Spectral asymmetry on an open space

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The spectral asymmetry is evaluated for a family of Dirac operators interacting with a topological background field and defined on an open infinite space. For these operators the spectral asymmetry is given by an integral over a local quantity that relates only to the homotopy properties of the background field. ζ-function regularization is employed and a possible simple pole in the limit where the regulator is removed is shown to vanish. The spectral asymmetry can be computed in a closed form in specific models. This is exemplified in various cases involving solitons, vortices, magnetic monopoles, and instantons as background fields.

I. INTRODUCTION

Various index theorems[1] have been applied to the analysis of both the classical and quantum structures of field theories. In particular the index of a Dirac operator has been used extensively to obtain relationships between the analytical and topological aspects of these theories. Typically, an index theorem relates the number of zero modes of a Dirac operator to a number that characterizes the background-field topology and hence it can yield useful information on the quantum field theory if the classical structure of the theory is known.

An important application of index theorems arises in the study of instantons, where the Atiyah-Singer index theorem[1] is used to enumerate the number of independent parameters that are needed to characterize the most general multi-instanton configuration with a given Pontryagin index.[1,2] Another widely studied application, with consequences both in particle physics and condensed matter physics, arises from the analysis of fermion number fractionization[3-5] in field-theory models with a conjugation symmetry such as charge conjugation. In these models the fermion number of a topological soliton is related to the number of zero modes of the pertinent Dirac Hamiltonian, which in turn can be obtained by the use of an index theorem.[5] However, in the absence of a conjugation symmetry, as in the field-theory models studied in Ref. 6, the use of index theorems to analyze fermion number fractionization is inadequate and a more profound approach is necessary. In the general case the fermion number

$$N=\tfrac{1}{2}\int d^Dx\langle[\psi^\dagger,\psi]\rangle \tag{1.1}$$

is a transcendental function of the parameters of the theory and is mathematically related to the Atiyah-Patodi-Singer η invariant[1,7] of the pertinent Dirac Hamiltonian, formally[8-10]

$$N=-\tfrac{1}{2}\eta=-\tfrac{1}{2}\sum_{\lambda_\mu}\operatorname{sgn}(\lambda_\mu)\,,$$
$$\operatorname{sgn}\lambda=\begin{cases}+1 & \text{for } \lambda\geq 0\\ -1 & \text{for } \lambda<0\,,\end{cases} \tag{1.2}$$

where λ_n are the energy eigenvalues. In the presence of a conjugation symmetry the positive and negative eigenvalues are paired, and only the zero modes can contribute to (1.2). But in the general case where no such symmetry exists, all eigenvalues contribute. In particular if the Dirac Hamiltonian operates on an open infinite space so that the spectrum admits a continuum part, this continuum must be included in a properly regulated form of (1.2).[5]

In the mathematics literature,[1,7] the Atiyah-Patodi-Singer η invariant in (1.2) is usually defined as the $s\to 0$ limit of the spectral asymmetry[11]

$$\eta(s)=\sum_{\lambda_k}\operatorname{sgn}(\lambda_k)\,|\,\lambda\,|^{-s}\,. \tag{1.3}$$

If the operator under consideration has a continuum spectrum, the summation in (1.3) becomes an integral over the continuum portion.

The properties of $\eta(s)$ have been studied extensively on compact manifolds with and without a boundary, and it has been established that $\eta(s)$ is in general a meromorphic function of s with a finite $s\to 0$ limit.[7]

The purpose of this paper is to analyze the spectral asymmetry $\eta(s)$ of a Dirac operator which is defined on an open infinite space of arbitrary dimensionality. We consider operators that are of the general type studied in Refs. 5 and 8–10 and we shall prove that for these operators $\eta(s)$ is given by an integral over a local quantity. This integral relates only to the topological (=global) properties of the background and can be readily evaluated in specific models. In the $s\to 0$ limit we obtain an expression for the fermion number (1.1) and (1.2) which agrees with the previously derived results[3-6,8-10] and in a conjugation-symmetric limit our formula for (1.2) reproduces the results derivable from index theorems.

While the original motivation for this work is in the study of fermion fractionization, our formalism should also have applications in other field-theory problems. Indeed, if we interpret (1.3) as a moment problem for the spectral density function, we can solve for its odd part provided $\eta(s)$ can be evaluated.[12] $\eta(s)$ is also closely related to the Riemann ζ function of the Dirac operator which is known to contain much information about the quantum theory,[13] and while it has not yet been clarified what information is contained in $\eta(s)$ [except for the connection to the fermion number (1.2)], we expect that our formalism and results will be useful especially in connection with anomalies, effective actions, and index theorems.[10]

We shall evaluate the spectral asymmetry $\eta(s)$ for various different Dirac operators. We shall also establish close relationships between problems that are traditionally considered distinct—in particular we obtain the Atiyah-Singer index theorem,[1] the Callias index theorem,[14] and the index theorem studied by Weinberg[15] as special cases of our formalism.

In Sec. II we introduce the operators that we study and give a formal derivation for an integral expression of the spectral asymmetry. Our technique here parallels that used by Callias[14] in the context of index theorems. Sections III—VI are devoted to various applications that are chosen to illuminate the use and different aspects of our formalism, and concluding remarks in Sec. VII summarize our results.

II. CALCULATING SPECTRAL ASYMMETRY

The spectral asymmetries that we are interested in arise from D-dimensional Euclidean-space Dirac operators that are of the general form

$$H_\kappa = \begin{bmatrix} \kappa & D \\ D^\dagger & -\kappa \end{bmatrix} . \tag{2.1}$$

Here κ is a positive constant, D an operator of the form

$$D = iP_i\partial_i + Q(x) , \tag{2.2}$$

and $D^\dagger$ the Hermitian conjugate of D. The P_i are constant matrices that satisfy

$$\begin{aligned} &P_i^\dagger P_j + P_j^\dagger P_i = 2\delta_{ij}1 , \\ &P_i P_j^\dagger + P_j P_i^\dagger = 2\delta_{ij}1 . \end{aligned} \tag{2.3}$$

