Effective Lagrangian for $\lambda\phi^4$ theory in curved spacetime with varying background fields: Quasilocal approximation

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By means of a Riemann normal coordinate expansion for the metric and a momentum-space representation for the Green's function, we derive in analytic form the one-loop effective Lagrangian for a $\lambda\phi^4$ theory in curved spacetime which is exact to all orders in λ and includes variation of the background field up to the second order. Ultraviolet divergences are removed by small-proper-time expansion and dimensional regularization. We obtain a generalized expression for the a_2 Minakshisundaram-DeWitt coefficient of the scalar wave operator with spacetime-dependent background field. A set of renormalization-group equations for the coupling constants of the theory is obtained, which can be used for analyzing their curvature and energy dependence. An alternative derivation of the effective Lagrangian is presented via the heat-kernel technique for anisotropic harmonic oscillators. Our result is useful for the study of quantum processes in the early universe or black holes under conditions where spacetime curvature and dynamical field effects are important. When suitably generalized, the effective Lagrangian obtained here in the form of a quasipotential should provide an improvement over the flat-space Coleman-Weinberg potential assumed in most discussions of the new inflationary universe. Possible directions for developing our method for more general problems are also discussed.

I. INTRODUCTION

Developing theories of interacting quantum fields in curved spacetime[1] is useful for the study of problems related to quantum processes in the early universe and near black holes, as well as to the unification of elementary-particle interactions with gravity. This includes multiparticle production,[2] phase transitions in cosmological[3] and black-hole spacetimes,[4] and the renormalization-group analysis of the behavior of coupling constants[5] in grand unified theories at high curvatures. Recent theoretical inquiries into curved-space interacting field theory have focused on the renormalizability of abelian theories such as $\lambda\phi^4$ or scalar electrodynamics, and non-abelian gauge theories, where noncovariant, state-dependent counterterms problematically appear.[6–12] Methods employed range from the use of the interaction picture[6,7] and operator-product expansion[8] in canonical quantization, to loop-expansion,[9] momentum-space techniques,[10] renormalization-group analysis,[11] and background-field methods[12] in path-integral quantization. While perturbative expansion in the coupling constants is useful for examining the divergences up to a finite-loop order, the knowledge of an exact Lagrangian is essential in treating problems of a nonperturbative nature or those which involve nonlocal properties of spacetime, such as instanton solutions and symmetry-breaking processes. Most previous calculations where exact (to one-loop) forms of an effective Lagrangian are derived have been for static (e.g., Einstein universe[13]) or maximally symmetric spacetimes (e.g., de Sitter universe[14]), whereas problems in cosmology deal almost exclusively with dynamic spacetimes.

Part of our aim in constructing an effective Lagrangian incorporating the effect of time-dependent background fields is to study quantum processes in the early universe from the grand unification (GU) time ($t_{\rm GU}\sim 10^{-35}$ sec) when the strong and electroweak forces are believed to be unified to the Planck time ($t_{\rm Pl}\sim 10^{-43}$ sec) when quantum gravitational effects become important. Dynamics of the background spacetime (with metric $g_{\mu\nu}$, as governed by the Einstein equation) dictates a corresponding change in the background field $\hat{\phi}$ (via the wave equation) which acts like a time-dependent order parameter. In this regard, it is perhaps noteworthy that much of the previous work on GU phase transitions[15] is inconsistent because its premise is an effective potential [e.g., the Coleman-Weinberg (CW) potential in the new inflationary universe[16]] derived from flat-space quantum field theory, yet the background spacetime remains dynamic (be it de Sitter during, or Robertson-Walker before or after, inflation). In particular, the background Higgs field $\hat{\phi}$ is usually assumed to be constant, which enables one to work with an effective potential. At best, this can be regarded as an approximation to the true picture at the GU time. Its validity has to be examined more closely with realistic cosmological solutions and field parameters. While close to the Planck time, it is almost certain that the flat-space result would not hold in general, as the changing background field would contribute more to altering the symmetry behavior of the system. This regime may include many important symmetry-breaking processes associated with possible quantum gravity,[17] induced gravity[18] (at $\sim 10^{19}$ GeV), Kaluza-Klein[19] or supersymmetry[20] ($\sim 10^{17}$ GeV) theories. At the Planck time, phase transitions will be complicated further by "dissipative" processes like particle production,[21] requiring a rather different treatment. The quasilocal techniques studied here, which account for a slowly varying background field, would have to be augmented or replaced by nonperturbative methods similar to those necessary in critical dynamics.

