Finite-energy sum rules for heavy quarkonia

A. Nicolaidis and G. J. Gounaris
Theoretical Physics Department, University of Thessaloniki, Thessaloniki, Greece
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We show that at large energies the parton model and the vector-dominance model give identical results for e^+e^- annihilation into hadrons. We derive finite-energy sum rules for heavy quarkonia which connect perturbative results to resonance parameters. The sum rules are satisfied extremely well by the Υ system. From the available masses and leptonic widths of heavy resonances we construct a quantity $M_{\rm tr}(T)$, which is the truncated version of a Green's function $M(T)$. We remark that as the time T goes to infinity $M(T)$ approaches $M_{\rm tr}(T)$ and we calculate via Monte Carlo techniques the full Green's function $M(T)$, in the simple case of a linear potential.

I. INTRODUCTION

By now it is an accepted dogma that quantum chromodynamics (QCD) describes the strong interactions. Due to asymptotic freedom, the coupling constant α_s at large energies is rather small and in such kinematical domains we can study sufficiently inclusive quantities (such as the e^+e^- annihilation total cross section, deep-inelastic scattering, and jet production in hadronic collisions) within perturbation theory. The perturbative results for the above quantities are expressed in terms of quark-gluon parameters (coupling constant α_s, quark masses, etc.). On the other hand, little progress has been made in explaining and understanding the actual hadronic properties (hadronic masses, widths of resonances, decay constants, etc.). Our main tools in confronting such problems are lattice gauge theories and different models (e.g., vector-meson dominance, potential models).

In principle, the hadronic properties should be determined in terms of the fundamental parameters of QCD. Since we are rather far from this goal, it is worthwhile to look for methods which would allow us to establish connections between the world of quarks and gluons and the world of hadrons. Duality[1-3] equates a suitable energy average of the physical cross section for $e^+e^-\to$hadrons, to the same energy average of the perturbative cross section for $e^+e^-\to$quarks, gluons. Duality has been shown to hold in the WKB approximation both nonrelativistically and relativistically.[4] In Sec. II we provide a precise formulation of nonrelativistic duality. We smear the total cross section $\sigma(E)$ with an exponential function e^{-ET} and we show that at large energies (or equivalently for short time intervals) the parton model and vector-dominance model give identical results.[5]

Another fruitful approach in correlating QCD parameters with hadronic parameters is the sum-rules approach.[6-9] In Sec. III we derive finite-energy sum rules which relate QCD perturbative calculations to parameters of heavy resonances. Using finite-energy sum rules, we avoid any guess about the high-energy behavior of the cross section. Furthermore, we use exponential moments, and it has been shown that the exponential moments suppress the contribution of higher-energy states, thus providing a fast convergence.[10] We consider the finite-energy sum rules for the c- and b-quark sectors. The sum rules are satisfied extremely well by the Υ system. We suggest that nonperturbative terms are important in the case of charmonium.[11]

From the masses M_n and the leptonic widths Γ_n of radially excited S states ($Q\bar{Q}$ bound states) we can construct a quantity $M(T)$, which is related to the logarithmic derivative with respect to time T of the nonrelativistic propagator of the heavy quark Q. In Sec. IV we point out that it is feasible to calculate $M(T)$ via Monte Carlo techniques and therefore extract information about the interquark potential. Finally, in Sec. V we summarize our results and compare our work with similar approaches.

II. BOUND STATES VS PARTON MODEL

Consider the experimentally measured ratio $R(E)=\sigma(e^+e^-\to\text{hadrons})/\sigma(e^+e^-\to\mu^+\mu^-)$. If we focus our attention on the contribution to $R(E)$ of a single heavy flavor, then at low energies we observe resonance peaks, while at high energies $R(E)$ approaches a constant value. The asymptotic flatness of $R(E)$ is explained by the parton model or QCD. A perturbative series for $R(E)$ gives

$$R(E)=3e_q{}^2(1+c_1\alpha_s+c_2\alpha_s{}^2+\cdots)\,, \qquad (1)$$

i.e., $R(E)$ approaches from above the constant value $3e_q{}^2$. The resonance structure can be understood through the vector-dominance model. The contribution of a single vector meson to $R(E)$ is given by

$$R_V(E)=\frac{9\pi}{2\alpha^2}\Gamma(V\to e^+e^-)\delta(E-M)\,, \qquad (2)$$

where nonrelativistically

$$\Gamma(V\to e^+e^-)=\frac{4\pi\alpha^2 e_q{}^2}{m_q{}^2}\,|\Psi(0)|^2\,. \qquad (3)$$

