PHYSICAL REVIEW D VOLUME 30, NUMBER 4 15 AUGUST 1984

Supersymmetric kinks and the Witten-Olive bound

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(Received 11 May 1984)

The Witten-Olive bound on the kink mass in supersymmetric $(1+1)$-dimensional $\lambda\phi^4$ theory is shown to be saturated under lowest-order $[O(\hbar)]$ quantum fluctuations.

Several years ago it was shown by Witten and Olive[1] that topologically nontrivial classical field configurations (kinks, solitons, etc.) of supersymmetric field theories lead to a modification of the supersymmetry algebra: certain surface terms, usually ignored vis-a-vis the vacuum sector, appear as central charges in the algebra. An immediate physical consequence of this is a lower bound on the mass $M(\phi_k)$ of the topologically nontrivial field configuration (kink) $\phi_k(x)$ in terms of the appropriate topological (central) charge

$$M(\phi_k)\geq \tfrac{1}{2}|T(\phi_k)| \quad . \tag{1}$$

It was further pointed out by these authors that the above bound is saturated at the classical $[O(\hbar^0)]$ level.

Recently, Kaul and Rajaraman[2] and Schonfeld[3] have reexamined an earlier claim by D'Adda and Di Vecchia[4] that in $(1+1)$-dimensional supersymmetric spinor-scalar theories the lowest-order $[O(\hbar)]$ quantum fluctuations do not contribute to $M(\phi_k)$. It turns out that although the bosonic and fermionic $O(\hbar)$ fluctuations about the kink have identical continuum eigenspectra,[4] the respective densities of normal modes are actually different,[2,3] when periodic boundary conditions are used,[5-7] leading to a primitively log-divergent $O(\hbar)$ contribution to $M(\phi_k)$. This ultraviolet divergence is renormalized away by the vacuum-sector counterterm, resulting in a renormalization-scheme-dependent expression for the finite $O(\hbar)$ contribution to $M(\phi_k)$ in terms of the renormalized mass 2μ.

One is thus led to address the question as to whether the Witten-Olive bound (1) remains saturated at the $O(\hbar)$ level. To this end, observe that $T(\phi)$, to this order, should include the effect of quantum fluctuations around the classical kink configuration at the spatial boundaries. This fluctuation contribution, computed using periodic boundary conditions, has a logarithmic divergence which, however, is canceled by the contribution of the mass counterterm to $T(\phi)$. The resulting expression for $T(\phi)$ ensures the saturation of the bound (1), as we now proceed to demonstrate.

We consider the supersymmetric version of the theory in $1+1$ dimensions. In the notation of Ref. 2,

$$\mathscr{L}=\tfrac{1}{2}[(\partial_\mu\phi)^2-S^2(\phi)+i\bar{\psi}\partial\!\!\!/\psi - \bar{\psi}(dS/d\phi)\psi+\delta\mu^2 S(\phi)] \quad , \tag{2}$$

where

$$S(\phi)=\left(\frac{\lambda}{2}\right)^{1/2}(\phi^2-\mu^2/\lambda) \quad , \tag{3}$$

and the last term is the boson-mass counterterm. Around the classical vacuum $\langle\phi\rangle=\mu/\lambda^{1/2}$, $\mathscr{L}$ can be rewritten in terms of the shifted field $\eta\equiv\phi-\mu/\lambda^{1/2}$,

$$\mathscr{L}=\tfrac{1}{2}(\partial_\mu\eta)^2-\mu^2\eta^2+\tfrac{1}{2}\bar{\psi}(i\partial\!\!\!/-\sqrt{2}\mu)\psi-\tfrac{1}{4}\lambda\eta^4-\mu\lambda^{1/2}\eta^3 - (\tfrac{1}{2}\lambda)^{1/2}\eta\bar{\psi}\psi+\tfrac{1}{2}\delta\mu^2(\eta^2-2\mu\eta/\lambda^{1/2}) \quad . \tag{4}$$

