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Search for higher-dimensional cosmologies

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Cosmologies in six-dimensional Einstein-Maxwell theory and Englert's solution for eleven-dimensional supergravity theory are studied. Spontaneous compactification is shown to occur in which the three-space has preferred expansion over the extra dimensions. An effective four-dimensional cosmological constant is calculated for each possible solution.

I. INTRODUCTION

If our four-dimensional universe has its roots in a space of extra dimensions[1] then cosmology is the ideal place to look for consistency. The isotropy in the three spatial dimensions suggests that the extra dimensions may also have evolved isotropically before compactification into the four and $D-4$ dimensions. We assume that the evolution of the D dimensions before compactification occurred from a common radius. The natural scale of gravity in four dimensions is the Plank mass M_P. After compactification, the size of the extra dimensions is expected to be of order $1/M_P$. In this paper we examine the cosmological solutions in the six-dimensional Einstein-Maxwell theory with a monopole in the extra two dimensions[2] and in the eleven-dimensional supergravity theory with the fourth-rank antisymmetric field required by supersymmetry taking values in the extra seven dimensions.[3] We assume the metric to be diagonal and time dependent in $D-1$ dimensions. Furthermore, it is taken to be of the Robertson-Walker form in the $1+3$ and $D-4$ dimensions,

$$g_{MN}=\begin{bmatrix} -1 & 0 & 0 \\ 0 & {R_3}^2(t)\tilde{g}_{ij}(x^i) & 0 \\ 0 & 0 & {R_{D-4}}^2(t)\tilde{g}_{mn}(y^p) \end{bmatrix}, \quad (1)$$

where $M,N=0,\ldots,D-1$; $i,j=1,2,3$; $m,n=4,\ldots,D-1$; and $\tilde{g}_{ij}$ and $\tilde{g}_{mn}$ are the maximally symmetric metrics for the three- and $(D-4)$-dimensional spaces. This form of the metric gives the following components for the Ricci tensor R_{MN}:[4]

$$R_{00}=3\frac{\ddot{R}_3}{R_3}+(D-4)\frac{\ddot{R}_{D-4}}{R_{D-4}},$$

$$R_{ij}=-\left[\frac{2\kappa_3}{{R_3}^2}+\frac{d}{dt}\left[\frac{\dot{R}_3}{R_3}\right]+\left[3\frac{\dot{R}_3}{R_3}+(D-4)\frac{\dot{R}_{D-4}}{R_{D-4}}\right]\frac{\dot{R}_3}{R_3}\right]g_{ij}, \quad (2)$$

$$R_{mn}=-\left[\frac{(D-5)\kappa_{D-4}}{{R_{D-4}}^2}+\frac{d}{dt}\left[\frac{\dot{R}_{D-4}}{R_{D-4}}\right]+\left[3\frac{\dot{R}_3}{R_3}+(D-4)\frac{\dot{R}_{D-4}}{R_{D-4}}\right]\frac{\dot{R}_{D-4}}{R_{D-4}}\right]g_{mn}.$$

Our interest in the six-dimensional Einstein-Maxwell model is based on the fact that, although it is not a realistic model (as remarked by the authors), it has several features that are of interest for any realistic theory to be developed. In particular, the coupling of the SU(2) Yang-Mills four-vector to massless fermions is chiral, due to the presence of the magnetic monopole that automatically circumvents the Atiyah-Hirzebruch theorem as recently emphasized by Witten.[5]

II. SIX-DIMENSIONAL EINSTEIN-MAXWELL THEORY

The six-dimensional Einstein-Maxwell action including a cosmological constant is given by

$$S=-\int d^6z(-g^{(6)})^{1/2}\left[\frac{R}{\kappa^2}+\tfrac{1}{4}F_{MN}F^{MN}+\lambda\right], \quad (3)$$

where $M,N=0,1,2,3,5,6$; $\mu,\nu=0,\ldots,3$; $m,n=5,6$, and the signature is $(-+++;++)$. Then Z^M is given by $Z^M=(x^\mu,y^m)$. R, λ, and κ are the six-dimensional curvature scalar, cosmological constant, and gravitational coupling constant, respectively, and the Maxwell field is defined by $F_{MN}=\partial_M A_N-\partial_N A_M$.