We write (2.1) in the form

$$H_\kappa = H + \kappa\Gamma^c = i\Gamma_i\partial_i + K(x) + \kappa\Gamma^c , \tag{2.4}$$

where

$$\Gamma_i = \begin{bmatrix} 0 & P_i \\ P_i^\dagger & 0 \end{bmatrix}, \quad \Gamma^c = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \tag{2.5}$$

and

$$K(x) = \begin{bmatrix} 0 & Q(x) \\ Q^\dagger(x) & 0 \end{bmatrix} . \tag{2.6}$$

The matrices (2.5) satisfy the algebra

$$\begin{aligned} &\{\Gamma_i,\Gamma_j\} = 2\delta_{ij}1 , \quad \{\Gamma^c,\Gamma_i\} = 0 , \\ &(\Gamma^c)^2 = 1 , \end{aligned} \tag{2.7}$$

and the operator H anticommutes with Γ^c

$$\{H,\Gamma^c\} = 0 . \tag{2.8}$$

As a consequence

$$H_\kappa{}^2 = H^2 + \kappa^2 \geq \kappa^2 \tag{2.9}$$

and all eigenvalues of H_κ

$$H_\kappa\psi_n = \lambda_n\psi_n \tag{2.10}$$

are nonzero. Since H_κ is a Dirac operator it has both positive and negative eigenvalues, and in general its spectrum is asymmetric around $\lambda=0$. We wish to evaluate the spectral asymmetry of H_κ (Ref. 16)

$$\eta_\kappa(s) = \sum_n \text{sgn}(\lambda_n)\,|\,\lambda_n\,|^{-s} . \tag{2.11}$$

We expect that (2.11) is a meromorphic function on the complex-s plane with isolated simple poles on the real axis.[7] An apparent pole occurs at $s=0$, but for the class of models studied here the residue of this pole vanishes. Hence $\eta_\kappa(s)$ is regular at $s=0$ and we can define

$$\eta_\kappa = \lim_{s\to 0+} \eta_\kappa(s) . \tag{2.12}$$

Formally, (2.12) is a measure of the difference between the number of positive and negative eigenvalues of the operator H_κ,

$$\eta_\kappa = \sum_{\lambda_n>0} 1 - \sum_{\lambda_n<0} 1 . \tag{2.13}$$

We are also interested in the $\kappa\to 0+$ limit of the spectral asymmetry $\eta_\kappa(s)$. In this limit H_κ becomes H. From (2.8) it follows that the spectrum of H is symmetric,[17] i.e., if

$$H\psi = E\psi \tag{2.14}$$

then

$$H(\Gamma^c\psi) = -E(\Gamma^c\psi) . \tag{2.15}$$

Hence only the zero modes of H can contribute to the $\kappa\to 0+$ limit. For $E=0$ the eigenvalue equation for H,

$$H\psi = \begin{bmatrix} 0 & D \\ D^\dagger & 0 \end{bmatrix} \begin{bmatrix} \psi_+ \\ \psi_- \end{bmatrix} = E\psi \tag{2.16}$$

becomes

$$D\psi_- = 0, \quad D^\dagger\psi_+ = 0 , \tag{2.17}$$

and since the zero modes of D and $D^\dagger$ are eigenmodes of Γ^c,

$$\Gamma^c\psi_\pm = \pm\psi_\pm \tag{2.18}$$

we conclude that in the $\kappa\to 0+$ limit (2.12) yields the index of H

$$\lim_{\kappa\to 0+} \eta_\kappa = -\mathrm{index}(H) = \mathrm{Dim\,Ker}(D^\dagger) - \mathrm{Dim\,Ker}(D) \,. \qquad (2.19)$$

We shall now explain how $\eta_\kappa(s)$ can be evaluated. We first introduce the Mellin transform

$$\mathrm{sgn}(\lambda)\,|\lambda|^{-s} = \frac{2}{\pi}\cos\left[s\frac{\pi}{2}\right]\int_0^s d\omega\,\omega^{-s}\frac{\lambda}{\lambda^2+\omega^2} \,. \qquad (2.20)$$

This yields

$$\eta_\kappa(s) = \frac{2}{\pi}\cos\left[s\frac{\pi}{2}\right]\sum_n\int_0^\infty d\omega\,\omega^{-s}\frac{\lambda_n}{\lambda_n{}^2+\omega^2} \,. \qquad (2.21)$$

We change the order of summations and get

$$\eta_\kappa(s) = \frac{2}{\pi}\cos\left[s\frac{\pi}{2}\right] \times \int_0^\infty d\omega\,\omega^{-s}\int d^Dx\,\mathrm{tr}\left\langle x\left|\frac{H_\kappa}{H_\kappa{}^2+\omega^2}\right|x\right\rangle \,. \qquad (2.22)$$

We now show that the spectral asymmetry density in (2.22) is a total divergence. For this we first observe that

$$\mathrm{tr}\left\langle x\left|\frac{H_\kappa}{H_\kappa{}^2+\omega^2}\right|y\right\rangle = \kappa\,\mathrm{tr}\left\langle x\left|\Gamma^c\frac{1}{H^2+\kappa^2+\omega^2}\right|y\right\rangle = i\frac{\kappa}{\sigma}\,\mathrm{tr}\left\langle x\left|\Gamma^c\frac{1}{H+i\sigma}\right|y\right\rangle \,, \qquad (2.23)$$

where we have defined

$$\sigma = (\omega^2+\kappa^2)^{1/2} \,. \qquad (2.24)$$

Now consider

$$\mathrm{tr}\left\langle x\left|i\Gamma^i\partial_i\Gamma^c\frac{1}{H+i\sigma} - \Gamma^c\frac{1}{H+i\sigma}i\Gamma^i\partial_i\right|y\right\rangle \,. \qquad (2.25)$$

Using (2.4), (2.7), and (2.8) we find that this equals

$$\mathrm{tr}\left\langle x\left|[H-K(x)+i\sigma-i\sigma]\Gamma^c\frac{1}{H+i\sigma} - \Gamma^c\frac{1}{H+i\sigma}[H-K(y)+i\sigma-i\sigma]\right|y\right\rangle$$
$$= 2i\sigma\,\mathrm{tr}\left\langle x\left|\Gamma^c\frac{1}{H+i\sigma}\right|y\right\rangle + \mathrm{tr}\left[[K(y)-K(x)]\left\langle x\left|\Gamma^c\frac{1}{H+i\sigma}\right|y\right\rangle\right] \,. \qquad (2.26)$$