Equation (4) has been written in terms of *renormalized* (not necessarily physical) parameters; the scalar-mass counterterm is assumed to be $O(\hbar)$ and therefore does not contribute classically. Note that in this manner of writing the fermion and scalar masses are degenerate classically (with respect to the supersymmetric classical vacuum $\langle\eta\rangle=0$),

$$m_b^c=m_f^c=2^{1/2}\mu \quad . \tag{5}$$

The theory in Eq. (2) admits the well-known kink solution

$$\phi_k(x)=(\mu/\lambda^{1/2})\tanh(\mu x/2^{1/2}) \quad , \tag{6}$$

which has the classical energy

$$M_k^c\equiv M^c(\phi_k)=(\tfrac{2}{3})^{3/2}(\mu^3/\lambda) \quad . \tag{7}$$

The corresponding central charge that appears in the modified superalgebra is given by

$$T=2\int_{-\infty}^{\infty}dx\,(\partial\phi/\partial x)S(\phi) \tag{6}$$

$$=2[W(\phi)]_{-\infty}^{\infty} \quad , \tag{7}$$

where

$$S(\phi)=dW/d\phi \quad . \tag{8}$$

With $S(\phi)$ given by Eq. (3) and $\phi_k=\phi_k(x)$, one obtains $|T_c|=(\tfrac{2}{3})^{5/2}(\mu^3/\lambda)$, showing that the bound in Eq. (1) is classically saturated.

The semiclassical quantization of the kink in this model has been discussed in Refs. 2–4. A proper evaluation of the difference in the density of normal modes of the bosonic and fermionic fluctuations around the kink yields the following result for the $O(\hbar)$ contribution to the kink mass[2]:

$$M_k\equiv M(\phi_k)=M_k^c+\hbar\left(-(\mu/2^{1/2})\int_{-\infty}^{\infty}(dk/2\pi)(k^2+2\mu^2)^{-1/2}+\sqrt{2}\delta\mu^2(\mu/\lambda)\right) \quad . \tag{9}$$

Following Ref. 2, we take

$$\delta\mu^2=\hbar\lambda(B+C) \ , \tag{10}$$

where

$$B=\frac{1}{2}\int_{-\infty}^{\infty}(dk/2\pi)(k^2+2\mu^2)^{-1/2} \ , \tag{11}$$

and C is a renormalization-scheme-dependent finite constant.[8] Note that the form (10) for $\delta\mu^2$ suffices to eliminate all vacuum-sector divergences. In the kink sector we now obtain

$$M_k(\mu)=M_k^c(\mu)+2^{1/2}\hbar\mu C \ . \tag{12}$$

The ambiguous quantity C can be eliminated from the expression for $M_k(\mu)$ by recasting (12) in terms of the common physical fermion/boson mass m, as obtained from the poles of the respective renormalized one-loop propagators:[2]

$$m_b^2=m_f^2\equiv m^2=2\mu^2+\hbar\lambda[2C+12\mu^2A(\mu^2)] \ , \tag{13}$$

where

$$A(p^2)\equiv i\int\frac{d^2k}{(2\pi)^2}(k^2-2\mu^2)^{-1}[(k-p)^2-2\mu^2]^{-1} \ . \tag{14}$$

This leads to the following expression for M_k:

$$M_k(m)=\frac{1}{3}(m^3/\lambda)-3\hbar m^3A(m^2) \ . \tag{15}$$

The topological central charge $T(\phi)$ is defined as[1]

$$T(\phi)=2[W(\phi(+\infty))-W(\phi(-\infty))] \ . \tag{16}$$

Now write $\phi(x,t)=\phi_k(x)+\xi(x,t)$, where $\xi(x,t)$ are the fluctuations of the bosonic field around the classical kink solution $\phi_k(x)$. Clearly, as $x\rightarrow\pm\infty$, the classical kink solution $\phi_k(x)$ goes to one or the other classical minima of the scalar potential, $\phi_k(\pm\infty)=\pm\mu/\lambda^{1/2}$. But since $\xi(x,t)$ must obey periodic boundary conditions, $\xi(+\infty,t)=\xi(-\infty,t)\equiv\xi(t)$. With these substitutions, we obtain

