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New approach to unified gauge theories

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A new approach to unified gauge theories, based upon "generalized Dirac" equations, is developed. Consideration of the symmetries of such equations is shown to lead to a general framework within which we may construct unified gauge theories of elementary particles. A symmetry-breaking mechanism is given which incorporates general relativity into this framework.

INTRODUCTION

In recent times the belief has grown among physicists that underlying all physical laws there are certain universal symmetries whose appearance is masked by "symmetry breaking" (nonzero vacuum expectation values, Higgs mechanisms, and so on). It is thought that at very high energy densities (in the very early stages of the Universe, for example), all fields become massless and the "hidden" symmetries reveal their true universal nature. Such grand unified theories[1] are realized, mathematically, as gauge theories for some compact gauge group. Which group one chooses in constructing such a theory has been, to a large extent, a matter of personal preference. The character of such models is determined by essentially two factors, their fermion content (leptons, quarks, and so on) with associated group transformation properties and their symmetry-breaking mechanisms.

In the present work we will restrict ourselves to classical theory—we deal only with the classical (first quantized) Dirac equation and its generalizations. We find that a noninteracting system represented by such a generalized Dirac equation is characterized by a certain (usually noncompact) global invariance group. A local gauge theory can then be constructed with the gauge group being a subgroup, at each point in space-time, of the global invariance group.

Before presenting the general model, we first study, in Sec. I, what may be considered the prototype for such models, the Dirac equation. In Sec. II we examine the symmetries of the general model, Sec. III gives a symmetry-breaking mechanism which incorporates gravitation, and finally in Sec. IV we look at an example.

I. THE DIRAC EQUATION[2]

We wish to discuss the Dirac equation in terms which are applicable to the curved space-times of general relativity. Consequently, the initial part of this section will be peppered with the language of differential geometry. The equations which we write down, however, should be familiar to readers not acquainted with the obsessions of relativists.

Consider a space-time possessing a bispinor structure,[3] so that the Dirac equation may be defined throughout the manifold. In a coordinate basis we have

$$\gamma^\alpha\psi_{;\alpha}+im\psi=0\ , \tag{1.1}$$

where α $(=0,1,2,3)$ are coordinate indices (summation convention assumed). The semicolon indicates space-time covariant derivative. The electron rest mass is m (units $\hbar=c=1$). ψ is a section of the bispinor bundle—a four-complex-dimensional vector bundle, with bundle group Pin(1,3) the twofold cover of the Lorentz group. The γ^α are the curved—space-time Dirac matrices satisfying

$$\gamma_\alpha\gamma_\beta+\gamma_\beta\gamma_\alpha=2g_{\alpha\beta}1 \tag{1.2}$$

with $g_{\alpha\beta}$ the space-time metric [signature $(+1,-1,-1,-1)$].

If the space-time is also space- and time-orientable,[4] then the bundle of bispinors can be decomposed into a direct sum of the two-spinor bundle[5] and the conjugate dual two-spinor bundle. This decomposition takes place under the action of the identity-connected component of Pin(1,3), $\mathrm{Spin}_+(1,3)$, which is isomorphic to the group $SL(2,C)$, the double cover of the proper orthochronous Lorentz group. In flat space-time (special relativity) this decomposition is the γ_5-diagonal representation,[6] which leads to the van der Waerden decomposition of the Dirac equation.[7] In our curved space-times the decomposition is (using the two-spinor notation popularized by Penrose[8])

$$\psi=\begin{bmatrix}u_A\\ \bar{v}^{\dot{B}}\end{bmatrix}\ , \tag{1.3}$$

where A and B are two-spinor indices $=0,1$, u_A is a section of the two-spinor bundle, and $\bar{v}^{\dot{B}}$ a section of the conjugate dual two-spinor bundle. The Dirac matrices are

$$\gamma^\alpha=\sqrt{2}\begin{bmatrix}0 & \sigma^\alpha{}_{B\dot{B}}\\ \sigma^{\alpha A\dot{A}} & 0\end{bmatrix}\ , \tag{1.4}$$