We then obtain the trace identity

$$\mathrm{tr}\left\langle x\left|\frac{H_\kappa}{H_\kappa{}^2+\omega^2}\right|y\right\rangle = \frac{\kappa}{2\sigma^2}\left[\frac{\partial}{\partial x_i}+\frac{\partial}{\partial y_i}\right]\mathrm{tr}\left\langle x\left|i\Gamma^i\Gamma^c\frac{1}{H+i\sigma}\right|y\right\rangle + \frac{\kappa}{2\sigma^2}\,\mathrm{tr}\left[[K(x)-K(y)]\left\langle x\left|\Gamma^c\frac{1}{H+i\sigma}\right|y\right\rangle\right] \,. \qquad (2.27)$$

We recognize that for D even, (2.27) is formally equivalent to the standard axial anomaly equation for a Dirac operator H_κ defined in a D-dimensional Euclidean space.[18] Consequently, when we consider the $y\to x$ limit of (2.27) we need to discuss two cases separately. First, if background gauge fields are not present or the space dimension D is odd the $y\to x$ limit of the left-hand side in (2.27) yields the integrand of (2.22) while the second term on the right-hand side vanishes since there are no anomalies in this case. The integrand becomes

$$\mathrm{tr}\left\langle x\left|\frac{H_\kappa}{H_\kappa{}^2+\omega^2}\right|x\right\rangle = \frac{\kappa}{2\sigma^2}\partial^i\,\mathrm{tr}\left\langle x\left|i\Gamma^i\Gamma^c\frac{1}{H+i\sigma}\right|x\right\rangle \,. \qquad (2.28)$$

However, if the space dimension D is even and background gauge fields are present, the second term in (2.27) in general does not vanish in the $y\to x$ limit. Since (2.27) is now formally equivalent to the ordinary D-dimensional axial anomaly equation[18] in the representation (2.5) of the Γ matrices, in order to preserve manifest gauge invariance we must include the appropriate axial anomaly term in the $y\to x$ limit of (2.25) and (2.26).[19] In this way we get from (2.27) the integrand

$$\mathrm{tr}\left\langle x\left|\frac{H_\kappa}{H_\kappa{}^2+\omega^2}\right|x\right\rangle = \frac{\kappa}{2\sigma^2}\left[\partial^i\,\mathrm{tr}\left\langle x\left|i\Gamma^i\Gamma^c\frac{1}{H+i\sigma}\right|x\right\rangle + \text{anomaly}\right] \,. \qquad (2.29)$$

Substituting (2.28) and (2.29) into (2.22) we get

$$\eta_\kappa(s) = \frac{1}{\pi}\cos\left[s\frac{\pi}{2}\right]\int_0^\infty d\omega\,\omega^{-s}\frac{\kappa}{\omega^2+\kappa^2}\left[2T_D + \oint_{R=\infty} dS^i\,\mathrm{tr}\left\langle x\left|i\Gamma^i\Gamma^c\frac{1}{H+i\sigma}\right|x\right\rangle\right] \,. \qquad (2.30)$$

Here T_D is the Pontryagin index of the background gauge fields that arises from the space integral of the anomaly term in (2.29). Since the surface term also relates to the topological (=asymptotic) properties of the background fields, we conclude that $\eta_\kappa(s)$ itself is a topological quantity in the sense that it is invariant under local variations of the background and (as we shall see) vanishes if the background field has a trivial topological structure.

We observe that if the space dimension is even and the background is a pure gauge field so that the explicit surface term in (2.30) is absent, the limit $s\to 0+$ yields the Atiyah-Singer index theorem:[1]

$$\eta_\kappa = -\text{index}(H) = T_D \ . \tag{2.31}$$

Notice that η_κ is now independent of κ. A nontrivial κ dependence can only arise from the explicit surface term in (2.30). This term is present only if the space is open. In such a case the spectrum of H_κ has both a discrete part and a continuum part, both of which contribute to η_κ.

In the following sections we shall evaluate (2.30) in various field-theoretical models. In Sec. III we consider the open-space problem of a (1 + 1)-dimensional fermion-soliton system. In this case there are no anomalies and $\eta_\kappa(s)$ arises entirely from the surface term in (2.30). In Sec. IV we exemplify the Atiyah-Singer index theorem by evaluating (2.30) for a four-dimensional spinor in an instanton background. In Sec. V we analyze the (3 + 1)-dimensional fermion—magnetic-monopole system. In the $s\to 0$ limit we reproduce the known[6,9] results for the fermion number (1.2) and in the $s\to 0$, $\kappa\to 0+$ limit of (2.30) yields the Callias index theorem,[11] and in Sec. VI we study a (2 + 1)-dimensional fermion-vortex system as an example of the case where both the anomaly contribution and the surface term in (2.30) are present.

III. A (1 + 1)-DIMENSIONAL MODEL

In this section we consider a (1 + 1)-dimensional field theory involving a scalar field Φ and a spinor field ψ. We assume that the spinor part of the Lagrangian has the form

$$L = \bar\psi(i\partial\!\!\!/ - \Phi - i\kappa\gamma^5)\psi \tag{3.1}$$

and we treat the scalar field Φ as a classical background field with a soliton profile,

$$\varphi_+ = \Phi(\infty) \neq \Phi(-\infty) = \varphi_- \ . \tag{3.2}$$

The (1 + 1)-dimensional Dirac algebra

$$\{\gamma^\mu,\gamma^\nu\} = 2g^{\mu\nu} \ , \quad \{\gamma^\mu,\gamma^5\} = 0 \ , \quad (\gamma^5)^2 = 1 \tag{3.3}$$

is represented by $\gamma^0=\sigma^1$, $\gamma^1=i\sigma^3$, and $\gamma^5=\gamma^0\gamma^1=\sigma^2$ where the σ's are the Pauli matrices. The Dirac equation is

$$\left[i\gamma^5\frac{\partial}{\partial x} - \gamma^0\Phi - i\kappa\gamma^1\right]\psi(xt) = -i\frac{\partial}{\partial t}\psi(xt) \ . \tag{3.4}$$

We separate the time variable by

$$\psi(xt) = \psi(x)e^{iEt} \tag{3.5}$$

and write (3.4) as an eigenvalue equation for $\psi(x)$

$$H_\kappa\psi = \begin{bmatrix} \kappa & D \\ D^\dagger & -\kappa \end{bmatrix}\begin{bmatrix} \psi_+ \\ \psi_- \end{bmatrix} = E\psi \ . \tag{3.6}$$

Here D is a first-order differential operator

$$D = \frac{\partial}{\partial x} - \Phi(x) \tag{3.7}$$

and $D^\dagger$ is the adjoint of D. We wish to evaluate the spectral asymmetry of H_κ.