In this paper we construct an effective Lagrangian for $\lambda\phi^4$ theory in curved spacetime which takes into account the lowest-order radiative correction (one-loop approximation, first order in $\hbar$) up to the second order in the variation of the background field $L(\hat{\phi},\partial\hat{\phi},\partial^2\hat{\phi})$, but exact to all orders in the coupling constant λ. Generally, the result is valid for slowly varying background spacetimes and background fields. (The exact range of validity in realistic cosmological situations will depend on the functional form of the background metric and on the time the observer adopts—cosmic, conformal, or otherwise.) In Sec. II we use the Schwinger-DeWitt proper-time formalism[22,23] for the derivation of the effective Lagrangian. The field is expanded in a Taylor series around an arbitrary spacetime point and the Green's function is sought from a solution of the inhomogeneous wave equation. For homogeneous spaces where the field changes only in time, this is related to the quasi-adiabatic expansion method which has been used for free[24] and interacting[7] fields at zero and finite temperature.[25] In this expansion if only field variations up to the second order are kept, it is observed that the effective Lagrangian can be cast in an exact analytic form. This is made possible by a reduction-to-quadrature procedure for the momentum variables in the functional integral. First used by Brown and Duff[26] and Iliopoulos, Itzykson, and Martin[27] in flat-space field theory, this procedure can easily be extended to curved space with local momentum-space techniques. We use a Riemann normal coordinate expansion for the metric and the momentum-space representation for the Green's function (in the manner of Bunch and Parker[10]) to derive the one-loop effective Lagrangian [Eq. (2.29) and (2.30)].

Processes involving quantum fields in a dynamic spacetime can usually be analyzed using the prototype model of a time-dependent harmonic oscillator. In Sec. III we present an alternative derivation of the effective Lagrangian by working with the heat kernel of such a system.[23,28] Making use of the known characteristic functions (Hermite polynomials) of the harmonic oscillator, one can derive the "exact" form when variation of the background field only to the second order is included. The zero mode of the spectrum is easily identified in this way, thus facilitating study of the infrared properties of the system for symmetry-breaking problems. With the well-studied mathematical properties of the harmonic oscillator, various methods from quantum mechanics can be used as a guide to getting better analytic results with realistic cosmological parameters. Using this method for anharmonic oscillators one can also extend the present quasilocal approximation to higher orders, thus encompassing more rapidly varying fields in homogeneous cosmologies.

In Sec. IV, we discuss the removal of ultraviolet divergences by the standard method of small-proper-time expansion and dimensional regularization. We are able to derive a generalized expression for the a_2 Minakshisundaram-DeWitt coefficient[23,29] of the Laplace-Beltrami operator with a spacetime-dependent background scalar field. From the counterterms we obtain a set of renormalization-group (RG) equations for the coupling constants and various parameters of the theory.[5] This includes all the equations we obtained earlier based on a ζ-function regularization[28] calculation for the Einstein universe.[13] It also yields the RG equation for the coupling constant of the conformal-tensor-squared term, as the background space considered here is completely general. Since the result is insensitive to the large-scale properties of the background field, it can also be obtained by any local expansion method. Solutions to these RG equations which govern the strength of the coupling constants as a function of energy and curvature should be useful for the analysis of spacetime-curvature effects on elementary field interactions. The final result for the renormalized one-loop effective Lagrangian is displayed in Eq. (4.22). In Sec. V, we discuss the possible extension of our method and the application of our result to related problems.

II. EFFECTIVE LAGRANGIAN

Consider a massive (m) scalar field $\tilde{\phi}$ with quartic self-interaction (λ) coupled to a background spacetime with metric $g_{\mu\nu}$ and scalar curvature R. It is described by the Lagrangian density

$$L[\tilde{\phi},g_{\mu\nu}]=-\tfrac{1}{2}\tilde{\phi}\left[\Box+(1-\xi)\frac{R}{6}+m^2\right]\tilde{\phi}-\lambda\frac{\tilde{\phi}^4}{4!}\ , \tag{2.1}$$

where $\Box$ is the Laplace-Beltrami operator in n dimensions and the constant $\xi=0,1$ denotes conformal and minimal coupling, respectively. This action has a minimum at $\tilde{\phi}=\hat{\phi}$, which satisfies the classical equation of motion

$$\left[\Box+(1-\xi)\frac{R}{6}+m^2+\lambda\frac{\hat{\phi}^2}{6}\right]\hat{\phi}=0\ . \tag{2.2}$$