where $\sigma^\alpha{}_{B\dot{B}}$ are the van der Waerden symbols[8] (in flat space-time $\sigma^\alpha{}_{B\dot{B}}$ are, for $\alpha=1,2,3$, $1/\sqrt{2}$ times the Pauli matrices) satisfying

$$\begin{aligned}&\sigma^{\alpha A\dot{A}}\sigma^\beta{}_{A\dot{A}}=g^{\alpha\beta}\quad(\text{summation on } A \text{ and } \dot{A})\ ,\\ &\sigma^\alpha{}_{A\dot{A}}\sigma_{\alpha B\dot{B}}=\epsilon_{AB}\epsilon_{\dot{A}\dot{B}}\ ,\\ &\sigma^\alpha{}_{A\dot{A};B}=0\ .\end{aligned} \tag{1.5}$$

Here, ϵ_{AB} is the 2×2 Levi-Civita symbol with inverse ϵ^{AB} ($\epsilon^{01}=\epsilon_{01}=1$, $\epsilon^{10}=\epsilon_{10}=-1$, and $\epsilon^{00}=\epsilon^{11}=\epsilon_{00}=\epsilon_{11}=0$). The ϵ_{AB} and ϵ^{AB} are used to raise and lower indices according to the prescription $\epsilon^{AB}\xi_B=\xi^A$ and $\epsilon_{AB}\xi^A=\xi_B$.[8]

With (1.3), (1.4), and (1.5), the Dirac equation (1.1) splits into a pair of two-spinor equations

$$u_A{}^{;A\dot{A}}+\frac{im}{\sqrt{2}}\bar{v}^{\dot{A}}=0\ ,$$
$$v_A{}^{;A\dot{A}}+\frac{im}{\sqrt{2}}\bar{u}^{\dot{A}}=0\ , \tag{1.6}$$

where $\xi_A{}^{;B\dot{B}}\equiv\xi_{A;\alpha}\sigma^{\alpha B\dot{B}}$.

As is well known, associated to any Dirac field there is a conserved current, the Dirac probability current. In our two-spinor notation this vector is

$$j^\alpha=\sigma^\alpha{}_{A\dot{A}}(u^A\bar{u}^{\dot{A}}+v^A\bar{v}^{\dot{A}})\ . \tag{1.7}$$

It is a simple matter to check that j^α is conserved, i.e., $j^\alpha{}_{;\alpha}=0$. The question is are there any other conserved currents? The answer is yes.[2] The complex vector s^α, given by

$$s^\alpha=\sigma^\alpha{}_{A\dot{A}}u^A\bar{v}^{\dot{A}}\ , \tag{1.8}$$

is divergence-free.

The three real vectors j^α, $s^\alpha+\bar{s}^\alpha$, and $i(\bar{s}^\alpha-s^\alpha)$ are, in fact, the only divergence-free vectors which can be constructed bilinearly from the Dirac field. The vector s^α does not seem to rate mention in any of the standard texts on the Dirac equation[9] so it is perhaps worth rewriting it in terms of the Dirac bispinor. The expression is

$$s^\alpha=\frac{1}{2\sqrt{2}}\psi_C^\dagger\gamma^\alpha\psi\ ,$$

where

$$\psi=\begin{bmatrix}u_A\\ \bar{v}^{\dot{B}}\end{bmatrix}$$

and the dagger indicates Dirac adjoint $\psi^\dagger=(v^A,\bar{u}_{\dot{B}})$, ψ_C is the charge-conjugate bispinor

$$\psi_C=C\bar{\psi}=\begin{bmatrix}v_A\\ \bar{u}^{\dot{B}}\end{bmatrix}$$

with the charge-conjugation matrix C given by

$$C=\begin{bmatrix}0 & \epsilon_{BA}\\ \epsilon^{\dot{B}\dot{A}} & 0\end{bmatrix}.$$