In order to apply (2.30) we identify

$$\Gamma^c = -i\gamma^1 \ , \quad \Gamma^1 = \gamma^5 \ . \tag{3.8}$$

The trace identity (2.28) then becomes

$$\text{tr}\left\langle x \left| \frac{H_\kappa}{H_\kappa{}^2+\omega^2} \right| x \right\rangle = -\frac{\kappa}{2\sigma^2}\partial^i \text{tr}\left\langle x \left| \gamma^0 \frac{1}{H+i\sigma} \right| x \right\rangle \tag{3.9}$$

and the spectral asymmetry is now

$$\eta_\kappa(s) = -\frac{1}{\pi}\cos\left[s\frac{\pi}{2}\right]\int_0^\infty d\omega\,\omega^{-s}\frac{\kappa}{\omega^2+\kappa^2}\left[\text{tr}\left\langle \infty \left| \gamma^0\frac{1}{H+i\sigma} \right| \infty \right\rangle - \text{tr}\left\langle -\infty \left| \gamma^0\frac{1}{H+i\sigma} \right| -\infty \right\rangle\right] . \tag{3.10}$$

Consider

$$\text{tr}\left\langle \pm\infty \left| \gamma^0\frac{1}{H+i\sigma} \right| \pm\infty \right\rangle = \text{tr}\left\langle \pm\infty \left| \frac{1}{i\gamma^1\partial - \Phi + i\gamma^0\sigma} \right| \pm\infty \right\rangle = 2\left\langle \pm\infty \left| \frac{\Phi}{\partial^2 - \Phi^2 - \sigma^2} \right| \pm\infty \right\rangle = -2\phi_\pm \int \frac{dk}{2\pi}\frac{1}{k^2+\varphi_\pm{}^2+\sigma^2} \ . \tag{3.11}$$

Substituting this into (3.10) we obtain the spectral asymmetry

$$\eta_\kappa(s)=\frac{1}{\pi}\cos\left[s\frac{\pi}{2}\right]\int_0^\infty d\omega\,\omega^{-s}\frac{\kappa}{\kappa^2+\omega^2}\left[\frac{\varphi_+}{(\varphi_+{}^2+\omega^2+\kappa^2)^{1/2}}-\frac{\varphi_-}{(\varphi_-{}^2+\omega^2+\kappa^2)^{1/2}}\right]$$
$$=\frac{1}{2\pi}\cos\left[s\frac{\pi}{2}\right]\kappa^{-s}B\left[\frac{1-s}{2},\frac{2+s}{2}\right]$$
$$\times\left[\frac{\varphi_+}{(\varphi_+{}^2+\kappa^2)^{1/2}}{}_2F_1\left[\frac{1}{2},\frac{1-s}{2};\frac{3}{2};\frac{\varphi_+{}^2}{\varphi_+{}^2+\kappa^2}\right]-\frac{\varphi_-}{(\phi_-{}^2+\kappa^2)^{1/2}}{}_2F_1\left[\frac{1}{2},\frac{1-s}{2};\frac{3}{2};\frac{\varphi_-{}^2}{\varphi_-{}^2+\kappa^2}\right]\right]. \tag{3.12}$$

Here B is the beta function and ${}_2F_1$ is Gauss's hypergeometric function.

We now discuss various limits of (3.12). In the $s\to 0+$ limit we get

$$\eta_\kappa=\frac{1}{\pi}\left[\arctan\left[\frac{\varphi_+}{\kappa}\right]-\arctan\left[\frac{\varphi_-}{\kappa}\right]\right] \tag{3.13}$$

and in the $\kappa\to 0+$ limit we obtain the index of H

$$\text{index}(H)=-\tfrac{1}{2}[\text{sgn}(\varphi_+)-\text{sgn}(\varphi_-)]\ . \tag{3.14}$$

This result agrees with that given in Refs. 3 and 14.

If either φ_+ or φ_- vanishes, (3.14) is ill defined. However, if. e.g., $\varphi_-=0$, (3.13) yields formally

$$\text{index}(H)=-\tfrac{1}{2}\,\text{sgn}[\varphi_+]\ , \tag{3.15}$$

i.e., a half-integer. This simply means that the continuum part of the spectrum extends to zero and contributes to the index.[20]

IV. SPECTRAL ASYMMETRY AND THE ATIYAH-SINGER INDEX THEOREM

We shall now use our technique to evaluate the spectral asymmetry $\eta_\kappa(s)$ of the four-dimensional Dirac operator (one flavor)

$$(i\gamma^\mu\partial^\mu+\gamma^\mu A^\mu+\kappa\gamma^5)\psi_n=\lambda_n\psi_n\ . \tag{4.1}$$

The background gauge field $A_a^\mu T^a$ is a Hermitian matrix that takes values in the Lie algebra of a simple compact Lie group G. We normalize

$$F^{\mu\nu}=\partial^\mu A^\nu-\partial^\nu A^\mu-i[A^\mu,A^\nu] \tag{4.2}$$

and represent the Dirac algebra (2.7),

$$\{\gamma^\mu,\gamma^\nu\}=2\delta^{\mu\nu},\ \ \{\gamma^5,\gamma^\mu\}=0,\ \ (\gamma^5)^2=1\ , \tag{4.3}$$

by the following Hermitian matrices

$$\gamma^0=\begin{bmatrix}0&1\\1&0\end{bmatrix},\ \ \gamma^k=\begin{bmatrix}0&-i\sigma^k\\i\sigma^k&0\end{bmatrix},\qquad \gamma^5=\begin{bmatrix}1&0\\0&-1\end{bmatrix}. \tag{4.4}$$

Equation (2.10) then becomes

$$H_\kappa\psi_n=\begin{bmatrix}\kappa&D\\D^\dagger&-\kappa\end{bmatrix}\begin{bmatrix}\psi_+\\\psi_-\end{bmatrix}=\lambda_n\psi_n\ , \tag{4.5}$$

where

$$D=i\partial^0+\sigma^k\partial^k+A^0-i\sigma^kA^k \tag{4.6}$$

and $D^\dagger$ is the Hermitian conjugate of D.