The field equations derived from this action are

$$R_{MN}-\tfrac{1}{2}g_{MN}R=-\frac{\kappa^2}{2}(T_{MN}-\lambda g_{MN}), \quad (4)$$

$$\frac{1}{(-g^{(6)})^{1/2}}\partial_M[(-g^{(6)})^{1/2}F^{MN}]=F^{MN}{}_{;M}=0, \quad (5)$$

where the semicolon denotes covariant differentiation with respect to our ground-state metric and the energy-momentum tensor T_{MN} is defined as $T_{MN}=F_{ML}F^L{}_N-\tfrac{1}{4}g_{MN}F^2$.

We take polar coordinates in the two-sphere with $g_{55}={R_2}^2$, $g_{56}=0$, $g_{66}={R_2}^2\sin^2\theta$, where $x^5=\theta$, and $x^6=\phi$ as usual. If the magnetic monopole potential (taken, of course, to be rotation invariant) is given by[2] $A_\mu(y)dy=(m/2e)(\cos\theta\mp 1)d\phi$, where m is a positive integer and e is the Maxwell-field coupling constant, we can show that the energy-momentum tensor components are

$$T_{MN}=\epsilon\frac{m^2}{8e^2R_2{}^4}g_{MN}, \quad \text{with } \epsilon=\begin{cases}-1 & \text{if } M,N=0,\ldots,3\\ +1 & \text{if } M,N=5,6 .\end{cases}$$

Before we start to consider cosmological solutions to this model, it is worthwhile to mention that, in our notation, a compact space should have a negative curvature scalar. Following Freund and Rubin,[6] we see that the presence of a rank-two antisymmetric tensor (the Maxwell field) will preferentially split the six-dimensional space-time into the desired product of a four-dimensional and a two-dimensional space with opposite signs for the curvature scalar. As we, of course, want the time to be in the four-dimensional space, the two-dimensional one should be compact in order to avoid closed time loops in the ordinary space-time. If we contract Einstein's equations (4), with the value of the energy-momentum tensor given above, we find that the four- and two-dimensional curvature scalars are given by

$$R_4=\kappa^2\left[\frac{m^2}{8e^2R_2{}^4}-\lambda\right], \quad R_2=-\frac{\kappa^2}{2}\left[\frac{3m^3}{8e^2R_2{}^4}+\lambda\right] . \tag{6}$$

As soon as $\lambda\geq 0$ the condition for a compact two-sphere is trivially satisfied. But if $\lambda<0$ we must have

$$\frac{3m^2}{8e^2R_2{}^4}>|\lambda| , \tag{7}$$

and this condition puts an upper bound on the value of the six-dimensional cosmological constant. This is very big for the radius of the two-sphere being of the order of the Planck length, as is usual in Kaluza-Klein-type theories.

We now look for cosmological solutions as we have discussed in the Introduction so that our metric has the form mentioned at the beginning, keeping the form of the monopole potential in the two-sphere. When inserted into the field equations (4) and (5) we obtain

$$3\frac{\ddot{R}_3}{R_3}+2\frac{\ddot{R}_2}{R_2}=\frac{\kappa^2}{4}\left[\lambda-\frac{m^2}{8e^2R_2{}^4}\right] , \tag{8}$$

$$\frac{\kappa_2}{R_2{}^2}+\frac{d}{dt}\left[\frac{\dot{R}_2}{R_2}\right]+\left[3\frac{\dot{R}_3}{R_3}+2\frac{\dot{R}_2}{R_2}\right]\frac{\dot{R}_2}{R_2} =\frac{\kappa^2}{4}\left[\lambda+\frac{3m^2}{8e^2R_2{}^4}\right] , \tag{9}$$

$$\frac{2\kappa_3}{R_3{}^2}+\frac{d}{dt}\left[\frac{\dot{R}_3}{R_3}\right]+\left[3\frac{\dot{R}_3}{R_3}+2\frac{\dot{R}_2}{R_2}\right]\frac{\dot{R}_3}{R_3} =\frac{\kappa^2}{4}\left[\lambda-\frac{m^2}{8e^2R_2{}^4}\right] . \tag{10}$$

We may divide the solutions into two classes, class I being for a nonzero cosmological constant and class II for a zero cosmological constant. In both cases we find, among the possible solutions, the familiar one with an oscillating anti—de Sitter universe and a static two-sphere.