M is the mass of the vector meson, m_q is the mass of the heavy quark, and $\Psi(0)$ is the wave function at the origin.

The duality concept[1-3] suggests that a summation of bound states can simulate the parton-model results. To

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explore this idea we define $R_{\rm BS}(T)$ as the Laplace transform of the contribution of bound states to $R(E)$,

$$R_{\rm BS}(T)=\int\sum_{r} R_{V_n}(E)e^{-ET}dE\ . \tag{4}$$

The heaviness of the quark allows us to use nonrelativistic quantum mechanics to study the bound-state properties. Recalling that $M_n=2m_q+E_n$, Eqs. (2), (3), and (4) give

$$\begin{aligned}R_{\rm BS}(T)&=18\pi^2\frac{e_q{}^2}{m_q{}^2}\sum_n|\Psi_n(0)|^2e^{-M_nT}\\&=18\pi^2\frac{e_q{}^2}{m_q{}^2}e^{-2m_qT}\langle\vec{r}=0,t=T\,|\,\vec{r}=0,t=0\rangle\ ,\end{aligned}\tag{5}$$

where $\langle\vec{r}=0,t=T\,|\,\vec{r}=0,t=0\rangle$ is the nonrelativistic three-dimensional propagator (in imaginary time). Equation (5) expresses $R_{\rm BS}(T)$ in terms of a compact object (propagator), which we know how to manipulate. Considering the radial part of the wave function

$$\Psi_n(0)=\frac{1}{\sqrt{4\pi}}\lim_{r\to0}\frac{u_n(r)}{r}\ , \tag{6}$$

Eq. (5) can be rewritten

$$R_{\rm BS}(T)=\frac{9\pi e_q{}^2}{2m_q{}^2}e^{-2m_qT}\lim_{r\to0}\frac{\langle r,t=T\,|\,r,t=0\rangle_B}{r^2}\ . \tag{7}$$

The propagator $\langle r,t=T\,|\,r,t=0\rangle_B$ satisfies the one-dimensional Schrödinger equation describing the radial motion.

This propagator can also be defined for negative r by assuming that for $r<0$ the potential is an infinite barrier. To proceed further we use the following lemma:

$$\begin{aligned}&\langle r,t=T\,|\,r,t=0\rangle_B\\&\quad=\langle r,t=T\,|\,r,t=0\rangle-\langle -r,t=T\,|\,r,t=0\rangle\ ,\end{aligned}\tag{8}$$

where $\langle r,t=T\,|\,r',t=0\rangle$ is the one-dimensional propagator for any r,r' (positive or negative).

Equation (8) reminds us of the image method of electrostatics. To make Eq. (8) look plausible we notice the following.

(i) When the barrier is present, there are two classical paths; the direct and the reflected. The first (second) term on the right-hand side of Eq. (8) corresponds to the direct (reflected) path.

(ii) We know that $\langle r,t=T\,|\,r,t=0\rangle_B=0$ at $r=0$. Equation (8) guarantees this boundary condition.