Equation (2.23) reads

$$\text{tr}\left\langle x\left|\frac{H_\kappa}{H_\kappa{}^2+\omega^2}\right|x\right\rangle=i\frac{\kappa}{\sigma}\,\text{tr}\left\langle x\left|\gamma^5\frac{1}{H+i\sigma}\right|x\right\rangle \tag{4.7}$$

and the trace identity (2.27) has now an anomaly contribution. We observe that the right-hand side term in (4.7) satisfies the four-dimensional axial anomaly equation[18]

$$\frac{i}{2}\partial^\mu\,\text{tr}\left\langle x\left|\gamma^\mu\gamma^5\frac{1}{H+i\sigma}\right|x\right\rangle$$
$$=i\sigma\,\text{tr}\left\langle x\left|\gamma^5\frac{1}{H+i\sigma}\right|x\right\rangle-\frac{1}{16\pi^2}\,\text{tr}[{}^*F^{\mu\nu}F^{\mu\nu}]\ . \tag{4.8}$$

Combining (4.7) and (4.8) we obtain the trace identity (2.29),

$$\text{tr}\left\langle x\left|\frac{H_\kappa}{H_\kappa{}^2+\omega^2}\right|x\right\rangle$$
$$=\frac{\kappa}{2\sigma^2}\left[i\partial^\mu\,\text{tr}\left\langle x\left|\gamma^\mu\gamma^5\frac{1}{H+i\sigma}\right|x\right\rangle+\frac{1}{8\pi^2}\,\text{tr}[{}^*F^{\mu\nu}F^{\mu\nu}]\right]. \tag{4.9}$$

We substitute this into (2.22) and use the fact that the space integral of the first term on the right-hand side of (4.9) vanishes. This yields for (2.30)

$$\eta_\kappa(s)=\frac{1}{\pi}\cos\left[s\frac{\pi}{2}\right]\int_0^\infty d\omega\,\omega^{-s}\frac{\kappa}{\omega^2+\kappa^2}[2T_4]$$
$$=\cos\left[s\frac{\pi}{2}\right]\kappa^{-s}T_4\ , \tag{4.10}$$

where

$$T_4=\frac{1}{16\pi^2}\int d^4\text{tr}[{}^*F^{\mu\nu}F^{\mu\nu}] \tag{4.11}$$

is the Pontryagin index of the background field. Setting $s\to 0+$ we obtain $\eta_\kappa=T_4$ independent of κ. Hence

$$\text{index}(H)=-T_4 \tag{4.12}$$

which is the four-dimensional spinor index theorem.[1]

V. MAGNETIC MONOPOLES AND THE CALLIAS INDEX THEOREM

We consider a (3 + 1)-dimensional Dirac equation for a fermion interacting with a background Yang-Mills-Higgs system:

$$H\psi(xt)=[i\alpha^k\partial^k+\alpha^kA^k-\beta(m-i\kappa\gamma^5+\Phi)]\psi(xt) = -i\partial^0\psi(xt)\ . \tag{5.1}$$

Here $A^\mu=A^\mu_aT^a$ is a static Hermitian gauge field based on a compact simple gauge group G, assumed to be purely magnetic so that $A^0=0$, and $\Phi=\varphi^aT^a$ is a static Hermitian Higgs field. The following representation of the Dirac algebra is convenient

$$\beta=\gamma^0=\begin{bmatrix}0 & -i\\ i & 0\end{bmatrix},\quad \alpha^k=\beta\gamma^k=\begin{bmatrix}0 & \sigma^k\\ \sigma^k & 0\end{bmatrix},\quad \gamma^5=\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}. \tag{5.2}$$

We separate the time variable as in (3.5) and obtain

$$H_\kappa\psi=\begin{bmatrix}\kappa & D\\ D^\dagger & -\kappa\end{bmatrix}\begin{bmatrix}\psi_+\\ \psi_-\end{bmatrix}=E\psi\ , \tag{5.3}$$

where

$$D=i\sigma^k\partial^k+\sigma^kA^k+i(m+\Phi) \tag{5.4}$$

and $D^\dagger$ is the Hermitian conjugate of D. We shall now calculate the spectral asymmetry (2.30) with the present technique. For this we define $\Gamma^c=i\beta\gamma^5$ and we also identify $\alpha^k=\Gamma^k$. The trace identity (2.28) then becomes

$$\mathrm{tr}\left\langle x\left|\frac{H_\kappa}{{H_\kappa}^2+\omega^2}\right|x\right\rangle=\frac{\kappa}{2\sigma^2}\partial^k\,\mathrm{tr}\left\langle x\left|\gamma^k\gamma^5\frac{1}{H+i\sigma}\right|x\right\rangle . \tag{5.5}$$

Notice that there are no anomalies in three dimensions. We substitute (5.5) into (2.22) and get

$$\eta_\kappa(s)=\frac{1}{\pi}\cos\left[s\frac{\pi}{2}\right]\int_0^\infty d\omega\,\omega^{-s}\frac{\kappa}{\omega^2+\kappa^2}\oint_{R=\infty}dS^i\,\mathrm{tr}\left\langle x\left|\gamma^i\gamma^5\frac{1}{H+i\sigma}\right|x\right\rangle . \tag{5.6}$$