For class-I solutions we try initially the power-law behavior $R_i(t)=r_it^{\alpha_i}$ $(i=2,3)$ with r_i and α_i constants, the latter to be determined from Einstein's equations (8)—(10). The solutions are in the following table:

Solution	α_3	α_2	$\lambda=$const	κ_3	κ_2
(a)	0	0	Yes	0	C
(b)	1	0	Yes	AdS	C
(c)	0	1	No	0	C^*
(d)	1	1	No	AdS	C^*

where C is the compact space; C^* is the condition for compactification, achieved by fine tuning of parameters; AdS = anti—de Sitter space-time.

Solution (a) is the trivial static solution of Ref. 2. Solutions (b), (c), and (d) involve expansion either in the three-space or in the S^2 or in both. We must realize that the only possible solution is (b) since solutions (c) and (d) can only exist for a time-dependent cosmological constant and this time dependence will violate the Bianchi identities. Thus, *expansion in the extra dimensions is ruled out by geometrical arguments.* An interesting point is that in the framework adopted here it is also impossible to find solutions with shrinking extra dimensions. If instead of a power-law behavior we try an exponential one, we can easily find a solution with expansion in the space-time but with static and compact internal space. This "inflationary" solution is (as in our previous one) a direct consequence of the presence of λ.[7]

If we now try an oscillating behavior for the four-dimensional space-time with a static two-sphere, i.e., if we write $R_3(t)=r_3\cos\alpha t$ and $R_2=r_2$, we are able to find another solution for $\lambda\neq m^2/8e^2r_2{}^4$. [For $\lambda=m^2/8e^2r_2{}^4$ we obtain solution (a) as above.] From Eqs. (8)—(10) we get solution (e),

$$\alpha^2=\frac{1}{12}\left[\frac{m^2\kappa^2}{8e^2r_2{}^4}-\kappa^2\lambda\right] ,$$

$$\kappa_2=\frac{r_2{}^2}{4}\left[\frac{3m^2\kappa^2}{8e^2r_2{}^4}+\kappa^2\lambda\right], \quad \kappa_3=-\alpha^2r_3{}^2 .$$

This is an oscillating anti—de Sitter universe with period $\tau=2\pi/\alpha$ and a compact two-sphere. These seem to be the only possible solutions of this form for class-I models. We will discuss the relation between the period and size of the compact two-sphere later when we discuss this problem for the eleven-dimensional situation, since these are similar.

For class II, we obviously can get an oscillatory solution if we just take $\lambda=0$ in the solution (e) above. This solution is very important since, in eleven-dimensional supergravity theories, one usually starts without a cosmological constant and thus we should expect to get this kind of solution in the round seven-sphere[6] and in Englert's solution.[3] Freund has already shown[4] that this is the case for the round seven-sphere. We later show that this will also be a possibility for the seven-sphere with torsion. The

Einstein equations for the two cases are identical except for numerical coefficients.

III. ELEVEN-DIMENSIONAL SUPERGRAVITY

The motivation for studying the previous model resides basically in the fact that we have the Maxwell field taking values only on the two-sphere. This interesting feature of this model could help us to find possible cosmological solutions for the round seven-sphere with torsion obtained by Englert for the compactification of eleven-dimensional supergravity.[3] For this the fourth-rank antisymmetric tensor also depends on the extra dimensions being related to the torsion used to "flatten" the seven-sphere as in the Cartan-Schouten formalism.[8] We here give a brief review. More details of the $(\mathrm{AdS})_4\times S^7$ solution can be found in Ref. 3.

The bosonic part of the action is given by

$$S=-\int d^{11}z(-g^{(11)})^{1/2}\left[\frac{R}{2}+\tfrac{1}{48}F_{MNPQ}F^{MNPQ}-\frac{\sqrt{2}}{6(4!)^2}\frac{\epsilon^{M_1\cdots M_{11}}}{(-g^{(11)})^{1/2}}F_{M_1\cdots M_4}F_{M_5\cdots M_8}A_{M_9\cdots M_{11}}\right], \tag{11}$$

where now, $M,N=0,\ldots,10$; $\mu,\nu=0,\ldots,3$; $m,n=4,\ldots,10$, and the signature is $(-++\ldots+)$. Z^M is given again by $Z^M=(x^\mu,y^m)$, R is the eleven-dimensional curvature scalar, and $F_{MNPQ}=4!\partial_{[M}A_{NPQ]}$. The field equations are

$$R_{MN}-\tfrac{1}{2}g_{MN}R=-\tfrac{1}{48}(8F_{MPQR}F_N{}^{PQR}-g_{MN}F_{SPQR}F^{SPQR}), \tag{12}$$

$$F^{MNPQ}{}_{;M}=\frac{-\sqrt{2}}{2(4!)^2}\frac{\epsilon^{M_1\cdots M_8NPQ}}{(-g^{(11)})^{1/2}}F_{M_1\cdots M_4}F_{M_5\cdots M_8}. \tag{13}$$

