Testing chiral anomalies with hadronic currents

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Chiral anomalies are calculated using an effective-Lagrangian technique introduced for anomalies by Wess and Zumino and recently reformulated by Witten. Anomalous amplitudes for vector currents decaying into three pseudoscalars are tested by comparison with K_{l4} decay, η and $\eta' \to \pi^+\pi^-\gamma$, and strong decays of vector mesons. The agreement with experiment for K_{l4} is an impressive verification of the anomaly in the vector current. For η and η' decay, the results are excellent, and for the strong decays, good. Since the electromagnetic and strong amplitudes have been extrapolated to higher momenta with a final-state-interaction approximation, it is not surprising that the agreement is less good here, where, indeed, further dynamical assumptions are needed. A number of new predictions are made for hadronic decays of ρ', ω', and ϕ'.

I. INTRODUCTION

Even before QCD was established as the correct theory of hadrons, aspects of its low-energy behavior were well known: its chiral symmetry[1] is spontaneously broken, with the appearance of an octet of Goldstone-Nambu particles; current-algebra methods describe low-energy interactions of these particles with each other and with currents that participate in weak and electromagnetic interactions; and these low-energy results can be conveniently realized by effective Lagrangians[2] whose tree diagrams are equivalent to the current-algebra and PCAC (partial conservation of axial-vector current) predictions of chiral symmetry. All effective-Lagrangian realizations give equivalent results at low energy, but among the more convenient are the nonlinear[3] ones, in which the pseudoscalar octet is unaccompanied by scalar partners.

An outstanding problem with these methods was the one solved by Adler, Bell and Jackiw, and Bardeen,[4] concerning "anomalies" of the current algebra. Naive application of current algebra led to the false theorem that π^0 does not decay into two photons in the chiral limit, whereas simple perturbation theory at the quark level yields a nonzero result from the triangle graph, in which photons couple to two vertices and the pion (mediated by the axial-vector current) to the third vertex. At the naive effective-Lagrangian level this meant there was no $\pi^0\gamma\gamma$ vertex when minimal electromagnetic couplings are introduced in the Lagrangian. Adler, Bell and Jackiw, and Bardeen showed how correct treatment of divergences in the current algebra restored the perturbative result of the triangle diagram. Meanwhile, Wess and Zumino[5] showed how the current-algebra anomalies were realized in an effective-Lagrangian approach, but the scale of the anomaly, basically the strength of $\pi^0 \to 2\gamma$, had to be put in by hand.

Recently Witten[6] has reformulated anomalies in the nonlinear-effective-Lagrangian framework. Observing that the naive effective action is too symmetric, forbidding processes allowed by QCD (e.g., $K^+K^- \to \pi^+\pi^-\pi^0$), Witten adds to the action a five-dimensional integral over a volume whose surface is physical four-dimensional space. This is the lowest-dimension operator that avoids the spurious conservation law. Topological arguments require this extra anomalous term to be quantized, its strength proportional to an integer n. This term, when gauged, yields all anomalous couplings, including $\pi^0 \to 2\gamma$. Matching this last amplitude to the classical anomaly then reveals $n = N_c$, where N_c is the number of colors.

Witten's elegant formulation is apparently equivalent in its results to the Wess-Zumino approach and has the merit of simplicity and convenience. In this paper we develop some consequences of Witten's Lagrangian and apply them to several processes involving anomalies. The formalism developed will be useful in a variety of other calculations, including τ decays. The applications presented test the anomalous parts of the vector current in weak, electromagnetic, and strong interactions. The results are generally in good agreement with experiment, and some new predictions are made.

Here we should mention that at times we augment the octet of currents and pseudoscalar mesons with a ninth meson and a ninth current. Our predictions, it must therefore be stressed, have two quite different levels of logical status. Those involving only the octet currents and mesons are bona fide low-energy theorems in QCD. Those including the ninth pseudoscalar and the ninth current are based on stronger symmetry assumptions involving approximate nonet structure of meson amplitudes.

In Sec. II we present useful general results on anomalous parts of currents and interactions in the Witten framework. In Sec. III we test the vector-current anomaly in K_{l4} decay. In Sec. IV we test the anomaly in η, $\eta' \to \pi^+\pi^-\gamma$ decay. In Sec. V we use a field-current identity to relate strong decays of vector mesons into three pseudoscalars to anomalies in the vector current.

Our philosophy is that the chiral nonlinear Lagrangian describes threshold processes, that is amplitudes of vanishing pseudoscalar momentum. When necessary we extrapolate these amplitudes to higher momenta with final-state interactions, dominating two pseudoscalar channels with vector mesons. No new scales or couplings enter the problem, apart from the known vector-meson masses and widths. The low-energy behavior is fully controlled by the effective Lagrangian involving pseudoscalars only.

Our result for K_{l4} decay is a dramatic verification of the anomaly in the vector current. The extrapolations here are slight, with the vector-dominance model and final-state enhancements playing no role. The result for $\eta \rightarrow \pi^+\pi^-\gamma$ decay, however, relies somewhat more heavily on these phenomenological ideas. When we come to the purely hadronic decays, we must regard our results largely as vector-dominance-model phenomenology, not derivable from QCD fundamentals in any clear way, but scaled by low-energy results of chiral anomalies.

Our results for anomalies complement the non-anomalous-current work of Fischer, Wess, and Wagner[7] (FWW) and Fischer, Kluver, and Wagner[8] (FKW). Our paper is in the spirit of FWW, who also introduced final-state interactions in terms of resonances. A different approach, concerning the status of the spin-1 mesons and how their effects are incorporated in the decay amplitudes, is considered by FKW. These authors add spin-1 mesons to the Lagrangian by a local gauge principle, break the local gauge invariance by mass terms, adjust parameters according to some reasonable assumptions, and then calculate amplitudes with spin-1 mesons and pseudoscalars on equal footing. The low-energy behavior is then controlled by the sum of all these effects. While these methods have a rich history in the development of chiral Lagrangians prior to QCD, we prefer the simpler approach, employing only pseudoscalars in the effective Lagrangian, and describing vector mesons (and for that matter any higher-spin and -mass mesons) as final-state strong interactions, outside of the strict chiral limit.

In Sec. VI we comment on relations between F_π, strong-interaction parameters, and the vector-dominance model (VDM).[9] Finally, in Sec. VII, we conclude with a summary of our results.

II. GENERAL RESULTS

The effective Lagrangian for the interaction of the octet π^a of pseudoscalars is given in terms of the matrix

$$U=\exp\left[\frac{2i}{F_\pi}\lambda^a\pi^a\right] . \tag{1}$$

We use the notation

$$\partial_\mu\pi^a=\pi^a_\mu, \quad \langle\ \rangle=\text{trace} ,$$
$$U^L_\mu=\partial_\mu U U^{-1}, \quad U^R_\mu=U^{-1}\partial_\mu U . \tag{2}$$

The effective Lagrangian is associated with an action $T=T_0+T_a$, where

$$T_0=-\frac{F_\pi{}^2}{16}\int d^4x\langle\partial_\mu U\,\partial_\mu U^{-1}\rangle=\frac{F_\pi{}^2}{16}\int d^4x\langle U^R_\mu U^R_\mu\rangle ,$$
$$T_a=\lambda\int_Q d^5x\,\epsilon_{ijklm}\langle U^R_i U^R_j U^R_k U^R_l U^R_m\rangle , \tag{3}$$
$$\lambda=\frac{N_c i}{240\pi^2},\quad N_c=3 .$$

T_0 and T_a behave[6] differently against the transform $U\rightarrow U^{-1}$, characterizing whether a process has an even or odd number of pseudoscalars: $(-1)^{N_B}$. T_0 is a scalar under this symmetry and develops amplitudes involving only an even number (N_B) of pseudoscalars. Witten points out that this is surely not a symmetry of QCD, thus motivating the "anomalous" form T_a, which contributes to processes involving an odd number of pseudoscalars. Here Q is a five-dimensional disk whose boundary is normal four-dimensional space.