The fact that the Dirac equation possesses three real conserved currents suggests that the Dirac equation possesses a three-parameter symmetry group. It is a simple matter to check that the Lagrangian[10] for (1.6),

$$L=i(u_A{}^{;A\dot{A}}\bar{u}_{\dot{A}}-v_A{}^{;A\dot{A}}\bar{v}_{\dot{A}})+\frac{m}{\sqrt{2}}(u^Av_A+\bar{u}^{\dot{A}}\bar{v}_{\dot{A}})\ , \tag{1.9}$$

is invariant under the group of transformations given by

$$\begin{bmatrix}u_A\\ v_A\end{bmatrix}\rightarrow\begin{bmatrix}a & b\\ \bar{b} & \bar{a}\end{bmatrix}\begin{bmatrix}u_A\\ v_A\end{bmatrix},\quad a,b\in C,\quad a\bar{a}-b\bar{b}=1\ .$$

Such transformations give a matrix representation of the group SU(1,1). As we noted in a previous paper,[2] the electromagnetic U(1) is a subgroup of this SU(1,1) symmetry group which leads to the quantization of charge for fields obeying the Dirac equation. The U(1) [maximal compact subgroup of SU(1,1)] transformations are

$$\psi\rightarrow e^{i\theta}\psi$$

or

$$\begin{bmatrix}u_A\\ v_A\end{bmatrix}\rightarrow\begin{bmatrix}e^{i\theta} & 0\\ 0 & e^{-i\theta}\end{bmatrix}\begin{bmatrix}u_A\\ v_A\end{bmatrix},\quad \theta\text{ real}\ .$$

If we now attempt to construct a local gauge theory based on the Dirac equation, we immediately come upon a problem, the noncompactness of SU(1,1). Any gauge theory based upon a noncompact group will encounter problems with negative energy densities and indefinite probabilities. These problems appear to vanish when we go to second-quantized theory, the anticommutation relations for the quantum Dirac field seem to destroy the SU(1,1) invariance.[9] Second quantization, it seems, singles out the maximal compact subgroup U(1) of SU(1,1) as the gauge group.

II. GENERALIZED DIRAC EQUATIONS

In our γ_5-diagonal representation the Dirac bispinor

$$\psi=\begin{bmatrix}u_A\\ \bar{v}^{\dot{B}}\end{bmatrix}$$

can be decomposed into its left- and right-handed parts as

$$\begin{bmatrix}u_A\\ 0\end{bmatrix}$$

and

$$\begin{bmatrix}0\\ \bar{v}^{\dot{B}}\end{bmatrix},$$

respectively, so that the two-spinor representing the left-handed part of ψ, u_A, appears in the Lagrangian (1.9) with a positive sign and the two-spinor representing the right-handed part, v_A, appears with a negative sign. In the following we will use the following definition of handedness.

Definition. A two-spinor field u_A will be referred to as left-handed if its kinetic energy term $(iu_A{}^{;A\dot{A}}\bar{u}_{\dot{A}})$ appears in the Lagrangian with a positive sign, and right-handed if it appears with a negative sign.

Now the bispinors representing any fermion field can always be decomposed into a set of two-spinor fields. For example, a lepton is given by one left-handed and one right-handed two-spinor (Sec. I); or a quark, as a triplet of bispinors, can be decomposed into six two-spinors—three left-handed and three right-handed. This means (if we accept the current understanding of particle physics) that all elementary particles can be described either as a set of interacting two-spinor fields or as the gauge particles associated with such interactions. If the interactions were "switched off," we would simply be left with a set of Weyl neutrino fields (mass being neglected). In the massless limit the free Dirac field consists of a pair of Weyl neutrino fields.

Let us consider a noninteracting system in its massless limit. The system will be described by n Weyl neutrino fields u_{aA} $(a=1,2,\ldots,n)$, p of which are left-handed and q of which are right-handed, $p+q=n$. These fields satisfy the Weyl neutrino equation

$$u_{aA}{}^{;A\dot{A}}=0 \; .$$

The Lagrangian will be

$$L_F=i\left[\sum_{a=1}^{p}\bar{u}_{\dot{a}\dot{A}}u_{aA}{}^{;A\dot{A}}-\sum_{a=p+1}^{n}\bar{u}_{\dot{a}\dot{A}}u_{aA}{}^{;A\dot{A}}\right] . \tag{2.1}$$

Clearly the invariance group (of linear transformations acting on the u_{aA}) of L_F is just $U(p,q)$—the group of linear transformations preserving the pseudo-Hermitian form

$$\sum_{a=1}^{p} z_a\bar{z}_{\dot{a}}-\sum_{a=p+1}^{n} z_a\bar{z}_{\dot{a}}$$

with $(z_a)\in C^n$.