If we then look for solutions in the form of a product of a four-dimensional space-time and a compact seven-sphere we find, following Englert,[3] that $R_{mn}=\gamma g_{mn}$; $R_{\mu\nu}=\gamma' g_{\mu\nu}$, $R_{m\mu}=0$, with $\gamma<0$ to have a compact space. We then have a solution where

$$F^{\mu\nu\rho\sigma}=\frac{\epsilon^{\mu\nu\rho\sigma}}{(-g^{(4)})^{1/2}}\frac{f}{(4!)^{1/2}},\quad f=\text{const} \tag{14}$$

plus a nonvanishing expectation value for F^{MNPQ} in the seven-sphere. Then (13) and (14) give

$$F^{mnpq}{}_{;m}=\left[\frac{-2}{(4!)^3}g^{(7)}\right]^{1/2}f\epsilon^{mpqrstu}F_{rstu}.$$

If we now use the parallelizability property of the seven-sphere, we can relate the torsion tensor $S^m{}_{np}$ used to flatten the sphere to the field F^{mnpq} by

$$F^{mnpq}=\lambda S_{\pm[npq,m]},$$

where $S_{\pm[npq,m]}$ are the two possible values for this torsion tensor.[3] Englert has shown that a solution of this kind is possible if

$$\pm f^2=-328,\quad F_{mpqr}F_m{}^{pqr}=\tfrac{2}{3}\lambda^2\gamma^2 g_{mn},\quad \lambda^2=-\frac{12}{\gamma}. \tag{15}$$

In order to look for cosmological solutions, the most natural ansatz is to take the constant f appearing in (14) to be time dependent. From (15) this will immediately give a time dependence for the field strength F^{mnpq}. We will see moreover that no mixed field strength on metric components arise with this modification and the metric being of the form assumed at the beginning of the paper. If we try the power-law behavior $R_i(t)=r_it^{\alpha_i}$ $(i=3,7)$, we get for Einstein's equations,

$$\begin{aligned}
&[3\alpha_3(\alpha_3-1)+7\alpha_7(\alpha_7-1)]t^{-2}=-\tfrac{5}{96}f^2(t),\\
&(-\alpha_3+3\alpha_3{}^2+7\alpha_3\alpha_7)t^{-2}=-\tfrac{5}{96}f^2(t)-\frac{2\kappa_3}{r_3{}^2t^{2\alpha_3}},\\
&(-\alpha_7+7\alpha_7{}^2+3\alpha_3\alpha_7)t^{-2}=\tfrac{1}{32}f^2(t)-\frac{6\kappa_7}{r_7{}^2t^{2\alpha_7}}.
\end{aligned} \tag{16}$$

Additionally, we have from (13) and (14) that $(d/dt)(R_7{}^7f)=0$.

We can easily see that a solution of this form is possible for Eqs. (16) similar to the one found by Freund[5] for the round S^7. Another important feature of this model is the absence of an eleven-dimensional cosmological constant in the initial action. If we look back in our six-dimensional model we see that the only solution with zero cosmological constant is the oscillating one. We expect then that, if the analogy prevails, this will again be the case for the present model. In fact, Einstein's equations are now

$$\begin{aligned}
&3\frac{\ddot R_3}{R_3}+7\frac{\ddot R_7}{R_7}=-\tfrac{5}{96}f^2,\\
&2\frac{\kappa_3}{R_3{}^2}+\frac{d}{dt}\left[\frac{\dot R_3}{R_3}\right]+3\left[\frac{\dot R_3}{R_3}\right]^2+7\frac{\dot R_7}{R_7}\frac{\dot R_3}{R_3}=-\tfrac{5}{96}f^2,\\
&6\frac{\kappa_7}{R_7{}^2}+\frac{d}{dt}\left[\frac{\dot R_7}{R_7}\right]+3\frac{\dot R_3}{R_3}\frac{\dot R_7}{R_7}+7\left[\frac{\dot R_7}{R_7}\right]^2=\tfrac{1}{32}f^2.
\end{aligned} \tag{17}$$