(iii) Only paths with $r\geq0$ contribute to the left-hand side of Eq. (8), while the paths contributing to the propagators of the right-hand side have arbitrary r values. However when we take the difference on the right-hand side of Eq. (8) the forbidden trajectories with $r<0$ are eliminated.[12]

Generally $R_{\rm BS}(T)$ depends upon the interquark potential $V(r)$. When $T\to0$, short distances are involved and the uncertainty principle implies that the kinetic energy of the particle exceeds considerably the potential energy, provided the potential is nonsingular at the origin. Therefore for short times the propagator coincides with the free-particle propagator.[10] As $T\to0$ we have

$$\langle r,T\,|\,r,0\rangle_B\underset{T\to0}{\longrightarrow}\langle r,T\,|\,r,0\rangle_F-\langle -r,T\,|\,r,0\rangle_F\ , \tag{9}$$

where

$$\langle x_b,T\,|\,x_a,0\rangle_F=\left[\frac{\mu}{2\pi T}\right]^{1/2}\exp\left[-\frac{\mu(x_b-x_a)^2}{2T}\right] \tag{10}$$

and $\mu=m_q/2$ is the reduced mass. Using Eqs. (7), (8), (9), and (10) we obtain

$$R_{\rm BS}(T)\underset{T\to0}{\longrightarrow}\frac{9e_q{}^2}{4}\left[\frac{\pi}{m_q}\right]^{1/2}e^{-2m_qT}T^{-3/2}\ . \tag{11}$$

The potential-dependent corrections to this result have been studied in detail by Whitenton *et al.*[5]

Now we turn to the parton model. To calculate $R_{\rm PM}(T)$, we use the standard parton-model expression

$$R_{\rm PM}(E)=3e_q{}^2v\left[\frac{3-v^2}{2}\right]\theta(E-2m_q)\ , \tag{12}$$

$$v=\left[1-\frac{4m_q{}^2}{E^2}\right]^{1/2}\ . \tag{13}$$

To compare $R_{\rm PM}(T)$ to $R_{\rm BS}(T)$ we consider nonrelativistic energies and velocities ($E\simeq2m_q$, $v\to0$). In this limit we can approximate Eq. (13) by

$$v\simeq\left[\frac{E-2m_q}{m_q}\right]^{1/2}\ . \tag{14}$$

Ignoring v^2 terms, we obtain

$$R_{\rm PM}(E)\cong\frac{9e_q{}^2}{2}\left[\frac{E-2m_q}{m_q}\right]^{1/2}\theta(E-2m_q)\ . \tag{15}$$

Using the relation

$$\int_b^\infty(t-b)^\nu e^{-pt}dt=\Gamma(\nu+1)p^{-\nu-1}e^{-bp} \tag{16}$$

we find that

$$\begin{aligned}R_{\rm PM}(T)&=\int R_{\rm PM}(E)e^{-ET}dE\\&=\frac{9e_q{}^2}{4}\left[\frac{\pi}{m_q}\right]^{1/2}e^{-2m_qT}T^{-3/2}\ .\end{aligned}\tag{17}$$

Comparing Eqs. (11) and (17) we observe that when $T\to0$, that is, at large energies, the parton model and summation of bound states give identical results.[5] When we include v^2 terms[13] in the leptonic width [Eq. (3)] and follow the same procedure we again obtain identical results, this time including the v^2 terms of the parton model. We feel that this kind of duality (which implies identical results from the parton model and summation of bound states at large energies) is a permanent feature and goes beyond the nonrelativistic approximation we have used.[14]

III. FINITE-ENERGY SUM RULES

The vacuum-polarization amplitude $\Pi(E)$, far from threshold, can be reliably computed in perturbation theory, while near physical threshold $\Pi(E)$ is saturated by resonances and continuum states. Relying on the analyticity of $\Pi(E)$ we can construct sum rules which connect the resonances to QCD parameters. Shifman, Vainshtein, and Zakharov fully exploited this idea and using the machinery of the renormalization group and aspects of nonperturbative physics, they extracted much information about low resonances ($J/\psi,\rho,\omega$, glueballs,...).[6,15] In what follows we derive finite-energy sum rules for heavy quarkonia and apply them to the charmonium and Υ systems.