The problem now is to construct a gauge theory for the interacting system based upon the Lagrangian (2.1). We known that at any point of the space-time the gauge group G must be a subgroup of the global invariance group $U(p,q)$. In a second-quantized theory the currents associated with the noncompact generators of $U(p,q)$ would vanish identically—the noncompact degrees of freedom mix left- and right-handed fields. This would seem to suggest that the largest possible gauge group is just the maximal compact subgroup of $U(p,q)$. Choosing the maximal compact subgroup as the gauge group has considerable aesthetic appeal. Interactions of the system would proceed via what might be called an elementary particle "equivalence principle": all global symmetries of the noninteracting system become local gauge symmetries of the interacting (quantum) system. This is a direct analogy of Einstein's equivalence principle: In the presence of gravitational interactions, global Lorentz invariance of a system becomes local Lorentz invariance (invariance of the system under Lorentz gauge transformations of the tetrad[11]).

The interacting system for (2.1) will be described, mathematically, by an associated vector bundle of two-spinor n-tuples (u_{aA}), with bundle group G, over the space-time manifold. The covariant derivative may be written as

$$\nabla_\alpha u_{aA}=u_{aA;\alpha}-\phi^b{}_{a\alpha}u_{bA} \; , \tag{2.2}$$

where the connection matrices $\phi^b{}_{a\alpha}$ take their values in the Lie algebra of G, and so we may write, as usual,

$$\phi^b{}_{a\alpha}=i\frac{g}{2}T^b{}_{aj}A^j{}_\alpha \; , \tag{2.3}$$

where g is the coupling constant, $A^j{}_\alpha$ the (real) gauge potentials, and $T_j=(T^b{}_{aj})$, $j=1,2,\ldots,m\le p^2+q^2$, are the generators of G in a basis compatible with the $U(p,q)$ invariance of (2.1) and appropriately normalized.

We introduce an invariant pseudo-Hermitian metric $H_{\dot{a}a}$ (inverse $H^{a\dot{a}}$) and pseudo-Hermitian conjugate of u_{aA} as follows:

$$(H_{\dot{a}a})=\mathrm{diag}(+1,+1,\ldots,+1,-1,-1,\ldots,-1) \; , \tag{2.4}$$

where there are p "+ 1" entries and q "−1" entries, and $\bar{u}^a{}_{\dot{A}}=H^{a\dot{a}}\bar{u}_{\dot{a}\dot{A}}$ (summation on $\dot{a}$) ($\dot{a}$ means the index transforms as the complex conjugate).

With (2.4) the generators T_j must satisfy [remember G is a subgroup of $U(p,q)$]

$$H_{\dot{a}b}T^b{}_{aj}-T^{\dot{b}}{}_{\dot{a}j}H_{\dot{b}a}=0 \; . \tag{2.5}$$

As usual, the field intensities (components of the curvature) may be written as

$$F^j{}_{\mu\nu}=A^j{}_{\nu;\mu}-A^j{}_{\mu;\nu}+\frac{g}{2}C^j{}_{kl}A^k{}_\mu A^l{}_\nu \tag{2.6}$$

with $C^j{}_{kl}$ the structure constants of G in the given basis.

The fully gauge-covariant Lagrangian for the fermion fields is now

$$L_F=i\bar{u}^a{}_{\dot{A}}\nabla^{A\dot{A}}u_{aA} \quad \text{(summation on all repeated indices)}$$

$$=i\bar{u}^a{}_{\dot{A}}u_{aA}{}^{;A\dot{A}}+\frac{g}{2}(\sigma^{\alpha A\dot{A}}\bar{u}^a{}_{\dot{A}}u_{bA}T^b{}_{aj})A^j{}_\alpha \; . \tag{2.7}$$

The fermions, described by (2.7) alone, are massless. To give them mass we need to introduce a symmetry-breaking mechanism, which is our next topic of discussion.