$$T(\phi)=2[W(\mu/\lambda^{1/2}+\xi)-W(-\mu/\lambda^{1/2}+\xi)] \ . \tag{16}$$

Taylor expansion of the right-hand side (RHS) of Eq. (16) about $\pm\mu/\lambda^{1/2}$ yields

$$\begin{aligned}\tfrac{1}{2}T(\phi)=&\,[W(\mu/\lambda^{1/2})-W(-\mu/\lambda^{1/2})]\\&+[S(\mu/\lambda^{1/2})-S(-\mu/\lambda^{1/2})]\xi\\&+[S'(\mu/\lambda^{1/2})-S'(-\mu/\lambda^{1/2})]\xi^2+O(\xi^3) \ .\end{aligned}\tag{17}$$

Since our aim is to compute the expectation value of $T(\phi)$ between one-kink states, viz., $\langle k|T(\phi)|k\rangle$, through $O(\hbar)$, we must retain all contributions to Eq. (18) which are of this order. It is thus convenient ro redefine $S(\phi)$, taking into account the effect of the mass counterterm, as

$$S(\phi)=(\tfrac{1}{2}\lambda)^{1/2}(\phi^2-\mu^2/\lambda)-\delta\mu^2/(2\lambda)^{1/2} \ . \tag{18}$$

It follows that

$$W(\phi)=(\tfrac{1}{2}\lambda)^{1/2}(\tfrac{1}{3}\phi^3-\mu^2\phi/\lambda)-[\delta\mu^2/(2\lambda)^{1/2}]\phi \ . \tag{19}$$

However, since $\langle k|\xi(t)|k\rangle$ and $\langle k|\xi^2(t)|k\rangle$ are $O(\hbar)$, it is possible to neglect the counterterm contribution to the second and third terms of the RHS of Eq. (17). Moreover, to this order it suffices to use the "small fluctuation" approximation which amounts to neglecting terms $O(\xi^3)$ in Eq. (17). Thus, to $O(\hbar)$, one obtains

$$\begin{aligned}\tfrac{1}{2}\langle k|T(\phi)|k\rangle=&-(\tfrac{2}{3})^{3/2}(\mu^3/\lambda)-2^{1/2}\mu\delta\mu^2/\lambda\\&+2^{1/2}\mu\langle k|\xi^2(t)|k\rangle \ ,\end{aligned}\tag{20}$$

where $\delta\mu^2$ is given by Eq. (10).

The next step is the computation of $\langle k|\xi^2(t)|k\rangle$. Observe that, to $O(\hbar)$,

$$\begin{aligned}\langle k|\xi^2(t)|k\rangle&=\lim_{x\rightarrow\pm\infty}\ \lim_{x't'\rightarrow xt}\ \langle k|T\{\xi(x,t)\xi(x't')\}|k\rangle\\&=\lim_{x\rightarrow\pm\infty}\ \lim_{x't'\rightarrow xt}\ \hbar G_B(xx';tt') \ ,\end{aligned}\tag{21}$$

where $G_B(xx';tt')$ is just the bosonic propagator in the background kink field, which satisfies

$$[\Box+V''(\phi_k)]G_B(xx';tt')=-i\delta(x-x')\delta(t-t') \ , \tag{22}$$

where $V(\phi_k)=\frac{1}{2}S^2(\phi_k)$ is the classical scalar potential for the kink field. $G_B(xx';tt')$ can be expanded in the usual manner in terms of the eigenfunctions of the eigenvalue equation for the bosonic fluctuations $\xi_n(x,t)$. Recall that these latter satisfy

$$\left[-\frac{d^2}{dx^2}+V''(\phi_k)\right]\xi_n=\omega_n{}^2\xi_n \ . \tag{23}$$