Consider the hadronic part of the photon vacuum polarization $\Pi(E)$. At low energies $\Pi(E)$ is uncalculable, since it contains all the complexities of strong interactions. We only know that $\Pi(E)$, as an analytic function of energy, has a cut along the real axis (Fig. 1). In the deep Euclidean region (large imaginary energy) $\Pi(E)$ is calculable within perturbation theory and we denote this perturbative component by $\Pi_{\rm QCD}(E)$. Consider now the quantity $[\Pi(E)-\Pi_{\rm QCD}(E)]f(E)$ integrated along the contour shown in Fig. 1. If the weight function $f(E)$ has no singularities within the contour, then we have[9,11]

$$\int_c [\Pi(E)-\Pi_{\rm QCD}(E)]f(E)dE=0 \ . \tag{18}$$

If the radius $\bar{E}$ is large, then on the circumference we expect $\Pi(E)\simeq\Pi_{\rm QCD}(E)$. Recalling that the imaginary part of $\Pi(E)$ is proportional to $R(E)$ and taking $f(E)=\exp(-ET)$ we arrive at

$$\int_0^{\bar{E}} R(E)e^{-ET}dE=\int_0^{E} R_{\rm QCD}(E)e^{-ET}dE \ . \tag{19}$$

The above finite-energy sum rule (FESR) relates low-energy data [embodied in the experimentally measured $R(E)$] to a high-energy quantity $R_{\rm QCD}(E)$ calculable in perturbation theory. Consider the contribution to $R(E)$ of a heavy quark. If we take for $\bar{E}$ a value just below the threshold, only the narrow states with $M_n\le E$ contribute to $R(E)$. We have then

$$\frac{9\pi}{2\alpha^2}\sum_{M_n\le\bar{E}}\Gamma_n(e^+e^-)e^{-M_nT}=\int_0^{\bar{E}} R_{\rm QCD}(E)e^{-ET}dE \ . \tag{20}$$

Equation (20) is quite interesting, since it establishes a direct connection between resonance parameters (leptonic widths, masses) and perturbative QCD. Notice also that as $m_q\to\infty$, more and more narrow states are found below the threshold $\bar{E}$. Therefore in the limit $m_q\to\infty$ the left-hand side of Eq. (20) approaches $R_{\rm BS}(T)$. This is a consequence of our choice for the weight function $f(E)$. Taking derivatives of Eq. (20) with respect to T we obtain

$$\begin{aligned}\frac{9\pi}{2\alpha^2}&\sum_{M_n\le\bar{E}}\Gamma_n(e^+e^-){M_n}^k e^{-M_nT}\\ &\equiv R_k(T,\bar{E})\\ &=\int_0^{\bar{E}} R_{\rm QCD}(E)E^k e^{-ET}dE \quad k=0,1,2\dots \ .\end{aligned} \tag{21}$$

In what follows, for $R_{\rm QCD}(E)$ we use

$$R_{\rm QCD}(E)=3{e_q}^2 v\left[\frac{3-v^2}{2}\right][1+\tfrac{4}{3}\alpha_s f(v)]\theta(E-2m_q) \ , \tag{22}$$

$$f(v)=\frac{\pi}{2v}-\frac{(3+v)}{4}\left[\frac{\pi}{2}-\frac{3}{4\pi}\right] . \tag{23}$$

We have applied Eq. (21) to charmonium and Υ systems.

A. Charmonium

We use $\bar{E}=3.7$ GeV, and for the ψ and ψ' parameters we have used[16]

$$\begin{aligned}&M_\psi=3.096 \text{ GeV} \ ,\\ &\Gamma(\psi\to e^+e^-)=4.8\pm0.6 \text{ keV} \ ,\\ &M_{\psi'}=3.687 \text{ GeV} \ ,\\ &\Gamma(\psi'\to e^+e^-)=2.2\pm0.3 \text{ keV} \ .\end{aligned} \tag{24}$$

The left-hand side of Eq. (21) for $k=0$ is represented as a dashed area (due to the experimental uncertainties of the leptonic widths) in Fig. 2. For the corresponding right-hand side of Eq. (21) we have to specify the coupling constant α_s and the quark mass m_c. For the coupling constant α_s we have used $\alpha_s=0.2$. The quark mass, since it appears in a formula derived within perturbation theory, is the current quark mass. For $m_c=1.23$ GeV we obtained the solid line (Fig. 2) which is in reasonable agreement with the dashed area constructed from the experimental data. If we attempt to test Eq. (21) for $k=1,2,\dots$, we find out that the agreement deteriorates as k rises. We come back to this point later on.