III. SYMMETRY BREAKING

The model developed thus far is incomplete in three ways: (a) all fermions are massless, (b) all gauge bosons are massless, and (c) gravitation has not been included (although all our expressions are space-time covariant).

First, in relation to (a), we note that the masslessness of the fermions implies that the Lagrange density $\mathscr{L}_F=L\sqrt{-g}$ $[g=\det(g_{\alpha\beta})]$ is conformally invariant. This is, under the conformal rescalings $g_{\alpha\beta}\to\Omega^2g_{\alpha\beta}$, $\sigma_\alpha{}^{A\dot{A}}\to\Omega\sigma_\alpha{}^{A\dot{A}}$, $\sigma^{\alpha A\dot{A}}\to\Omega^{-1}\sigma^{\alpha A\dot{A}}$, and $u_{aA}\to\Omega^{-3/2}u_{aA}$ with $\Omega=\Omega(x^\alpha)$, the density $\mathscr{L}_F$ is invariant.[12]

Second, a mass term for the fermions must arise via a coupling to the indices ab in the expression $u_a{}^Au_{bA}$, which is skew-symmetric in ab.

To take account of these two observations, we introduce a (Higgs) "mass-field" $M^{ab}=-M^{ba}$, which satisfies a conformally invariant wave equation

$$\Box M^{ab}+\tfrac{1}{6}RM^{ab}+\lambda(\bar{M}_{cd}M^{cd})M^{ab}=0 \; , \tag{3.1}$$

where $\Box=\nabla_\alpha\nabla^\alpha$ (∇^α the covariant derivative of Sec. II), $\bar{M}_{cd}=H_{\dot{c}c}H_{\dot{d}d}\bar{M}^{\dot{c}\dot{d}}$, R is the Ricci scalar (with the conventions

$$R^{\alpha}{}_{\eta\beta\gamma}=\left\{\begin{matrix}\alpha\\ \beta\eta\end{matrix}\right\}_{,\gamma}-\left\{\begin{matrix}\alpha\\ \eta\gamma\end{matrix}\right\}_{,\beta}+\cdots$$

and $R_{\beta\gamma}=R^{\alpha}{}_{\beta\alpha\gamma}$), and λ is a constant—included here for completeness and to bring out the startling resemblance to the usual Higgs fields.

The Lagrangian for (3.1) may be taken as

$$L_M=\tfrac{1}{2}[(V_\alpha M^{ab})(\nabla^\alpha \overline{M}_{ab})-\tfrac{1}{6}R(M^{ab}\overline{M}_{ab}) -\tfrac{1}{2}\lambda(M^{ab}\overline{M}_{ab})^2]\ . \tag{3.2}$$

The action for (3.2) is conformally invariant with $M^{ab}\to\Omega^{-1}M^{ab}$, the Lagrange density $\mathscr{L}_M=L_M\sqrt{-g}$ actually changes by a divergence

$$\frac{\partial}{\partial x^\alpha}\left[g^{\alpha\beta}\left[\frac{2\ln\Omega}{\partial x^\beta}\right]\overline{M}_{ab}M^{ab}\sqrt{-g}\right]\ .$$

A conformally invariant Lagrangian (for a scalar field) has been introduced by Zee[13] in order to combine Einstein's theory of gravitation with gauge theories via a Higgs-type mechanism. This is, of course, our aim here. It is also worth mentioning that a theory of gravitation based on (3.2) is of the same type as the "Machian" theory introduced by Hoyle and Narlikar[14]—though it must be emphasized that with symmetry breaking our theory will reduce to that of Einstein's.

