,\chi_3)] .\end{aligned} \tag{33}$$

Adding up all the other contributions we get the leading contribution

$$\begin{aligned}\frac{d^3\Sigma^{q\bar{q}}}{d\Omega_1d\Omega_2d\Omega_3}=&\frac{1}{4}\frac{d\sigma}{d\Omega_1}[F_2(\pi-\chi_1,\pi-\chi_2,\chi_3)+F_1(\pi-\chi_1)F_1(\chi_2)+F_1(\pi-\chi_2)F_1(\chi_1)+F_2(\chi_1,\chi_2,\chi_3)]\\ &+\frac{1}{4}\frac{d\sigma}{d\Omega_2}[F_1(\pi-\chi_1)F_1(\chi_3)+F_2(\pi-\chi_1,\pi-\chi_3,\chi_2)+F_1(\pi-\chi_3)F_1(\chi_1)+F_2(\chi_1,\chi_3,\chi_2)]\\ &+\frac{1}{4}\frac{d\sigma}{d\Omega_3}[F_1(\pi-\chi_2)F_1(\chi_3)+F_1(\pi-\chi_3)F_1(\chi_2)+F_2(\pi-\chi_2,\pi-\chi_3,\chi_1)+F_2(\chi_2,\chi_3,\chi_1)] .\end{aligned} \tag{34}$$

Since the arguments of the F_1 and F_2 functions which enter Eq. (34) are away from the peaking regions, we can use the estimates $F_1 \propto 1/W$, $F_2 \propto 1/W^2$ together with the well known expression of $d\sigma/d\Omega_1$ to secure the result

$$\frac{1}{\sigma_{\text{tot}}}\frac{d^3\Sigma^{q\bar{q}}}{d\Omega_1 d\Omega_2 d\Omega_3} \propto \frac{1}{W^2} . \tag{35}$$

Similarly to the double-energy-correlation case, Eq. (35) does not imply that at present energies ($W \approx 30$ GeV) the correction is small. The only point we want to emphasize is the energy dependence of this correction.

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[5]See Appendix.

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