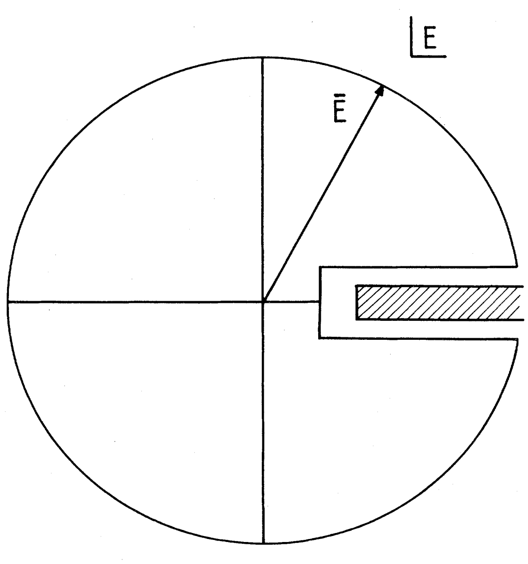

FIG. 1. Contour in the complex energy plane used in evaluation of Eq. (18).

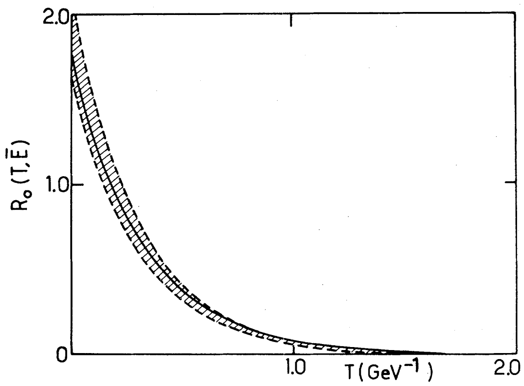

FIG. 2. The sum rule, Eq. (21) for $k=0$, in the case of charmonium.

B. Υ states

We use $\bar{E}=10.4$ GeV, so that three narrow states ($\Upsilon,\Upsilon',\Upsilon''$) contribute to the left-hand side of Eq. (21). For the Υ parameters we have used the Cornell Electron Storage Ring data[17]

$$\begin{aligned}
&M_\Upsilon=9.433\text{ GeV}\\
&\Gamma(\Upsilon\to e^+e^-)=1.125\text{—}0.725\text{ keV},\\
&M_{\Upsilon'}=10.0\text{ GeV},\\
&\Gamma(\Upsilon'\to e^+e^-)=0.578\text{—}0.358\text{ keV},\\
&M_{\Upsilon''}=10.32\text{ GeV},\\
&\Gamma(\Upsilon''\to e^+e^-)=0.368\text{—}0.208\text{ keV}.
\end{aligned}\tag{25}$$

The first (second) number for the leptonic widths corresponds to the upper (lower) experimental bound. For the QCD parameter α_s and the quark mass m_b we use $\alpha_s=0.15$ and $m_b=4.4$ GeV. Figure 3 shows the nice agreement between perturbative QCD (solid line) and resonance physics (dashed area) for $k=0$. What we found in the Υ case is that Eq. (21) is satisfied for all values of k (we checked for k values up to 30). Figure 4 shows this persistent agreement for $k=20$.

$R_{\rm QCD}(E)$ as given by Eq. (22), contains only perturbative terms. In general there are also nonperturbative

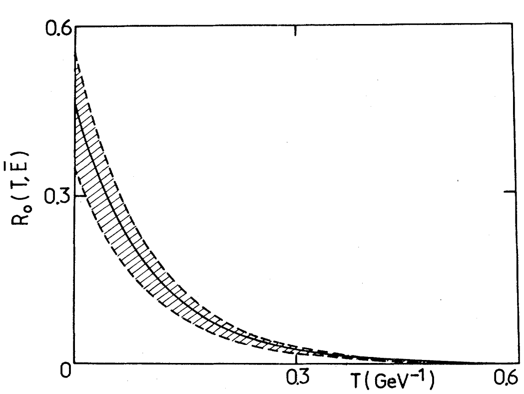

FIG. 3. The sum rule, Eq. (21) for $k=0$, in the case of the Υ system.