We may also introduce a gravitational Lagrangian with terms quadratic in the Riemann tensor. (Such terms are introduced into the present theory only for the sake of completeness and with one eye on a future renormalization program.) However, because of our conformal invariance the only such possibility is $kC^{\alpha}{}_{\beta\gamma\delta}C^{\beta}{}_{\alpha}{}^{\gamma\delta}$, where $C^{\alpha}{}_{\beta\gamma\delta}$ is the Weyl conformal curvature tensor and k is a (dimensionless) coupling constant prescribing the strength of the gravitational interaction.

We now need to introduce some idea of symmetry breaking into our scheme. However, we cannot appeal to the idea of nonzero vacuum expectation values, as in our curved space-times the usual notions of quantum field theory may not be well defined.[15] Trautman[16] has given a general description of a Higgs field (valid in general space-times) and it is his approach we shall use here. The definition, in the present case, simply says that the range of M^{ab} is an orbit of the gauge group G in $V\wedge V$ [where V is the bundle of vectors (v^a)]. Symmetry breaking is now just a matter of choosing a particular gauge. The reason why nature should "break" conformal, global, and gauge invariance in this manner is undoubtedly due, in some way, to nonzero vacuum expectation values—but the present-day theory of quantum fields in curved space-times is not sufficiently refined for us to use this appealing physical explanation.

Given the above definition of M we can now write

$$M\equiv(M^{ab})=SIS^t\ , \tag{3.3}$$

where S is a matrix of functions which takes its values in G/k, k the isotropy subgroup of G in $V\wedge V$ at $I=(I^{ab})=-I^t$ a constant skew matrix. By performing a gauge transformation we can thus take $M=I$. There will, however, be more than one I (and more than one group k) by which symmetry breaking can take place.

After the symmetry breaking $M\to I$, the full Lagrangian of the theory will be

$$L=L_F+L_I+L_M+L_G+L_E\ , \tag{3.4}$$

where

$$L_F=i\overline{u}^{a}{}_{\dot A}\nabla^{A\dot A}u_{aA},\quad \text{as in (2.7)}\ ,$$

$$L_I=\tfrac{1}{2}c(I^{ab}u_a{}^A u_{bA}+\overline{I}^{\dot a\dot b}u_{\dot a}{}^{\dot A}u_{\dot b\dot A})\ ,$$

c a coupling constant ,

$$L_M=\tfrac{1}{2}[\tfrac{1}{2}g^2(T^d{}_{ak}T^a{}_{cj}I^{cb}\overline{I}_{db}+T^a{}_{cj}T^d{}_{bk}I^{cb}\overline{I}_{ad})A_\alpha{}^jA^{k\alpha} -\tfrac{1}{6}(I^{ab}\overline{I}_{ab})R-\tfrac{1}{2}\lambda(I^{ab}\overline{I}_{ab})^2]\ ,$$

$$L_G=-\tfrac{1}{4}F^j{}_{\mu\nu}F^{k\mu\nu}(T^b{}_{aj}T^a{}_{bk})\ ,$$

and

$$L_E=kC^{\alpha}{}_{\beta\gamma\delta}C^{\beta}{}_{\alpha}{}^{\gamma\delta}\ .$$

In this form the conformal invariance of the theory is broken, gauge invariance is reduced to the gauge group k, and global invariance is broken to the subgroup of $U(p,q)$ which leaves I invariant.

For gravitation to be "attractive" we must have $I^{ab}\overline{I}_{ab}<0$. To identify $-(I^{ab}\overline{I}_{ab})$ with the inverse gravitational constant we must have a complete theory. The larger the $U(p,q)$ and G become, the more contributions there will be from I^{ab} to $I^{ab}\overline{I}_{ab}$.

Two points should be emphasized. First, the theory of gravitation that results (ignoring the contributions from L_E) is just standard general relativity. Secondly, after symmetry breaking $M\to I$, all components of the Higgs field are constant, so there are no elementary scalar fields in the theory.