Quantum fluctuations $\phi\equiv\tilde{\phi}-\hat{\phi}$ around the classical background $\hat{\phi}$ satisfies an equation of the form (to lowest order in ϕ)

$$(\Box+\mathscr{M}^2)\phi(x)=0 \tag{2.3}$$

where $\mathscr{M}^2=M^2+(1-\xi)R/6$ is an effective mass which depends on the background curvature R, the coupling ξ, and on the background field $\hat{\phi}$ via $M^2\equiv m^2+\frac{1}{2}\lambda\hat{\phi}^2$. When contributions from quantum fluctuations are included, the equation satisfied by the background field $\hat{\phi}$ contains an extra term due to the variance of the fluctuations, i.e.,

$$\left[\Box+(1-\xi)\frac{R}{6}+m^2+\lambda\frac{\hat{\phi}^2}{6}+\lambda\frac{\langle\phi^2\rangle_0}{2}\right]\hat{\phi}=0\ . \tag{2.4}$$

In the functional-integral perturbative approach, the effective action which is related to the effective Lagrangian $L_{\rm eff}$ by

$$\Gamma[\hat{\phi},g_{\mu\nu}]=\int d^4x\sqrt{-g}\,L_{\rm eff} \tag{2.5}$$

is expanded in powers of $\hbar$ as

$$\Gamma[\hat{\phi}]=S[\hat{\phi}]+\Gamma^{(1)}+\Gamma'\ , \tag{2.6}$$

where $S[\hat{\phi}]$ is the classical action

$$S=\int d^4x\sqrt{-g}L_{\phi}^{(0)} ,$$

$$L_{\phi}^{(0)}=-\theta\Box\hat{\phi}^2-\tfrac{1}{2}\hat{\phi}(x)\left[\Box+(1-\xi)\frac{R}{6}+m^2\right]\hat{\phi}(x)-\lambda\frac{\hat{\phi}^4}{4!} . \quad (2.7)$$

The term $\theta\Box\hat{\phi}^2$ contributes only for spacetimes with boundaries. It is introduced here in anticipation of the renormalization of the full theory. [See (4.18).] The one-loop effective action is given by

$$\Gamma^{(1)}=-\frac{i\hbar}{2}\ln[\mathrm{Det}(G)]\equiv\int d^4x\sqrt{-g}L^{(1)} \quad (2.8)$$

and Γ' denotes higher-loop contributions. G here is the (bare) Feynman Green's function, which is a solution of

$$\sqrt{-g}\left[\Box+(1-\xi)\frac{R}{6}+m^2+\lambda\frac{\hat{\phi}^2}{2}\right]G(x,x')=\delta(x,x') . \quad (2.9)$$

In a static homogeneous spacetime $\hat{\phi}$ is a constant field, in which case one can define an effective potential $V(\hat{\phi})=-(\mathrm{vol})^{-1}\Gamma(\hat{\phi})$, where vol denotes the spacetime volume. In dynamic or inhomogeneous spacetimes, $\hat{\phi}$ changes in time or space according to the scalar wave equation (2.2). Under such conditions the effective potential is ill defined. Instead, one has to work with the effective action $\Gamma(\hat{\phi})$. If the background field changes gradually, one can carry out a quasilocal expansion around any spacetime point x^μ, including its spatial or temporal derivatives, up to a finite order:

$$\hat{\phi}^2(x')=\hat{\phi}^2(x)+\hat{\phi}^2_{,\mu}(x)(x'-x)^\mu+\tfrac{1}{2}\hat{\phi}^2_{,\mu\nu}(x)(x'-x)^\mu(x'-x)^\nu+\cdots . \quad (2.10)$$

The effective Lagrangian will then be a functional of $\partial\hat{\phi}$, $\partial^2\hat{\phi}$, etc.,

$$L_{\mathrm{eff}}=L_{\mathrm{eff}}(\hat{\phi},\partial_\mu\hat{\phi},\partial_\mu\partial_\nu\hat{\phi},\ldots) . \quad (2.11)$$