It is sufficient to evaluate the integrand in (5.6) in an asymptotic expansion where we only keep terms of order $1/x^2$. For a magnetic monopole we can assume that, for $|x|\to\infty$,

$$\Phi(x)\simeq O(1)\ ,\quad A^k(x)\simeq O\left[\frac{1}{|x|}\right],\quad F^{ik}(x)\simeq O\left[\frac{1}{|x|^2}\right], \tag{5.7}$$

and $[D^k,\Phi]$ vanishes exponentially. Hence

$$\frac{1}{H+i\sigma}=\frac{1}{D^2+(m+\Phi)^2+\sigma^2}\gamma^0(\not{D}-m-\Phi-i\gamma^0\sigma) + \frac{1}{D^2+(m+\Phi)^2+\sigma^2}\frac{i}{4}[\gamma^k,\gamma^i]F^{ki}\frac{1}{D^2+(m+\Phi)^2+\sigma^2}\gamma^0(\not{D}-m-\Phi-i\gamma^0\sigma)+\cdots\ , \tag{5.8}$$

where we have dropped terms smaller than $1/x^2$, and we have also defined $D^k=i\partial^k+A^k$. We substitute (5.8) into (5.6) and evaluate the trace over γ matrices. The result is

$$\eta_\kappa(s)=\frac{4}{\pi}\cos\left[s\frac{\pi}{2}\right]\int_0^\infty d\omega\,\omega^{-s}\frac{\kappa}{\omega^2+\kappa^2}\oint_{R=\infty}dS^i\mathrm{tr}\left[\left\langle x\left|\frac{1}{[-\partial^2+(m+\Phi)^2+\omega^2+\kappa^2]^2}\right|x\right\rangle B^i(m+\Phi)\right] . \tag{5.9}$$

Here $B^i=\frac{1}{2}\epsilon^{ijk}F^{jk}$ is the magnetic field and the trace is over the group indices, and in the $s\to 0$ limit this agrees with the expression for the fermion number evaluated in Ref. 9.

The formula (5.9) allows the computation of $\eta_\kappa(s)$ for a wide class of models. As an application we shall study the Dirac equation for an isospin-T particle in the field of a static system of SU(2) magnetic monopoles. We evaluate the trace in (5.9) in the unitary gauge by first dividing the surface integral into integrations over two patches. Inside each patch we perform a nonsingular gauge transformation which aligns the Higgs field $\Phi=\varphi^aT^a$ along the 3-direction in isospin space. The T^a are now the generators of isospin rotations that satisfy

$$[T^a,T^b]=i\epsilon^{abc}T_c\ ,\quad T^aT^a=T(T+1)\ , \tag{5.10}$$

and we choose T^3 to be diagonal with eigenvalues $(T,T-1,\ldots,-T)$. Evaluating the trace in (5.9) we get

$$\eta_\kappa(s)=\frac{4}{\pi}\cos\left[s\frac{\pi}{2}\right]\int_0^\infty d\omega\,\omega^{-s}\frac{\kappa}{\omega^2+\kappa^2}\int\frac{d^3k}{(2\pi)^3}\oint_{\text{patches}}dS^iB^i_3\sum_{j=-T}^{T}\left[\frac{1}{[k^2+(m+j\varphi)^2+\omega^2+\kappa^2]^2}j(m+j\varphi)\right], \tag{5.11}$$

where $\varphi^2=\varphi^a\varphi^a$. We use the gauge-invariant result

$$\oint_{\text{patches}} dS^i B_3^i = \oint dS^i B_a^i \frac{\varphi^a}{\varphi} = 4\pi N \ , \tag{5.12}$$

where N is an integer. Integration over d^3k in (5.11) then yields

$$\eta_\kappa(s)=\sum_{j=-T}^{T}\frac{2N}{\pi}\cos\left[s\frac{\pi}{2}\right]\int_0^\infty d\omega\,\omega^{-s}\frac{\kappa}{\omega^2+\kappa^2}\frac{j(m+j\varphi)}{[\omega^2+\kappa^2+(m+j\varphi)^2]^{1/2}} \ . \tag{5.13}$$

This should be compared with (3.12). Integration over $d\omega$ gives the spectral asymmetry

$$\eta_\kappa(s)=\frac{N}{\pi}\cos\left[s\frac{\pi}{2}\right]\kappa^{-s}B\left[\frac{1-s}{2},\frac{2+s}{2}\right]\sum_{j=-T}^{T} j\frac{m+j\varphi}{[\kappa^2+(m+j\varphi)^2]^{1/2}}\,{}_2F_1\left[\frac{1}{2},\frac{1-s}{2};\frac{3}{2};\frac{(m+j\varphi)^2}{\kappa^2+(m+j\varphi)^2}\right] . \tag{5.14}$$

Various limits of (5.14) can now be discussed. In the $s\to 0+$ limit we get

$$\eta_\kappa=\frac{2}{\pi}N\sum_{j=-T}^{T} j\arctan\left[\frac{m+j\varphi}{\kappa}\right] \tag{5.15}$$

which gives the fermion number (1.2) in the present case, and the $\kappa\to 0+$ limit yields the index of H

$$-\text{index}(H)=[T(T+1)-\alpha(\alpha+1)]N \ , \tag{5.16}$$

where α is the largest value of j which is less than m/φ. The result (5.16) agrees with that found by Callias.[14]

VI. A (2 + 1)-DIMENSIONAL SUPERCONDUCTOR MODEL

In our previous examples the spectral asymmetry $\eta_\kappa(s)$ has arisen either from the anomaly term of the surface term in (2.30). We shall now consider an example where both the anomaly term and the surface term are present. This can only happen in an open even-dimensional space since in odd dimensions anomalies are absent.

The spectral asymmetry that we shall consider arises from a (2 + 1)-dimensional, two-component fermion field interacting with background gauge and Higgs fields according to the Lagrangian[15,21]

$$L=\bar\psi(i\partial\!\!\!/-eA\!\!\!/)\psi-\frac{i}{2}g\bar\psi\varphi\psi^c+\frac{i}{2}g\bar\psi^c\varphi^*\psi \ , \tag{6.1}$$

where $\psi_i^c=C_{ij}\bar\psi_j$ with C the charge-conjugation matrix. The Dirac equation that follows is

$$i\partial_0\psi=-i\alpha^k\partial_k\psi-e\alpha^kA_k\psi-g\varphi\sigma^2\psi^* \tag{6.2}$$

with α^k ($k=1,2$) being the pair of Pauli matrices (σ^1,σ^2). For static solutions this reduces to

$$H\psi=\begin{bmatrix}0 & D\\ D^\dagger & 0\end{bmatrix}\begin{bmatrix}\psi_+\\ \psi_-\end{bmatrix}=0 \ , \tag{6.3}$$

where ψ_+ and ψ_- are now real spinors and

$$D=(-\partial_1+i\sigma^2\partial_2)+e(-A_2-i\sigma^2A_1)+g(-\sigma^3\varphi_1-\sigma^1\varphi_2) \ , \tag{6.4}$$