This action leads to the equation of motion

$$0=-\partial_\mu\frac{F_\pi{}^2}{8}U^R_\mu+5\lambda\epsilon_{\mu\nu\alpha\beta}U^R_\mu U^R_\nu U^R_\alpha U^R_\beta . \tag{4}$$

With this equation of motion one may show that conserved right- and left-handed currents, generating SU(3)$\times$SU(3), are

$$j^{R,a}_\mu=\frac{iF_\pi{}^2}{8}\left\langle\frac{\lambda^a}{2}U^R_\mu\right\rangle-\frac{1}{16\pi^2}\epsilon_{\mu\nu\alpha\beta}\left\langle\frac{\lambda^a}{2}U^R_\nu U^R_\alpha U^R_\beta\right\rangle ,$$
$$j^{L,a}_\mu=\frac{-iF_\pi{}^2}{8}\left\langle\frac{\lambda^a}{2}U^L_\mu\right\rangle-\frac{1}{16\pi^2}\epsilon_{\mu\nu\alpha\beta}\left\langle\frac{\lambda^a}{2}U^L_\nu U^L_\alpha U^L_\beta\right\rangle . \tag{5}$$

To extract useful information from these currents, we form the vector and axial-vector combinations

$$V_\mu=j^L_\mu+j^R_\mu ,$$
$$A_\mu=j^L_\mu-j^R_\mu ,$$

and expand these in powers of π^a. We designate terms arising from T_0 as "normal" and those arising from T_a as "anomalous." The vector current has normal terms which are even in the number of pseudoscalars and anomalous terms which are odd in the number of pseudoscalars. In contrast, the axial-vector current has odd normal terms and even anomalous terms. (However the axial current has no bilinear term.)

Although we do not use it in what follows, the leading contribution to the axial-vector current A_μ is displayed for normalization purposes:

$$A^a_\mu=\frac{F_\pi}{2}\partial_\mu\pi^a+O(\pi^3) ,$$
$$F_\pi=186\text{ MeV} . \tag{6}$$

The leading contributions to the vector current are

$$V^d_\mu=v^d_\mu+\overline{V}^d_\mu$$
$$=f_{dab}\pi^a\partial_\mu\pi^b$$
$$-\frac{1}{F_\pi{}^3\pi^2}d_{dae}f_{bce}\epsilon_{\mu\nu\alpha\beta}\partial_\nu\pi^a\partial_\alpha\pi^b\partial_\beta\pi^c+O(\pi^4) . \tag{7}$$

The first term v^d_μ is normal and the second term $\overline{V}^d_\mu$ is anomalous. We concentrate on the second term (but will use the first in Sec. V to scale the strong decays). The anomalous three pseudoscalar parts of the vector current are

$$\bar{V}_\mu^{1-i2}=\frac{2i\epsilon_{\mu\nu\alpha\beta}}{\pi^2 F_\pi{}^3}\left[\sqrt{2}\pi_\nu^+(K_\alpha^+K_\beta^- +K_\alpha^0\bar{K}_\beta^0)+\frac{\eta_\nu}{\sqrt{3}}(4K_\alpha^+\bar{K}_\beta^0-\sqrt{2}\pi_\alpha^+\pi_\beta^0)\right],$$

$$\bar{V}_\mu^{3}=\frac{2i\epsilon_{\mu\nu\alpha\beta}}{\pi^2 F_\pi{}^3}\left[\pi_\nu^0(K_\alpha^+K_\beta^- +K_\alpha^0\bar{K}_\beta^0)+\frac{1}{\sqrt{3}}\eta_\nu(\pi_\alpha^+\pi_\beta^- +2K_\alpha^+K_\beta^- +2\bar{K}_\alpha^0K_\beta^0)\right],$$

$$\bar{V}_\mu^{4-i5}=\frac{2i\epsilon_{\mu\nu\alpha\beta}}{\pi^2 F_\pi{}^3}\left[K_\nu^+[\sqrt{2}\pi_\alpha^+\pi_\beta^- +\sqrt{2}\bar{K}_\alpha^0K_\beta^0+(\tfrac{2}{3})^{1/2}\pi_\alpha^0\eta_\beta]+2\left[\pi_\nu^0-\frac{\eta_\nu}{\sqrt{3}}\right]\pi_\alpha^+K_\beta^0\right], \qquad (8)$$

$$\bar{V}_\mu^{6-i7}=\frac{2i\epsilon_{\mu\nu\alpha\beta}}{\pi^2 F_\pi{}^3}\left[K_\nu^0[\sqrt{2}\pi_\alpha^-\pi_\beta^+ +\sqrt{2}K_\alpha^-K_\beta^+ +(\tfrac{2}{3})^{1/2}\eta_\alpha\pi_\beta^0]-2\left[\pi_\nu^0+\frac{\eta_\nu}{\sqrt{3}}\right]\pi_\alpha^-K_\beta^+\right],$$

$$\bar{V}_\mu^{8}=\frac{2i\epsilon_{\mu\nu\alpha\beta}}{\pi^2 F_\pi{}^3}[\sqrt{3}\pi_\nu^+\pi_\alpha^-\pi_\beta^0+\eta_\nu(K_\alpha^-K_\beta^+ +\bar{K}_\alpha^0K_\beta^0)]\,.$$

The anomalous three-pseudoscalar piece of the electromagnetic current is

$$j_\mu^{\rm EM}=\frac{2i}{\pi^2 F_\pi{}^3}\epsilon_{\mu\nu\alpha\beta}\left[(K_\nu^+K_\alpha^- +\pi_\nu^+\pi_\alpha^-)\left[\pi_\beta^0+\frac{\eta_\beta}{\sqrt{3}}\right]+K_\nu^0\bar{K}_\alpha^0(\pi_\beta^0-\sqrt{3}\eta_\beta)\right]. \qquad (9)$$

It is noteworthy that the following transitions are forbidden:

$$\bar{V}^3,\bar{V}^8,J^{\rm EM}\to\pi^+K^0K^-,\pi^-\bar{K}^0K^+\,,$$
$$\bar{V}^{1+i2}\to\pi^0K^-K^0\,. \qquad (10)$$

In these transitions the $K\bar{K}$ system has isotopic spin $I=1$. The $(K\bar{K})_{I=1}$ system has a G parity opposite to the $K\bar{K}$ exchange symmetry. Because of the antisymmetric tensor structure of the anomalous current, the $K\bar{K}$ system must thus have positive G parity. But then these $\pi(K\bar{K})_{I=1}$ combinations have negative G parity, and cannot couple to the positive-G-parity currents. Another way of saying this is that the anomaly terms do not develop second-class currents. For the same reason $\bar{V}^3$ does not couple to three pions.