For the case of the kink solution, the fluctuation eigenspectrum consists of a zero mode, a discrete state at $\omega_1=(\frac{3}{2})^{1/2}\mu$, and a continuum corresponding to frequencies $\omega(k)=(k^2+2\mu^2)^{1/2}$, $0\leq k<\infty$. Thus we obtain[9]

$$G_B(xx';tt')=i\sum_n(N_n)^{-1}\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}e^{i\omega(t-t')}\frac{\xi_n^*(x')\xi_n(x)}{\omega^2-\omega_n{}^2+i\epsilon}+i\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\int_{-\infty}^{\infty}dk\,(N_k)^{-1}\rho(k)e^{i\omega(t-t')}\frac{\xi_k^*(x')\xi_k(x)}{\omega^2-\omega_k{}^2+i\epsilon} \ , \tag{24}$$

where the sum (integral) over n (k) runs through discrete (continuum) frequencies, $\rho(k)$ is the density of continuum modes, and $N_{n,k}$ are normalization constants given by $N_{n,k}=\int dx\,|\xi_{n,k}|^2$.

Next we proceed to take the appropriate limits; it is trivial to see that

$$G_B(xx;tt)=i\sum_n(N_n)^{-1}\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}|\xi_n|^2/(\omega^2-\omega_n{}^2+i\epsilon)+i\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\int_{-\infty}^{\infty}dk\,(N_k)^{-1}\rho(k)|\xi_k(x)|^2/(\omega^2-\omega_k{}^2+i\epsilon) \ . \tag{25}$$

Now, in the limit $x\rightarrow\pm\infty$, the discrete modes $\xi_n(x)\rightarrow0$, while[2] the continuum mode $\xi_k(x)\rightarrow\exp\{i[kx\pm\frac{1}{2}\delta_B(k)]\}$. If we normalize the continuum modes in a box of length L, we obtain, in the limit $L\rightarrow\infty$, $N_k=L$, $\rho(k)=L/2\pi$, thus yielding

$$\langle k|\xi^2(t)|k\rangle = \lim_{x\to\pm\infty} \hbar G_B(xx;tt)$$
$$= i\hbar \int [d^2k/(2\pi)^2](k^2-2\mu^2+i\epsilon)^{-1} \quad . \qquad (26)$$

Recall that

$$B = i\int [d^2k/(2\pi)^2](k^2-2\mu^2+i\epsilon)^{-1} \quad ,$$

which implies

$$\langle k|\xi^2(t)|k\rangle = \hbar B \quad . \qquad (27)$$

Substitution into Eq. (21) gives us the result

$$\tfrac{1}{2}\langle k|T(\phi)|k\rangle = -(\tfrac{2}{3})^{3/2}(\mu^3/\lambda) - 2^{1/2}\mu\hbar C \quad . \qquad (28)$$

It follows immediately from Eqs. (7), (12), and (28) that

$$M_k(\mu) = \tfrac{1}{2}|\langle k|T(\phi)|k\rangle| \quad , \qquad (29)$$

to $O(\hbar)$.

Thus, the Witten-Olive bound has been shown to be saturated for the $(1+1)$-dimensional $\lambda\phi^4$ theory under lowest-order quantum fluctuations. The question of validity of this result for general $S(\phi)$ and also the issue of alternative boundary conditions will be reported elsewhere.[10]

Note added. After this work was completed, we received a report by C. Imbimbo and S. Mukhi [Ecole Normale Superieure, Laboratoire de Physique Theorique, Report No. 84/04 (unpublished)] in which $T(\phi)$ has been computed for general $S(\phi)$ using slightly different techniques, and a result identical to ours has been obtained. We thank S. Mukhi for sending us their report prior to publication.

We thank R. Rajaraman for extremely valuable discussions.

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[8]The logarithmically divergent integral B can be made well-defined by means of any supersymmetric regularization scheme like dimensional reduction [see, e.g., P. Majumdar, E. Poggio, and H. Schnitzer, Phys. Rev. D **21**, 2203 (1980)].
[9]In these operations, we have treated the zero mode in a rather cavalier fashion. However, we expect that a more careful treatment of the zero mode will not affect our $O(\hbar)$ result.
[10]For a discussion of the Witten-Olive bound as applicable to the appropriate soliton mass and central charge densities, see H. Yamagishi, Massachusetts Institute of Technology Report No. MIT-CTP-1158, 1984 (unpublished).