$D^\dagger$ is the Hermitian conjugate of D, and φ_1 and φ_2 are the real and imaginary parts of the Higgs field. We define

$$H_\kappa=H+\Gamma^c\kappa \ , \tag{6.5}$$

$$\Gamma^c=\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix} , \tag{6.6}$$

and we wish to calculate the spectral asymmetry $\eta_\kappa(s)$ of H_κ. For this we first introduce the following 4×4 matrices:

$$\Gamma^1=\begin{bmatrix}0 & i\\ -i & 0\end{bmatrix} , \quad \Gamma^2=\begin{bmatrix}0 & \sigma^2\\ \sigma^2 & 0\end{bmatrix} ,$$
$$\Gamma^3=\begin{bmatrix}0 & -\sigma^3\\ -\sigma^3 & 0\end{bmatrix} , \quad \Gamma^4=\begin{bmatrix}0 & -\sigma^1\\ -\sigma^1 & 0\end{bmatrix} . \tag{6.7}$$

These matrices satisfy the algebra (2.7) with

$$\Gamma^c=-\Gamma^1\Gamma^2\Gamma^3\Gamma^4 \ . \tag{6.8}$$

We also define the chirality operator

$$\hat\Gamma=\Gamma^1\Gamma^2=\begin{bmatrix}i\sigma_2 & 0\\ 0 & -i\sigma_2\end{bmatrix} . \tag{6.9}$$

With these definitions we then have

$$H_\kappa=H+\kappa\Gamma^c$$
$$=i\Gamma^1\partial_1+i\Gamma^2\partial_2+i\Gamma^c\epsilon^{ij}\Gamma^ieA_j+g\Phi+\kappa\Gamma^c \ , \tag{6.10}$$

where

$$\Phi=\Gamma^3\varphi_1+\Gamma^4\varphi_2 \ . \tag{6.11}$$

Consider now the trace identity (2.27). Since H_κ is a two-dimensional operator with background gauge fields we can expect an anomaly contribution to arise in the $x \to y$ limit. Furthermore, since background Higgs fields are also present we expect that the Higgs fields give a nonanomalous contribution in (2.30).

We shall now evaluate the anomaly contribution. The anomaly cannot depend on the Higgs field, hence it is enough to consider the operator

$$\hat{H}=i\partial\!\!\!/+i\Gamma^c\epsilon^{ij}\Gamma^i eA_j \ . \tag{6.12}$$

We first write

$$\mathrm{tr}\left\langle x \left| \Gamma^c \frac{1}{\hat{H}+i\sigma} \right| x \right\rangle = -i\sigma\,\mathrm{tr}\left\langle x \left| \Gamma^c \frac{1}{\hat{H}^2+\sigma^2} \right| x \right\rangle , \tag{6.13}$$

where

$$\hat{H}^2=-\partial^2+e\Gamma^c F^{12}-e\Gamma^c\hat{\Gamma}\partial^i A^i -2e\Gamma^c\hat{\Gamma}A^i\partial^i+e^2A_i^2 \ . \tag{6.14}$$

We then use in (6.13) the identity

$$\Gamma^c=\tfrac{1}{2}(1+\Gamma^c)-\tfrac{1}{2}(1-\Gamma^c) \tag{6.15}$$

to obtain

$$-i\sigma\,\mathrm{tr}\left\langle x \left| \frac{1}{-\partial^2+eF^{12}-ie\sigma_2\partial^iA^i-2ie\sigma_2A^i\partial^i+e^2A_i^2+\sigma^2} - \frac{1}{-\partial^2-eF^{12}-ie\sigma_2\partial^iA^i-2ie\sigma_2A^i\partial^i+e^2A_i^2+\sigma^2} \right| x \right\rangle . \tag{6.16}$$

We now use the identity

$$1=\tfrac{1}{2}(1+\sigma_2)+\tfrac{1}{2}(1-\sigma_2) \tag{6.17}$$

to find

$$\mathrm{tr}\left\langle x \left| \Gamma^c \frac{1}{\hat{H}+i\sigma} \right| x \right\rangle = \mathrm{tr}\left\langle x \left| \gamma^5 \frac{1}{i\partial\!\!\!/+eA\!\!\!/+i\sigma} \right| x \right\rangle - \mathrm{tr}\left\langle x \left| \gamma^5 \frac{1}{i\partial\!\!\!/-eA\!\!\!/+i\sigma} \right| x \right\rangle , \tag{6.18}$$

where $i\partial\!\!\!/=i\gamma^1\partial_1+i\gamma^2\partial_2$ and $\gamma^1,\gamma^2,\gamma^5=\gamma^1\gamma^2$ is a 2×2 representation of the two-dimensional Dirac algebra. The two-dimensional axial anomaly equation[18]

$$\frac{i}{2}\partial^i\,\mathrm{tr}\left\langle x \left| \gamma^i\gamma^5 \frac{1}{i\partial\!\!\!/m\pm eA\!\!\!/+i\sigma} \right| x \right\rangle = i\sigma\,\mathrm{tr}\left\langle x \left| \gamma^5 \frac{1}{i\partial\!\!\!/\pm eA\!\!\!/+i\sigma} \right| x \right\rangle \mp \frac{e}{4\pi}\epsilon^{ij}F^{ij} \tag{6.19}$$

together with (6.18) gives finally

$$\mathrm{tr}\left\langle x \left| \Gamma^c \frac{1}{\hat{H}+i\sigma} \right| x \right\rangle = \frac{1}{2\sigma}\partial^i\,\mathrm{tr}\left\langle x \left| \frac{1}{i\partial\!\!\!/+eA\!\!\!/+i\sigma} - \frac{1}{i\partial\!\!\!/-eA\!\!\!/+i\sigma} \right| x \right\rangle - i\frac{e}{2\pi\sigma}\epsilon^{ij}F^{ij} . \tag{6.20}$$