It is sometimes useful to extend these currents and particle states to a nonet structure. The augmented anomalous terms are

$$\bar{V}_\mu^{0}=3(\tfrac{2}{3})^{1/2}\frac{i\epsilon_{\mu\nu\alpha\beta}}{\pi^2 F_\pi{}^3}[2\pi_\nu^+\pi_\alpha^-\pi_\beta^0+\sqrt{2}(\pi_\nu^-K_\alpha^+\bar{K}_\beta^0+\pi_\nu^+K_\alpha^0K_\beta^-)+\pi_\nu^0(K_\alpha^+K_\beta^- -K_\alpha^0\bar{K}_\beta^0)+\sqrt{3}\eta_\nu(K_\alpha^+K_\beta^- +K_\alpha^0\bar{K}_\beta^0)]\,,$$

$$\bar{V}_\mu^{1-i2}=(\tfrac{2}{3})^{1/2}\frac{i\epsilon_{\mu\nu\alpha\beta}}{\pi^2 F_\pi{}^3}\eta'_\nu(2\sqrt{2}\pi_\alpha^0\pi_\beta^+ +2K_\alpha^+\bar{K}_\beta^0)\,,$$

$$\bar{V}_\mu^{3}=(\tfrac{2}{3})^{1/2}\frac{i\epsilon_{\mu\nu\alpha\beta}}{\pi^2 F_\pi{}^3}\eta'_\nu(2\pi_\alpha^+\pi_\beta^- +K_\alpha^+K_\beta^- -K_\alpha^0\bar{K}_\beta^0)\,,$$

$$\bar{V}_\mu^{4-i5}=(\tfrac{2}{3})^{1/2}\frac{i\epsilon_{\mu\nu\alpha\beta}}{\pi^2 F_\pi{}^3}\eta'_\nu[2\pi_\alpha^+K_\beta^0+\sqrt{2}\pi_\alpha^0K_\beta^+ +2(\tfrac{3}{2})^{1/2}\eta_\alpha K_\beta^+]\,, \qquad (11)$$

$$\bar{V}_\mu^{6-i7}=(\tfrac{2}{3})^{1/2}\frac{i\epsilon_{\mu\nu\alpha\beta}}{\pi^2 F_\pi{}^3}\eta'_\nu[2\pi_\alpha^-K_\beta^+ -\sqrt{2}(\pi_\alpha^0-\sqrt{3}\eta_\alpha)K_\beta^0]\,,$$

$$\bar{V}_\mu^{8}=(\tfrac{2}{3})^{1/2}\frac{i\epsilon_{\mu\nu\alpha\beta}}{\pi^2 F_\pi{}^3}\eta'_\nu(\sqrt{3}K_\alpha^+K_\beta^- +\sqrt{3}K_\alpha^0\bar{K}_\beta^0)\,.$$

Here η' is the SU(3) singlet.

We shall also have occasion to use the five-pseudoscalar amplitude that develops from T_a. The anomalous part of the Lagrangian density to leading order in π is

$$L_a = \frac{2}{5\pi^2 F_\pi^{\,5}} \tfrac{1}{8} \langle \lambda_a[\lambda_b,\{\lambda_c,[\lambda_d,\lambda_e]\}]\rangle \epsilon_{\mu\nu\alpha\beta}\pi^a\pi^b_\mu\pi^c_\nu\pi^d_\alpha\pi^e_\beta \,, \qquad (12)$$

$$= \frac{4}{5F_\pi^{\,2}} v^g_\mu \bar{v}^g_\mu \,.$$

With the notation

$$[\pi^a,\pi^b_\mu] = (\pi^a\pi^b_\mu - \pi^b\pi^a_\mu) \,, \qquad (13)$$

we find

$$v^{1-i2}_\mu = i(\sqrt{2}[\pi^+,\pi^0_\mu] - [K^+,\bar{K}^0_\mu]) \,,$$

$$v^3_\mu = i([\pi^-,\pi^+_\mu] + \tfrac{1}{2}[K^-,K^+_\mu] + \tfrac{1}{2}[K^0,\bar{K}^0_\mu]) \,,$$

$$v^{4-i5}_\mu = i\left[[K^0,\pi^+_\mu] + \frac{1}{\sqrt{2}}[K^+,\pi^0_\mu] + \frac{\sqrt{3}}{2}[K^+,\eta_\mu] \right] , \qquad (14)$$

$$v^{6-i7}_\mu = i\left[-[\pi^-,K^+_\mu] - \frac{1}{\sqrt{2}}[K^0,\pi^0_\mu] + \frac{\sqrt{3}}{2}[K^0,\eta_\nu] \right] ,$$

$$v^8_\mu = i\left[\frac{\sqrt{3}}{2}[K^-,K^+_\mu] + \frac{\sqrt{3}}{2}[\bar{K}^0,K^0_\mu] \right] .$$

We may use L_a to calculate the $K^+K^- \to \pi^+\pi^-\pi^0$ or $\eta\pi^+ \to \pi^-\pi^+\pi^-$ scattering amplitudes, with the result

$$T_a(K^+K^- \to \pi^+\pi^-\pi^0) = \frac{24}{\pi^2 F_\pi^{\,5}} \epsilon_{\mu\nu\alpha\beta} P_{1\mu} P_{2\mu} P^+_\alpha P^-_\beta \,, \qquad (15)$$

where $\pm$ refers to $K^\pm$ momenta and 1,2 to π^+,π^- momenta, and

$$T_a(\pi_1^+\pi_2^- \to \eta\pi_3^+\pi_4^-)$$

$$= \frac{16}{5\sqrt{3}} \frac{1}{\pi^2 F_\pi^{\,5}} \epsilon_{\mu\nu\alpha\beta} P^{(1)}_\mu P^{(2)}_\nu P^{(3)}_\alpha P^{(4)}_\beta \,. \qquad (16)$$

These amplitudes may be useful in interpreting multipion production in peripheral reactions, e.g., $K^+p \to K^+\pi^+\pi^-p$ with π^0 exchange.

III. K_{l4} DECAY

A direct test of the strangeness-changing vector-current anomaly is afforded by the decay $K^+ \to e^+\pi^+\pi^-\nu$, measuring the matrix element

$$\langle \pi^+\pi^- \mid V^{4+i5}_\mu \mid K^+ \rangle \equiv \frac{2H}{m_K^{\,3}} \epsilon_{\mu\nu\alpha\beta} P^\nu_K P^\alpha_{\pi^-} P^\beta_{\pi^+} \,. \qquad (17)$$

Using the anomalous current [Eq. (8)], evaluated in the soft-momentum limit, we find

$$\frac{2H}{m_K^{\,3}} = \frac{-2\sqrt{2}}{\pi^2 F_\pi^{\,3}} \,,$$

$$H = -2.68 \,. \qquad (18)$$

This is in excellent agreement with the recent data of Rosselet *et al.*,[10] who report

$$H = -2.68 \pm 0.68 \,.$$

These authors analyze the experimental data and extrapolate to zero-momentum transfer, incorporating the K^* form factor and final-state interactions in their analysis.