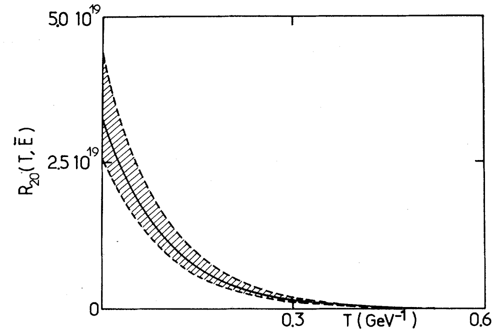

FIG. 4. The sum rule, Eq. (21) for $k=20$, in the case of the Υ system.

terms contributing to $R_{\rm QCD}(E)$ and for heavy quarks the most important is the gluon condensate, analyzed thoroughly in Ref. 6. The gluon-condensate term behaves as $1/m^4$. Therefore, while for the charmonium system the gluon-condensate term is expected to be rather important, for the Υ system is smaller by a factor $(m_b/m_c)^4\simeq 160$.[11] We feel this is the reason why [always using Eq. (22) for $R_{\rm QCD}(E)$], for the Υ system the FESR are satisfied amazingly well, while for the charmonium system the agreement is less impressive.

IV. MONTE CARLO ESTIMATES

Imagine a situation where the masses M_n and the leptonic widths Γ_n of radially excited S states ($Q\bar{Q}$ bound states) are known. We can construct then the following quantity:

$$\begin{aligned}
M(T)&=\frac{\sum_n \Gamma_n M_n e^{-M_nT}}{\sum_n \Gamma_n e^{-M_nT}}\\
&=2m_q+\frac{\sum_n |\psi_n(0)|^2E_ne^{-E_nT}}{\sum_n |\psi_n(0)|^2e^{-E_nT}}\\
&\equiv 2m_q+\langle E\rangle\,,
\end{aligned}\tag{26}$$

where $\langle E\rangle$ is the expectation value of the energy. $\langle E\rangle$ can be viewed also as the logarithmic derivative with respect to time of the nonrelativistic propagator of the quark Q. Following Feynman's recipe,[18] we can consider the expectation value of the energy as a statistical average over all possible trajectories starting and ending at the origin

$$\langle E\rangle=\frac{\int \mathscr{D}r(t)E[r(t)]e^{-S[r(t)]}}{\int \mathscr{D}r(t)e^{-S[r(t)]}}\,,\tag{27}$$

where $S[r(t)]$ is the classical action along a trajectory $r(t)$. Creutz and Freedman have shown[19] that a Monte Carlo evaluation of different quantities in one-dimensional quantum mechanics is feasible, if we use the Metropolis algorithm.[20] The Metropolis algorithm generates paths $r(t)$ with probability $\exp(-S[r(t)])$. Then Eq. (27) is reduced to an arithmetic average

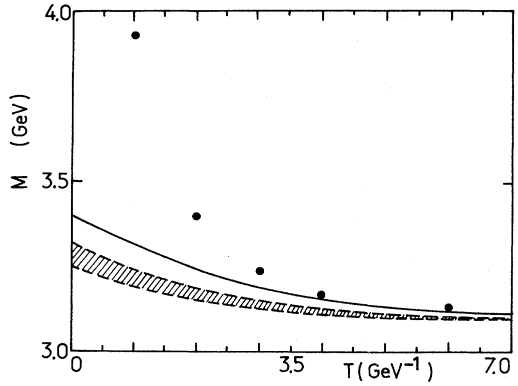

FIG. 5. $M_{tr}(T)$ and $M(T)$ as functions of T in the case of charmonium. Shaded area represents $M_{tr}(T)$ constructed from the experimental parameters of Ψ and Ψ'. Solid curve represents $M_{tr}(T)$ corresponding to a linear potential, while closed circles correspond to the Monte Carlo evaluation of $M(T)$.

$$\langle E \rangle = \frac{1}{N} \sum_{n=1}^{N} E[r_n(t)] , \tag{28}$$

where N is the total number of paths generated in the Monte Carlo sequence.