IV. AN EXAMPLE: THE WEAK INTERACTION

To construct a theory of the weak interactions we require one lepton (one right-handed two-spinor and one left-handed two-spinor) and one left-handed neutrino (one left-handed two-spinor). The global invariance group is U(2,1). We choose the maximal compact subgroup as gauge group, so $G=SU(2)\otimes U(1)$ which is, of course, the gauge group of the standard Glashow-Weinberg-Salam theory.[17]

The correct form of L_G is given when the generators $T^a{}_{bj}$ are appropriately normalized as generators of SU(2,1). We write

$$
(T^a{}_{bj}A^j{}_\alpha)=\begin{bmatrix} \frac{1}{\sqrt{2}}A^3_\alpha+\frac{1}{\sqrt{6}}A^4_\alpha & W^-_\alpha & 0 \\ W^+_\alpha & -\frac{1}{\sqrt{2}}A^3_\alpha+\frac{1}{\sqrt{6}}A^4_\alpha & 0 \\ 0 & 0 & -\frac{2}{\sqrt{6}}A^4_\alpha \end{bmatrix}, \tag{4.1}
$$

where $W^-_\alpha=\overline{W}^+_\alpha$.

The necessary symmetry breaking $SU(2)\otimes U(1)\rightarrow U^{em}(1)$ can be taken to be given by an I of the form

$$
I=\begin{bmatrix} 0 & 0 & v \\ 0 & 0 & 0 \\ -v & 0 & 0 \end{bmatrix}, \tag{4.2}
$$

where v can be taken as a real constant.

The Lagrangian L_F of (3.4) now takes the form

$$
L_F=i\bar{u}^a{}_{\dot{A}}u_{aA}{}^{;A\dot{A}}+\frac{g}{2}\left[\frac{1}{\sqrt{2}}A^3{}_\alpha\sigma^{\alpha A\dot{A}}(u_{1A}\bar{u}_{1\dot{A}}-u_{2A}\bar{u}_{2\dot{A}})+\frac{1}{\sqrt{6}}A^4{}_\alpha\sigma^{\alpha A\dot{A}}(u_{1A}\bar{u}_{1\dot{A}}+u_{2A}u_{2\dot{A}}+2u_{3A}\bar{u}_{3\dot{A}}) + W^-_\alpha\sigma^{\alpha A\dot{A}}\bar{u}_{1\dot{A}}u_{2A}+W^+_\alpha\sigma^{\alpha A\dot{A}}\bar{u}_{2\dot{A}}u_{1A}\right]. \tag{4.3}
$$

The theory is now formally the same as the standard theory in its particle content (except for the lack of Higgs scalars) if we make the following identifications:

electron bispinor: $\psi=\begin{bmatrix} u_{1A} \\ \bar{u}_3{}^{\dot{B}} \end{bmatrix}$,

left-handed neutrino: u_{2A} ,

boson of weak hypercharge: A^4_α .

The Weinberg angle[17] θ_W is given by

$$
\sin^2\theta_W=\frac{\left[\frac{1}{\sqrt{6}}g\right]^2}{\left[\frac{1}{\sqrt{6}}g\right]^2+\left[\frac{1}{\sqrt{2}}g\right]^2}=\frac{1}{4}\,.
$$

This is not a bad estimate considering that we have an incomplete theory (we have not included the strong interactions) and have not included any renormalization effects. The incompleteness of the theory means that we cannot identify $-\frac{1}{12}\bar{I}_{ab}I^{ab}=v^2/6$ with the inverse gravitational constant (the squared Planck mass in units $\hbar=c=1$).

CONCLUSIONS

The aim of this work was to put forward a coherent framework within which we could construct unified gauge theories. The underlying assumption of the work is the equivalence principle of Sec. II: global symmetries of the noninteracting (quantum?) system become local gauge symmetries of the interacting system. Clearly a more thorough investigation of the relationship between the classical symmetry groups $U(p,q)$ and the quantum symmetry groups (as the maximal compact subgroups) $S(U(p)\otimes U(q))$ (essentially the chiral groups) is required.

To construct a grand unified theory within this framework requires a considerable enlargement of the group structure. For example, to include just one massive lepton, one neutrino, and two quarks requires the global symmetry group U(8,7). Also, if we wish to get the correct gravitational constant, then some of the gauge bosons must acquire a mass of the order of e times the Planck mass.

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