There is another approach to K_{l4} decay in which the chiral Lagrangian scale (F_π) is related to strong-interaction parameters. Using the VDM via the field-current identity

$$V^a_\mu = \frac{m_V^{\,2}}{g_V} \rho^a_\mu \,, \qquad (19)$$

we may relate the matrix element in Eq. (17) to the matrix element of the source current for the corresponding vector meson, at zero-momentum transfer,

$$\langle \pi\pi \mid V^{4+i5} \mid K^+ \rangle = \frac{m_{K^*}^{\;2}}{g_{K^*}} \langle \pi^+\pi^- \mid K^{*4+i5}_\mu \mid K^+ \rangle$$

$$= \frac{1}{g_{K^*}} \langle \pi^+\pi^- \mid j^{4+i5}_\mu(K^*) \mid K^+ \rangle \,. \qquad (20)$$

The last matrix element is measured by the strong decay $K^{*+} \to K^+\pi^+\pi^-$. Following the method of Gell-Mann, Sharp, and Wagner[11] (GSW), we compute this rate by a vector-meson-dominated isobar model, shown in Fig. 1. The result is

$$\langle \pi^+\pi^- \mid V^{4+i5}_\mu \mid K^+ \rangle$$

$$= \frac{-\sqrt{2}}{g_{K^*}} \left[\frac{2g_{\rho\pi\pi}g_{K^*\rho K}}{m_\rho^{\,2} - S_{\pi\pi}} + \frac{2g_{K^*K^*\pi}g_{K^*K\pi}}{m_{K^*}^{\;2} - S_{K\pi}} \right]$$

$$\times \epsilon_{\mu\nu\alpha\beta} P^\nu_K P^\alpha_{\pi^-} P^\beta_{\pi^-} \,. \qquad (21)$$

Let us now evaluate this estimate of the matrix element at soft π and K momenta, and compare it with the prediction of the effective Lagrangian [Eqs. (17) and (18)]. We assume SU(3) coupling-constant symmetry for g_{VVP} and g_{VPP}, and $g_{K^*} = g_\rho = g_{\rho\pi\pi}$. Then we find

$$\langle \pi^+\pi^- \mid V^{4+i5} \mid K^+ \rangle$$

$$= -\sqrt{2} g_{\omega\rho\pi} \left[\frac{1}{m_\rho^{\,2}} + \frac{1}{m_{K^*}^{\;2}} \right] \epsilon_{\mu\nu\alpha\beta} P^\nu_K P^\alpha_{\pi^-} P^\beta_{\pi^+} \,. \qquad (22)$$

Setting this result equal to Eq. (21), we find

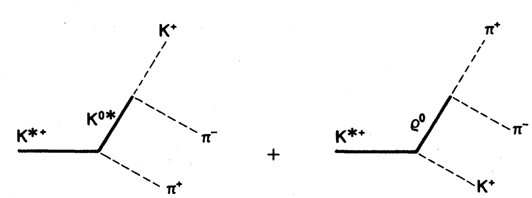

FIG. 1. Pole-enhancement diagrams for K_{l4} decay.

$$g_{\omega\rho\pi}=\frac{m_\rho^2}{\pi^2F_\pi^3}\left[\frac{1}{2}\left[1+\frac{m_\rho^2}{m_{K^*}^2}\right]\right]^{-1}$$
$$=10.7\ \mathrm{GeV}^{-1}\ . \tag{23}$$

The GSW-model result for $g_{\rho\omega\pi}$, obtained from $\omega\to3\pi$ decay, is 14.1 GeV^{-1}. Evidently this model does rather well in predicting the K_{l4} form factor at low momentum, but the effective Lagrangian does better. In the SU(3)-symmetric limit Eq. (23) yields

$$g_{\omega\rho\pi}=\frac{m_\rho^2}{\pi^2F_\pi^3}\ , \tag{24}$$

a result to be discussed in Secs. IV and VI.

We shall often use the VDM below to extrapolate the soft amplitude of the effective Lagrangian to higher momenta. In the language of final-state interaction, the final $K\pi$ and $\pi\pi$ channels resonate, resulting in an enhancement factor

$$F=\left|\frac{1}{2}\left[\frac{m_\rho^2}{m_\rho^2-S_{\pi^+\pi^-}}+\frac{m_{K^*}^2}{m_{K^*}^2-S_{K^+\pi^-}}\right]\right|^2\ , \tag{25}$$

multiplying the soft-pseudoscalar rate predicted by Eqs. (17) and (18). The factor is unity at the threshold $S_{\pi^+\pi^-}=S_{K^+\pi^-}=0$.

IV. ELECTROMAGNETIC DECAYS

The classic two-photon anomalies (e.g., $\pi^0\to2\gamma$, $\eta\to2\gamma$) are readily reproduced by the effective Lagrangian, as shown by Wess and Zumino,[5] and by Witten.[6] Here we concentrate on the single-photon, three-pseudoscalar processes

$$\eta\to\pi^+\pi^-\gamma,\ \ \eta'\to\pi^+\pi^-\gamma \tag{26}$$

with anomalous terms

$$L=eA_\mu j_\mu^{\mathrm{EM}}\ ,$$
$$=\frac{2i}{\sqrt{3}}\frac{e}{\pi^2F_\pi^3}\epsilon_{\mu\nu\alpha\beta}(\partial_\nu\eta+\sqrt{2}\partial_\nu\eta')\partial_\alpha\pi^-\partial_\beta\pi^+A_\mu\ ,$$
$$=i\frac{\epsilon_{\mu\nu\alpha\beta}}{\pi^2F_\pi^3}(G_\eta\partial_\nu\eta+G_{\eta'}\partial_\nu\eta')\partial_\alpha\pi^-\partial_\beta\pi^+A_\mu\ , \tag{27}$$

where we have used the currents in Eqs. (8) and (11).

If M is the decaying pseudoscalar mass, we have the partial width

$$\Gamma=\frac{G^2}{(2\pi)^3}\left[\frac{M}{2}\right]^7 I\ . \tag{28}$$

With the masses of the particles in the decay defined as $M\to m_1+m_2+m_1$, I is the dimensionless integral

$$I=\frac{1}{M^4}\int_{(m_1+m_2)^2}^{(M-m_1)^2}ds\int_{(m_1+m_2)^2}^{(M-m_1)^2}dt\ \Theta(\Delta)\Delta F\ . \tag{29}$$

Here Δ is the triangle function ($u=s_{\pi\pi}$),

$$\Delta=\frac{stu}{M^6}\left[\frac{m_1}{M}\right]^2\left[\frac{M^2-m_2^2}{M^2}\right]^2$$
$$+\frac{u}{M^2}\left[\left[\frac{m_1}{M}\right]^2-\left[\frac{m_2}{M}\right]^2\right]\left[\frac{M^2-m_1^2}{M^2}\right] \tag{30}$$

and F is a $\pi^+\pi^-$ final-state interaction factor, taking into account the ρ pole

$$F=\left|\frac{m_\rho^2-im_\rho\Gamma_\rho(u)}{m_\rho^2-im_\rho\Gamma_\rho(u)-u}\right|^2\ . \tag{31}$$

At $u=0$ this factor is unity so that the soft amplitude is given by Eq. (27). The widths $\Gamma_\rho(u)$ are taken to show the threshold behavior

$$\Gamma_\rho(u)=\left[\frac{p_{\pi\pi}}{\bar{p}_{\pi\pi}}\right]^3\theta(p_{\pi\pi}^2)\Gamma_\rho\ . \tag{32}$$

Here $\bar{p}_{\pi\pi}$ is the $\pi\pi$ momentum in on-shell ρ decay.