The anomaly contribution to $\eta_\kappa(s)$ is obtained by substituting (6.20) into (2.23) and evaluating (2.30). The first term on the right-hand side of (6.20) does not contribute and we get

$$\frac{e}{2\pi}\cos\left[s\frac{\pi}{2}\right]\kappa^{-s}\int d^2x\,\epsilon^{ij}F^{ij} \ . \tag{6.21}$$

We shall now evaluate the nonanomalous contribution to the spectral asymmetry $\eta_\kappa(s)$. For this we consider

$$\mathrm{tr}\left\langle x \left| \Gamma^i\Gamma^c \frac{1}{H+i\sigma} \right| x \right\rangle \tag{6.22}$$

which now appears in the surface integral of (2.30). It is sufficient to evaluate (6.22) in an asymptotic expansion where we only keep terms of order $1/|x|$. For a vortexlike background we can assume the following asymptotic behavior for $|x|\to\infty$:

$$\Phi(x)\simeq O(1) \ , \quad A^i(x)\simeq O\left[\frac{1}{|x|}\right] , \tag{6.23}$$

and $\varphi^2=\varphi_1{}^2+\varphi_2{}^2$ approaches a constant exponentially while F^{ik} vanishes exponentially. We obtain

$$\frac{1}{H+i\sigma}=-ig\frac{1}{-\partial^2+g^2\varphi^2+\sigma^2}\partial\!\!\!/\Phi\frac{1}{-\partial^2+g^2\varphi^2+\sigma^2}g\Phi -2ig\frac{1}{-\partial^2+g^2\varphi^2+\sigma^2}\Gamma^c\epsilon^{ij}\Gamma^i eA^j\Phi\frac{1}{-\partial^2+g^2\varphi^2+\sigma^2}g\Phi+\cdots , \tag{6.24}$$

where we have deleted terms that vanish faster than $1/|x|$ as $|x|\to\infty$ or that have already been accounted for in (6.21). We substitute (6.24) into (6.22) and evaluate the trace over Γ matrices. Retaining terms of order $1/|x|$ we have

$$\mathrm{tr}\left\langle x\left|\Gamma^i\Gamma^c\frac{1}{H+i\sigma}\right|x\right\rangle=-4ig\epsilon^{ij}\epsilon^{ab}(D^j\varphi)^a\varphi^b\left\langle x\left|\frac{1}{(-\partial^2+g^2\varphi^2+\sigma^2)^2}\right|x\right\rangle, \tag{6.25}$$

where

$$(D^i\varphi)^a=\partial^i\varphi^a+2eA^i\epsilon^{ab}\varphi^b \tag{6.26}$$

is the covariant derivative of the Higgs field. (Recall that the charge of the scalar is twice that of the fermion.) We substitute (6.25) into (2.30) and evaluate the remaining integrals. Combining the result with (6.21) we then get for the spectral asymmetry $\eta_\kappa(s)$ the expression

$$\begin{aligned}\eta_\kappa(x)=&\frac{e}{2\pi}\cos\left[s\frac{\pi}{2}\right]\kappa^{-s}\int d^2x\,\epsilon^{ij}F^{ij}\\&-\frac{g^2}{2\pi^2}\cos\left[s\frac{\pi}{2}\right]\frac{\kappa^{-s}}{g^2\varphi^2+\kappa^2}B\left[\frac{1-s}{2},\frac{3+s}{2}\right]{}_2F_1\left[1,\frac{1-s}{2};2;\frac{g^2\varphi^2}{g^2\varphi^2+\kappa^2}\right]\oint dl_i\epsilon^{ab}(D^i\varphi)^a\varphi^b\,.\end{aligned} \tag{6.27}$$

In the $s\to0+$ limit this yields

$$\eta_\kappa=\frac{e}{2\pi}\int d^2x\,\epsilon^{ij}F^{ij}-\frac{g^2}{2\pi}\frac{1}{g^2\varphi^2+\kappa^2-\kappa(g^2\varphi^2+\kappa^2)^{1/2}}\oint dl_i\epsilon^{ab}(D^i\varphi)^a\varphi^b \tag{6.28}$$

and in the $\kappa\to0+$ we obtain

$$\eta=\frac{e}{2\pi}\int d^2x\,\epsilon^{ij}F^{ij}-\frac{1}{2\pi}\oint dl_i\frac{\epsilon^{ab}(D^i\varphi)^a\varphi^b}{\varphi^2} \tag{6.29}$$

the index of the operator H. This result agrees with that found in Refs. 15 and 21.

VII. DISCUSSION

We have derived a general expression (2.30) for the spectral asymmetry $\eta(s)$ of a class of Dirac operators, and used our result to evaluate this spectral asymmetry in various field-theoretical models.

The spectral asymmetry $\eta(s)$ is closely related to the Riemann ζ function of the Dirac operator, and in the $s\to0$ limit $\eta(s)$ provides a natural generalization of the concept of an index for a Dirac operator; in a particular limit where conjugation symmetry retains, the $s\to0$ limit of our formula (2.30) reproduces various well-known index theorems such as the Atiyah-Singer index theorem and the Callias index theorem.

The examples that we have discussed have several common features. While the integrand of the surface integral in (2.30) is in general a nonlocal quantity, only a local part survives in the large-R limit and contributes to the surface integral. Consequently the evaluation of the surface integral is quite straightforward. In some models the integrand of the surface integral has an *a priori* divergent part, but for all models that we have discussed this divergent part does not survive the traces over the Γ matrices and the internal-symmetry matrices. Hence the only divergent quantity we actually encounter is the one arising from the axial anomaly equation in the $y\to x$ limit, but this divergence vanishes when we insist on gauge invariance.

Even though our analysis is restricted to a particular class of Dirac operators, we expect that some of the qualitative features persist for more general operators. In particular we conjecture that the spectral asymmetry quite generally relates, in addition to the explicit coupling constants in the Hamiltonian, only to the global homotopy properties of the background fields.

It would be very interesting to extend our analysis to gravitational backgrounds. The connection between the spectral asymmetry of a Dirac Hamiltonian and the fermion number of the background would then reveal the existence of gravitational objects with a nontrivial fermion number.

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