For $R_7=r_7=\text{const}$ and $R_3=r_3\cos\alpha t$, from (17) we find

$$f^2=\tfrac{288}{5}\alpha^2=\text{const};\quad \kappa_2=-\alpha^2r_3{}^2;\quad \kappa_7=3\alpha^2r_7{}^2/10.$$

This model allows as solution an oscillating four-dimensional anti–de Sitter universe with period $\tau=2\pi/\alpha$ and a static seven-sphere. Although the oscillatory solutions are an immediate consequence of the choice for the time dependence of R_3 and R_2 (or R_7) they provide some interesting results for the effective four-dimensional cosmological constant.

For the six-dimensional model we can write an expression for α provided that we normalize the six-dimensional gravitational coupling constant in terms of the four-dimensional one (K_4) as

$$\kappa_4 = \frac{\kappa}{V_2} = \frac{\kappa K_2}{4\pi r_2{}^2}$$

and that we take the class-II solution for simplicity. Then,

$$\alpha^2 = \frac{1}{6}\left[\frac{2m\pi\kappa_4}{3er_2{}^2}\right]^{2/3}$$

with the correct dimension $[\alpha]=L^{-1}$ (here $[e]=L$). In eleven dimensions, α will depend on the field strength f and on the eleven-dimensional gravitational coupling constant which involve one of the free parameters of the theory, the "size" when all dimensions were equivalent. Of course, the other free parameter is the time of compactification (t_0) that we have to include in our solutions $R_i(t)=r_i(t+t_0)^{\alpha_i}$ since they are valid from $t \geq t_0$ in principle. The mass gap of the theory is $\Delta m \sim 1/r_i{}^2$ ($i=2,7$) which is, as usual, taken to be of the order of (Planck length)$^{-2}$.

We can obtain an effective four-dimensional cosmological constant Λ from Eqs. (8) and (10) for the six-dimensional model and from the first two equations in (17) for the eleven-dimensional model. If we contract Einstein's equation we get, in four dimensions,

$$R_{\mu\nu} = \frac{-\kappa^2}{2}\Lambda g_{\mu\nu} .$$

By comparing with our solutions we have

$$\Lambda = \lambda/2 - m^2/16e^2R_2{}^4 :$$

$$\text{(a) and (b):}\quad \Lambda = 0 ,$$

$$\text{(e)}\quad \Lambda = \lambda/2 - m^2/16e^2r_2{}^4 \neq 0 ,$$

$$\text{(f)}\quad \Lambda = -m^2/16e^2r_2{}^4 = -24\pi^2/\kappa^2\tau^2 .$$

For the oscillating solution in eleven dimensions we have

$$\Lambda = -\tfrac{5}{96}f^2 = -3\alpha^2 = -\frac{12\pi^2}{\tau} .$$

In our units we must write

$$\Lambda = -12\pi^2 M_p{}^4\left[\frac{\tau_p{}^2}{\tau^2}\right] .$$

For $\tau = 1.5\times10^{10}$ yr$=10^{28}$ cm and $M_P=10^{-32}$ cm, $\Lambda \simeq 10^{-120} M_P{}^4$.

For solutions (a) and (b) we have imposed $\Lambda=0$ to solve Einstein's equations. Nevertheless it is an interesting feature of higher-dimensional Einstein-Maxwell theories that this can be achieved through a fine tuning of the D-dimensional cosmological constant. In other words, although the higher-dimensional universe admits a cosmological constant we, in four dimensions, do not see it. This suggests that higher-dimensional theories may eventually give a proper answer for the cosmological-constant puzzle. For the oscillatory solutions we have used $\tau=1.5\times10^{10}$ years as the period of oscillation. This will again provide a very small Λ which is in agreement with the current accepted value. Although our model is only valid for the radiation-dominated era, since we are not including fermionic matter in our analysis, we can say that the solutions presented here provide evidence that it is worthwhile to take the extra dimensions seriously as a tool for unified models of gravitational and gauge interactions.

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