In Eqs. (28)—(32) we have taken $m_\pi^2\neq0$, allowing explicit SU(3)$\times$SU(3) breaking in phase-space and final-state interaction. We are thus making the canonical assumption that the amplitude G, defined in Eq. (27) for massless pions, extrapolates smoothly to $m_\pi\neq0$.

We could now, in principle, test L at low-pion momenta ($p_\pi=0$) in the Dalitz plot, dispensing with the need for extrapolating the amplitude to energetic pions. Here we normalize the amplitude by requiring it to match the prediction of L at low-pion momenta, and then calculate the width using the ρ pole to describe the energy dependence of the amplitude. The results are

$$\Gamma(\eta\to\pi^+\pi^-\gamma)=36\ \mathrm{eV}\ \ (\text{experiment:}^{12}\ 41\pm1\ \mathrm{eV})\ ,$$
$$\Gamma(\eta'\to\pi^+\pi^-\gamma)=62\ \mathrm{keV}\ \ (\text{experiment:}^{12}\ 84\pm5\ \mathrm{keV})\ . \tag{33}$$

If an -11^0 singlet-octet mixing[12] is allowed the results are changed to

$$\Gamma(\eta\to\pi^+\pi^-\gamma)=(\cos\theta-\sqrt{2}\sin\theta)^2\Gamma_\eta(\theta=0)$$
$$=45\ \mathrm{eV}\ ,$$
$$\Gamma(\eta'\to\pi^+\pi^-\gamma)=[\cos\theta+(1/\sqrt{2})\sin\theta]^2\Gamma_{\eta'}(\theta=0)$$
$$=52\ \mathrm{eV}\ . \tag{34}$$

Considering the extrapolations involved, especially in the η' amplitude, and the implicit assumption[13] $F_\pi=F_\eta=F_{\eta'}$, the agreement with experiment is good.

We close this section on electromagnetic anomaly tests by deriving a relation which is of some interest in the following sections. The $\pi^0\to2\gamma$ anomaly *coupling* predicted by the effective Lagrangian is

$$g_{\pi^0\gamma\gamma}=\frac{e^2}{4\pi^2F_\pi}\ . \tag{35}$$

Let us compare this with the VDM prediction. With

$$j_\mu^{\rm EM}=\frac{m_v{}^2 e}{g_\rho}\left[\rho_\mu^0+\tfrac{1}{3}\omega_\mu-\frac{\sqrt{2}}{3}\phi_\mu\right], \tag{36}$$

we have

$$g_{\pi^0\gamma\gamma}=\frac{e^2}{3g_\rho{}^2}g_{\rho\pi\omega}. \tag{37}$$

However, in the previous section we found

$$g_{\omega\rho\pi}=\frac{m_\rho{}^2}{\pi^2F_\pi{}^3}.$$

Combining the two results, we find

$$F_\pi{}^2g_\rho{}^2=\tfrac{4}{3}m_\rho{}^2. \tag{38}$$

We shall return to this relation in Secs. V and VI.

V. STRONG DECAYS OF VECTOR MESONS

We consider the following strong decays:

$$\begin{aligned}&\rho\to 2\pi\ \text{(normalization)},\\&\omega\to 3\pi,\ \phi\to 3\pi,\ K^*\to K\pi\pi,\\&\rho'\to 2\pi\ \text{(normalization)},\\&\rho'\to K\bar{K}\pi,\ \rho'\to\pi\pi\eta,\\&\omega'\to K\bar{K}\pi,\ \phi'\to K\bar{K}\pi.\end{aligned} \tag{39}$$

Here we follow the methods outlined in Eqs. (19) and (20). The two pseudoscalars couple normally and the three pseudoscalars couple anomalously to the vector current.

A. $\omega\to 3\pi$, $K^*\to K\pi\pi$

In the spirit of the VDM, we generally put $g_\rho=g_{\rho\pi\pi}$: at zero-momentum transfer

$$\begin{aligned}&\langle\pi^+\pi^-\,|\,V_\mu^3\,|\,0\rangle=g_\rho{}^{-1}\langle\pi^+\pi^-\,|\,j_\mu(\rho^0)\,|\,0\rangle,\\&p_\mu^+-p_\mu^-=\frac{g_{\rho\pi\pi}}{g_\rho}(p_\mu^+-p_\mu^-),\\&g_\rho=g_{\rho\pi\pi}.\end{aligned} \tag{40}$$

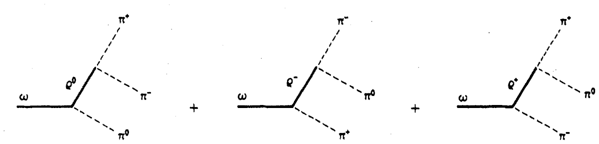

FIG. 2. Pole-enhancement diagrams for ω decay into pions.

Thus we find

$$\langle\pi^+\pi^-\pi^0\,|\,V_\mu^8\,|\,0\rangle=g_{\rho\pi\pi}{}^{-1}\langle\pi^+\pi^-\pi^0\,|\,j_\mu(\omega_8)\,|\,0\rangle, \tag{41}$$

where $j_\mu(\omega_8)$ is the source current for ω_8, whose matrix element is measured by the strong decay $\omega_8\to 3\pi$. We thus relate this strong decay to the chiral anomaly,

$$\begin{aligned}\langle\pi^+\pi^-\pi^0\,|\,j_\mu(\omega_8)\,|\,0\rangle&=g_{\rho\pi\pi}\langle\pi^+\pi^-\pi^0\,|\,V_\mu{}^8\,|\,0\rangle\\&=g_{\rho\pi\pi}\frac{6}{\sqrt{3}}\epsilon_{\mu\nu\alpha\beta}p_{\pi^+}^\nu p_{\pi^-}^\alpha p_{\pi^0}^\beta\\&\equiv G_{\omega_8\pi^+\pi^-\pi^0}\epsilon_{\mu\nu\alpha\beta}p_{\pi^+}^\nu p_{\pi^-}^\alpha p_{\pi^0}^\beta.\end{aligned} \tag{42}$$

Here we have used Eq. (8). Using the results of Eq. (11), we can make a similar determination of $G_{\omega_0\pi^+\pi^-\pi^0}$. For ideally mixed ω,ϕ, we find

$$\begin{aligned}&G_{\omega\pi^+\pi^-\pi^0}=\frac{1}{\sqrt{3}}G_{\omega_8\pi^+\pi^-\pi^0}+\frac{\sqrt{2}}{\sqrt{3}}G_{\omega_0\pi^+\pi^-\pi^0}=\frac{6g_{\rho\pi\pi}}{\pi^2F_\pi{}^3},\\&G_{\phi\pi^+\pi^-\pi^0}=0.\end{aligned} \tag{43}$$

In terms of this coupling G, the partial width for a vector meson decaying to three pseudoscalars is

$$\Gamma=\frac{1}{3}\frac{G^2}{(2\pi)^3}\left[\frac{M}{2}\right]^7 I, \tag{44}$$

where I is defined in Sec. IV. Here I is the integrated matrix element enhanced by ρ poles in all $\pi\pi$ channels, as shown in Fig. 2, yielding the enhancement factor

$$F=\frac{1}{9}\left|\frac{m_\rho{}^2-im_\rho\Gamma_\rho(S_{+-})}{m_\rho{}^2-im_\rho\Gamma_\rho(S_{+-})-S_{+-}}+\frac{m_\rho{}^2-im_\rho\Gamma_\rho(S_{-0})}{m_\rho{}^2-im_\rho\Gamma_\rho(S_{-0})-S_{-0}}+\frac{m_\rho{}^2-im_\rho\Gamma_\rho(S_{+0})}{m_\rho{}^2-im_\rho\Gamma_\rho(S_{+0})-S_{+0}}\right|^2. \tag{45}$$

The final result is

$$\Gamma(\omega\to 3\pi)=3.3\ \text{MeV}. \tag{46}$$

With 3.30^0 angular departure[12] from ideal mixing, the result is changed to

$$\Gamma(\omega\to 3\pi)=3.3\ \text{MeV},\quad \Gamma(\phi\to 3\pi)=0.33\ \text{MeV}. \tag{47}$$

The experimental partial widths are[12] $\Gamma(\omega\to 3\pi)=8.9$ MeV and $\Gamma(\phi\to 3\pi)=0.60$ MeV.