It is beyond the scope of this paper to generalize the work of Ref. 19 to three-dimensional quantum mechanics. We examine here the simple case of a linear potential $V(r)=kr$. For a linear potential the wave function at the origin is a constant independent of n and therefore $\langle E \rangle$ can be evaluated in one dimension (the radial one). In the actual case we do not have the infinite S states to construct $M(T)$. In the charmonium case we have only Ψ and Ψ'. Using Ψ and Ψ' only we can construct a truncated M, M_{tr}. In the limit $T \to \infty$ $M(T)$ is dominated by the lowest-energy states and we have

$$M(T) \underset{T\to\infty}{\longrightarrow} M_{tr}(T) . \tag{29}$$

For heavier quarks $(b,t,\ldots)$ we would have more resonances below threshold, more states contributing to M_{tr}, and therefore a larger region of overlap between $M(T)$ and $M_{tr}(T)$. In Fig. 5 the shaded area represents $M_{tr}(T)$ constructed from the experimental parameters of Ψ and Ψ'. The charmonium states have been studied phenomenologically using a linear potential in Ref. 21. Using the parameters of Ref. 21 we constructed the theoretical M_{tr} (solid line in Fig. 5) corresponding to the linear potential. The black points in Fig. 5 correspond to the Monte Carlo evaluation of $M(T)$. Notice that as T increases $M(T)$ approaches $M_{tr}(T)$.

V. CONCLUSIONS

In the present work we searched to establish relations between the quark-gluon world of QCD and the actual hadronic world. We examined the idea of duality[1–3] and we found out that it is not simply a conjecture but a true statement: at large energies summation of bound states and parton model give identical results. Our proof is based on the observation that the cross section for $e^+e^- \to$hadrons in potential models is directly related, through a Laplace transform, to the Feynman propagator.

In Sec. III we derived FESR's which provide a direct relation between masses and leptonic widths of heavy resonances on one hand and QCD parameters on the other hand. Our derivation rests upon the essential hypothesis that at large energies perturbative QCD is a valid theory. The amazing success of FESR's in the $b\bar{b}$ case is a strong indication that perturbative QCD (including only α_s corrections) does describe the $b\bar{b}$ system. In the $c\bar{c}$ case our analysis indicates that for a full description of charm systems we have to introduce nonperturbative terms (gluon condensate). It is worthwhile to point out the values we obtained for the quark current masses. If we accept that the quark constituent mass is one half the mass of the vector meson and if we further argue that the mass difference between constituent mass and current mass is around 300 MeV (as it is for u and d quarks), then we end up with $m_c \simeq 1.2$ GeV and $m \simeq 4.4$ GeV. These are precisely the values we have obtained from the FESR.

In Sec. IV we constructed, from available experimental data, a quantity $M_{tr}(T)$ which for large values of T is related to a Green's function. For the simple case of the linear potential we have shown that a Monte Carlo study is feasible. Clearly it is desirable to extend our analysis to three dimensions, so that any interquark potential can be examined and work along these lines is under way.

Recently it has been pointed out[22] that the approach outlined in Ref. 6 involves a variational approximation and consequently the value extracted for the gluon-condensate parameter is not trustworthy. We would like to remark that our derivation of the FESR's does not involve any approximation. Therefore our FESR's supplemented by the gluon-condensate term, can be used in the charmonium case to determine the value of the gluon-condensate. An analysis along these lines has been carried out by Miller and Olsson,[11] using power moments rather than exponential moments, and a reasonable value for the gluon condensate has been obtained.

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