We can compare with the Gell-Mann, Sharp, and Wagner (GSW) theory, as follows. It yields, for the diagrams of Fig. 2, using the notation of Eqs. (42) and (43),

$$G_{\omega\pi^+\pi^-\pi^0}=2g_{\omega\rho\pi}g_{\rho\pi\pi}\left[\frac{1}{m_\rho{}^2-S_{+-}}+\frac{1}{m_\rho{}^2-S_{-0}}+\frac{1}{m_\rho{}^2-S_{+0}}\right]. \tag{48}$$

This amplitude at soft momenta ($S_{+-}=S_{-0}=S_{+0}=0$) matches the Lagrangian prediction, Eq. (43), if

$$g_{\omega\rho\pi}=\frac{m_\rho{}^2}{\pi^2F_\pi{}^3}=9.3\ \text{GeV}^{-1}. \tag{49}$$

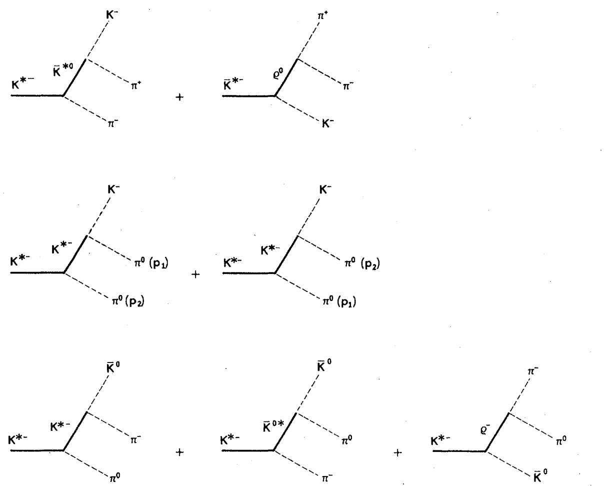

FIG. 3. Pole-enhancement diagrams for K^{*-} decay into three pseudoscalar mesons.

This should be compared with the updated GSW result, $g_{\omega\rho\pi}=14.1\ \text{GeV}^{-1}$. Assuming the extrapolation to low momenta is correctly described by the GSW model, the effective Lagrangian prediction is low by 35% in the amplitude.

A similar treatment of $K^{*-}\to K^-\pi^+\pi^-$, $K^-\pi^0\pi^0$, and $K^0\pi^-\pi^0$ is possible using the final-state enhancements of Fig. 3. Here some remarks are in order which will also be pertinent in ρ' decays (see below). The effective-Lagrangian prediction for $K^{*-}\to K^-\pi^0\pi^0$ [via the coupling of the strangeness-changing current to $K^-\pi^0\pi^0$, Eq. (8)] is zero because the identical pions need to be in an overall symmetric state, whereas the tensor structure of the anomalous-current coupling is antisymmetric, there being no further dynamical dependence on kinematic variables other than that of the currents, Eq. (8). (Here we assume the number of derivatives dictated by the anomalous current.) However, in the VDM process of Fig. 3, further dynamical dependence is present. Here the amplitude also has antisymmetric tensor structure in the pion momenta, but there is an enhancement factor that is antisymmetric in $S_{K\pi_1}$ and $S_{K\pi_2}$, hence the amplitude is overall symmetric in exchange of the two symmetric pions. In this case we use the antisymmetric enhancement factor and scale the amplitude so that pole residues match the amplitude of the other charge state, in which there is a rigorous chiral symmetry prediction. Note in any case that this amplitude is very small. The results of this procedure are

$$\begin{aligned}&\Gamma(K^{*-}\to K^-\pi^+\pi^-)=15\ \text{keV}\ ,\\ &\Gamma(K^{*-}\to K^-\pi^0\pi^0)=0.08\ \text{keV}\ ,\\ &\Gamma(K^{*-}\to \bar{K}^0\pi^-\pi^0)=31\ \text{keV}\ ,\\ &\Gamma(K^{*}\to K\pi\pi)=47\ \text{keV}\ .\end{aligned} \tag{50}$$

These are to be compared with the experimental limit[12]

$$\Gamma(K^*\to K\pi\pi)<35\ \text{keV (95\% C.L.)}\ .$$

If SU(3)-breaking effects[13] are included in Eq. (50), i.e., $F_K/F_\pi\approx 1.2$, then our prediction for $K^*\to K\pi\pi$ is roughly at its experimental upper limit.

B. ρ',ϕ',ω' decays

We treat these decays similarly to $\omega\to 3\pi$, but scaling here to $\rho'\to 2\pi$ rather than $\rho\to\pi\pi$. Thus, the absolute predictions depend on the poorly known coupling $g_{\rho'\pi\pi}$. The final-state enhancement factors correspond to the diagrams of Figs. 4 and 5.

As earlier, we match low-energy behavior to the effective-Lagrangian prediction. The final-state interaction factors are (here we suppress width factors):

$$\begin{aligned}
&F(\rho'\to K^+\pi^0K^-\ \text{or}\ K^0\pi^0\bar{K}^0)\\
&\qquad=\frac{1}{4}\left|\frac{m_{K^*}^2}{m_{K^*}^2-S_{K\pi}}+\frac{m_{K^*}^2}{m_{K^*}^2-S_{\bar{K}\pi}}\right|^2,\\
&F(\omega'\to\pi K\bar{K})\\
&\qquad=\frac{1}{16}\left|\frac{2m_\rho^2}{m_\rho^2-S_{K\bar{K}}}+\frac{m_{K^*}^2}{m_{K^*}^2-S_{K\pi}}+\frac{m_{K^*}^2}{m_{K^*}^2-S_{\bar{K}\pi}}\right|^2,\\
&F(\phi'\to K\bar{K}\pi)=\frac{1}{4}\left|\frac{m_{K^*}^2}{m_{K^*}^2-S_{K\pi}}+\frac{m_{K^*}^2}{m_{K^*}^2-S_{\bar{K}\pi}}\right|^2,\\
&F(\rho'\to\pi\pi\eta)=\left|\frac{m_\rho^2}{m_\rho^2-S_{\pi\pi}}\right|^2,\\
&F(\rho'\to K^0\pi^+K^-\ \text{or}\ \bar{K}^0\pi^-K^+)\\
&\qquad=\frac{1}{4}\left|\frac{m_{K^*}^2}{m_{K^*}^2-S_{\bar{K}\pi}}-\frac{m_{K^*}^2}{m_{K^*}^2-S_{K\pi}}\right|^2.
\end{aligned} \tag{51}$$

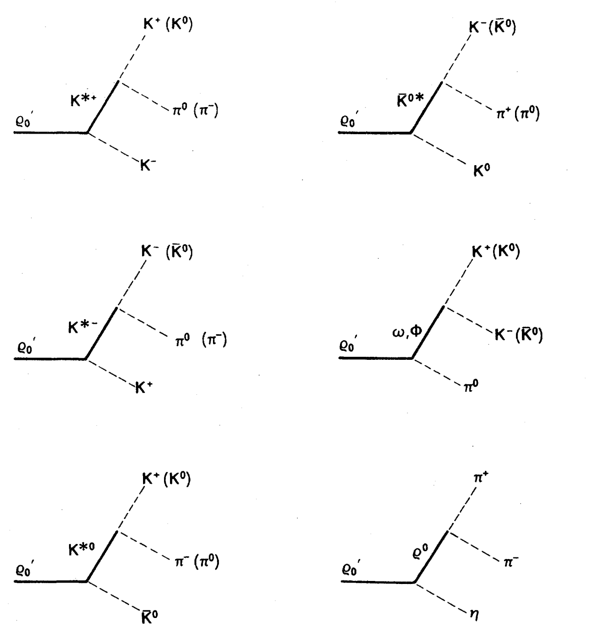

FIG. 4. Pole-enhancement diagrams for ρ_0' decay into three pseudoscalar mesons.

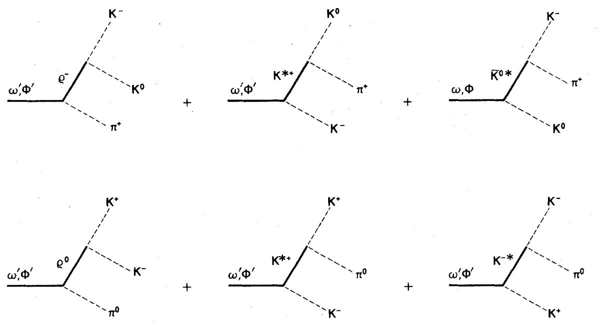

FIG. 5. Pole-enhancement diagrams for ω' and ϕ' decay into three pseudoscalar mesons.

The first four factors are unity at soft-pseudoscalar thresholds. The last factor is necessarily zero there because G parity and the tensor structure require it to be antisymmetric in the $K\bar{K}$ exchange. As before, we normalize this amplitude so that pole residues match other charge states in corresponding channels.

The ρ' decays are predicted to be

$$\Gamma(\rho'\to K^+K^-\pi^0)=\Gamma(\rho'\to K^0\bar{K}^0\pi)=\frac{g_{\rho'\pi\pi}^{\;2}}{4\pi}12.7\text{ MeV},$$

$$\Gamma(\rho'\to K^-K^0\pi^+)=\Gamma(\rho'\to K^+\bar{K}^0\pi^-)=\frac{g_{\rho'\pi\pi}^{\;2}}{4\pi}62.7\text{ MeV},$$

$$\Gamma(\rho'\to K\bar{K}\pi)=\frac{g_{\rho'\pi\pi}^{\;2}}{4\pi}150\text{ MeV}, \tag{52}$$

$$\Gamma(\rho'\to\pi^+\pi^-\eta)=\frac{g_{\rho'\pi\pi}^{\;2}}{4\pi}19\text{ MeV},$$

$$\Gamma(\rho'\to\pi^+\pi^-)=\frac{g_{\rho'\pi\pi}^{\;2}}{4\pi}127\text{ MeV}.$$

We thus have the relative rates

$$\frac{\Gamma(\rho_0'\to K\bar{K}\pi)}{\Gamma(\rho_0'\to\pi^+\pi^-)}=1.18,$$

$$\frac{\Gamma(\rho_0'\to\pi^+\pi^-\eta)}{\Gamma(\rho_0'\to\pi^+\pi^-)}=0.15, \tag{53}$$

$$\frac{\Gamma(\rho_0'\to\pi^+\pi^-\eta)}{\Gamma(\rho_0'\to K\bar{K}\pi)}=0.13.$$

Experimental results for ρ' branching ratios are sketchy.[12] However, we do predict much less $\eta\pi\pi$ than is currently reported.

The ideally mixed ω' decays are predicted to be

$$\Gamma(\omega'\to 3\pi)=\frac{g_{\rho'\pi\pi}^{\;2}}{4\pi}926\text{ MeV},$$

$$\Gamma(\omega'\to K^+K^-\pi^0)=\Gamma(\omega'\to K^0\bar{K}^0\pi)=\frac{g_{\rho'\pi\pi}^{\;2}}{4\pi}21\text{ MeV},$$

$$\Gamma(\omega'\to\pi^+K^-K^0)=\Gamma(\omega'\to\pi^-K^+\bar{K}^0)=\frac{g_{\rho'\pi\pi}^{\;2}}{4\pi}42\text{ MeV}, \tag{54}$$

$$\Gamma(\omega'\to K\bar{K}\pi)=\frac{g_{\rho'\pi\pi}^{\;2}}{4\pi}125\text{ MeV},$$

$$\frac{\Gamma(\omega'\to K\bar{K}\pi)}{\Gamma(\rho'\to K\bar{K}\pi)}=0.83,$$

$$\frac{\Gamma(\omega'\to K\bar{K}\pi)}{\Gamma(\omega'\to 3\pi)}=0.13.$$

The ideally mixed ϕ' decays are predicted to be

$$\Gamma(\phi'\to\pi^0K^+K^-)=\Gamma(\phi'\to\pi^0K^0\bar{K}^0)=\frac{g_{\rho'\pi\pi}^{\;2}}{4\pi}43\text{ MeV},$$

$$\Gamma(\phi'\to\pi^+K^-K^0)=\Gamma(\phi'\to\pi^-K^+K^0)=\frac{g_{\rho'\pi\pi}^{\;2}}{4\pi}86\text{ MeV}, \tag{55}$$

$$\Gamma(\phi'\to K\bar{K}\pi)=\frac{g_{\rho'\pi\pi}^{\;2}}{4\pi}260\text{ MeV},$$

$$\frac{\Gamma(\phi'\to K\bar{K}\pi)}{\Gamma(\rho'\to K\bar{K}\pi)}=1.7.$$

We can attempt absolute limits by placing a bound on $g_{\rho'\pi\pi}$:

$$\Gamma(\omega'\to 3\pi)+\Gamma(\omega'\to\pi KK)<\Gamma_\omega=166\text{ MeV},$$
$$\frac{g_{\rho'\pi\pi}^{\;2}}{4\pi}<0.16. \tag{56}$$

Thus we expect

$$\Gamma(\rho'\to K\bar{K}\pi)<24\text{ MeV},$$
$$\Gamma(\rho'\to\pi\pi\eta)<0.03\text{ MeV},$$
$$\Gamma(\omega'\to 2\pi)<20\text{ MeV},$$
$$\Gamma(\omega'\to 3\pi)<150\text{ MeV}, \tag{57}$$
$$\Gamma(\omega'\to K\bar{K}\pi)<20\text{ MeV},$$
$$\Gamma(\phi'\to K\bar{K}\pi)<41\text{ MeV}.$$

The experimental status on ω' and ϕ' branching ratios is sketchy, with some disagreements between experiments. Our results suggest the ω' should be seen strongly in photoproduction of pions $\gamma p\to\pi^+\pi^-\pi^0X$, consistent with reports of the Omega Photon Collaboration.[14] However, they do not see a $K\bar{K}\pi$ signal, in conflict with the DCI e^+e^- experiment data,[15] which indicates a strong ϕ signal in this final state. The analyses are in early stages with limited statistics. We hope our amplitudes, which indicate interesting zeros in $\rho'\to K^0\pi^+K^-$ and $K^0\pi^-K^+$ [Eq. (51)], are useful as we learn more about this interesting energy region.

It is interesting to note why radially excited vector mesons have such narrow widths despite the large phase space, e.g., $\omega'\to 3\pi<150$ MeV. The coupling $g_{\omega'\rho\pi}$ in our model is given by

$$g_{\omega'\rho\pi}=\frac{m_\rho^{\;2}}{\pi^2F_\pi^{\;3}}\frac{g_{\rho'\pi\pi}}{g_{\rho\pi\pi}}=g_{\omega\rho\pi}\frac{g_{\rho'\pi\pi}}{g_{\rho\pi\pi}},$$

which is small because $g_{\rho'\pi\pi}$ is small.

VI. COMMENT CONCERNING RELATIONS BETWEEN F_π, STRONG-INTERACTION PARAMETERS, AND THE VDM

In Sec. III we derived the relation

$$g_{\omega\rho\pi}=\frac{m_\rho^2}{\pi^2 F_\pi^3} \tag{24}$$

by requiring the GSW amplitude to match the chiral-Lagrangian anomaly at low-pseudoscalar momenta. (This interesting relation perhaps helps explain the unusually large scale of the dimensional coupling $g_{\omega\rho\pi}$—it is related to the smallness of the chiral-symmetry parameter F_π.) In Sec. IV we used this result in connection with the VDM to give $g_{\pi^0\gamma\gamma}$; matching that result with the chiral-Lagrangian determination then yielded

$$F_\pi^2 g_\rho^2=\tfrac{4}{3}m_\rho^2 \; . \tag{38}$$

Here we derive Eq. (38) in another way, namely, by requiring ρ exchange in $\pi\pi$ scattering to reproduce, at low energy, the amplitude predicted by the effective Lagrangian. (For simplicity we consider a specific charge state and do not project $I=1$, but the final result would be the same.) Restricted to pions the action of Eq. (3) yields the four-point function

$$L=\frac{2}{3F_\pi^2}[(\vec{\pi}\partial_\mu\vec{\pi})^2-\vec{\pi}^2(\partial_\mu\vec{\pi})^2] \; , \tag{58}$$

and the scattering amplitude

$$T(\pi_1^+\pi_2^-\to\pi_3^+\pi_4^-)=\frac{-8p_1p_4}{F_\pi^2} \; , \tag{59}$$

where we have used the soft-pion condition

$$0=p_3^2=2p_1p_2-2p_1p_4-2p_2p_4 \; .$$

The ρ-exchange amplitude for this process is

$$T_p=g_{\rho\pi\pi}^2\left[\frac{(p_1-p_2)(p_3-p_4)}{m_\rho^2-(p_1+p_2)^2}-\frac{(p_1+p_3)(p_2+p_4)}{m_\rho^2-(p_1-p_2)^2}\right] . \tag{60}$$

The leading soft-pion contribution is

$$T_\rho=\frac{-6g_{\rho\pi\pi}^2}{m_\rho^2}p_1p_4 \; . \tag{61}$$

Matching the VDM [Eq. (61)] and effective-Lagrangian prediction [Eq. (59)], we find again Eq. (38).

We note that Eq. (38) is similar but not identical to the Kawarabayashi-Suzuki-Riazuddin-Fayyazuddin[16] relation. [Here we should mention that Eq. (38) can also be derived by comparing $\pi^0\to 2\gamma$ with $\gamma\to 3\pi$, using vector dominance.[17]] It is interesting to observe this difference and question how stable this relation is, process to process; if it were stable there would apparently be no need for the effective Lagrangian: all results would be reproduced by the VDM. Happily, or unhappily, depending on the point of view, this is not the case, as we now demonstrate for the anomalous five-pseudoscalar amplitude. We consider $\eta\pi^+\to\pi^+\pi^+\pi^-$ scattering, for which there are only two VDM diagrams, as shown in Fig. 6. (The results we shall derive are identical for $K^+K^-\to\pi^+\pi^-\pi^0$, for which there are 11 diagrams.) From VDM (Fig. 5) we have the leading-momenta expansion

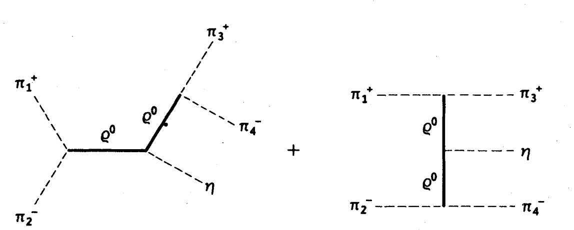

FIG. 6. Pole-enhancement diagrams for $\pi^+\pi^-\to\pi^+\pi^-\eta$.

$$T=\frac{2}{\sqrt{3}}\frac{g_{\rho\pi\pi}^2}{m_\rho^4}g_{\omega\rho\pi}\epsilon_{\mu\nu\alpha\beta}p_\mu^1 p_\nu^2 p_\alpha^3 p_\beta^4 \; . \tag{62}$$

Comparing this with the effective-Lagrangian result [Eq. (16b)] we find they match if

$$g_\rho^2 g_{\omega\rho\pi}=\frac{8}{5}\frac{m_\rho^4}{\pi^2 F_\pi^5} \; . \tag{63}$$

Using Eq. (24) we find

$$F_\pi^2 g_\rho^2=\frac{8}{5}m_\rho^2 \; , \tag{64}$$

somewhat different from Eq. (38). It is clear that while the VDM has the same tensor structure as the effective Lagrangian at low energy, and thus is suitable for the description of final-state interactions, it does not quantitatively lead to the same predictions as the more fundamental chiral Lagrangian. It is, however, possible that inclusion of tensor and axial-vector mesons could improve the VDM and restore consistency between Eqs. (38) and (64).

VII. SUMMARY AND CONCUSIONS

We have calculated chiral anomalies using an effective-Lagrangian technique introduced by Wess and Zumino and recently reformulated by Witten. A number of useful results have been presented, including all three-meson contributions to the anomalous part of the vector current. These anomalies were then tested in K_{l4} decay, η and $\eta'\to\pi^+\pi^-\gamma$, and strong decays of vector mesons. The agreement with experiment for K_{l4} is impressive, for η and η' decays, excellent, and for the strong decays, good. Since the amplitudes have been extrapolated to higher momenta with a final-state-interaction approximation, it is not surprising that the agreement is less good for the hadronic decays, where, indeed, further dynamical assumptions were made, involving vector dominance. We expect that our new predictions for decays of ρ', ω', and ϕ' should be of interest as new experimental data becomes available, not only in the absolute rate, but also in the Dalitz-plot distributions, where we predict some zeros at low momenta and at symmetric points in the decay distributions.

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