The effect of a slowly varying background field on quantum processes such as symmetry breaking can be derived from such an effective Lagrangian. For spatially homogeneous backgrounds as in cosmological situations where the field changes only in time, this Lagrangian is related to the adiabatic expansion method which has been used previously for free[24] and interacting fields[7] in zero-temperature and finite-temperature (FT) quantum theories.[25] The Taylor expansion used here and the adiabatic expansion are not always equivalent in that the definition of an adiabatic vacuum state (or adiabatic n-particle state in FT theory[25]) is devised such that there will be no particle present to within a definite adiabatic order. Although one can loosely say that terms with nth derivatives are of nth adiabatic order, the reverse is not always true. Terms of a definite adiabatic order, say, in the energy density, usually involve combinations of terms of the same derivative order and products of terms of lower derivative orders. The definition of particle states associated with adiabatic expansion has a precise physical meaning but not so for the simple Taylor expansion as in (2.11) beyond incorporating the nonlocal characteristics of the background field. In the latter, a choice of Riemann normal coordinates could be replaced by Fermi normal coordinates possibly resulting in a change in the definition of particle state. The range of validity of these expansions in realistic calculations will of course depend on the choice of time coordinates and the functional dependence of the background metric and the dynamics of the background field.

Previous analysis of interacting fields has largely relied upon perturbation expansions in order of the interaction constant. For example, Bunch and Parker[10] have given a momentum-space representation of Feynman propagators in curved space and derived the form of the Green's functions for some low-order terms in λ in a $\lambda\phi^4$ theory. In our present treatment we will use their momentum-space technique to solve for the Green's function, but show that an exact analytic form can be derived if only background-field variations up to the second order are included. Thus, in a local Riemann normal coordinate expansion around a spacetime point x, the metric can be written as

$$g_{\mu\nu}=\eta_{\mu\nu}+\tfrac{1}{3}R_{\mu\alpha\nu\beta}y^\alpha y^\beta+\tfrac{1}{6}R_{\mu\alpha\nu\beta;\gamma}y^\alpha y^\beta y^\gamma+\left(\tfrac{1}{20}R_{\mu\alpha\nu\beta;\gamma\delta}+\tfrac{2}{45}R_{\alpha\mu\beta\lambda}R^\lambda{}_{\gamma\nu\delta}\right)y^\alpha y^\beta y^\gamma y^\delta+\cdots , \quad (2.12)$$

where $\eta_{\mu\nu}$ is the Minkowski metric $(+1,-1,-1,-1)$ $y=x'-x$ and the coefficients in this expansion are evaluated at $y=0$. The Green's function $G(x,x')$ for the fluctuation field ϕ satisfies the inhomogeneous form of Eq. (2.3),

$$[\Box+\mathscr{M}^2(x)]G(x,x')=(-g)^{-1/2}\delta(x,x') . \quad (2.13)$$

Define

$$\bar{G}(x,x')=(-g)^{1/4}(x)G(x,x')(-g)^{1/4}(x')$$

and

$$H=(-g)^{+1/4}(\Box+\mathscr{M}^2)(-g)^{-1/4} . \quad (2.14a)$$

We can write

$$H=\partial_\mu g^{\mu\nu}\partial_\nu+V ,$$

where (2.14b)

$$V=(-g)^{-1/4}[\partial_\mu(-g)^{1/2}g^{\mu\nu}\partial_\nu(-g)^{-1/4}]+\mathscr{M}^2 .$$

The Lorentz invariance of the momentum-space representation of $\bar{G}$ implies that $g^{\mu\nu}$ in $\partial_\mu g^{\mu\nu}\partial_\nu$ becomes $\eta^{\mu\nu}$. Expressing V with the Riemann normal coordinate expansion for $g_{\mu\nu}$ gives the following equation to quadratic order in y [this agrees with Eq. (2.19) of Bunch and Parker[10]]:

$$(\eta^{\mu\nu}\partial_\mu\partial_\nu+\alpha^2+\beta_\mu y^\mu+\tfrac{1}{4}\gamma^2{}_{\mu\nu}y^\mu y^\nu)\bar{G}(y)=\delta(y) , \quad (2.15)$$

where

$$\alpha^2 \equiv \mathcal{M}^2 - \tfrac{1}{6}R = m^2 + \tfrac{1}{2}\lambda\hat{\phi}^2 - \tfrac{1}{6}\xi R \equiv m^2 + U \ , \tag{2.16a}$$

$$\beta_\mu \equiv (\tfrac{1}{2}\lambda\hat{\phi}^2 - \tfrac{1}{6}\xi R)_{;\mu} = U_{;\mu} \ , \tag{2.16b}$$

$$\tfrac{1}{4}\gamma^2{}_{\mu\nu} \equiv \tfrac{1}{2}U_{;\mu;\nu} + a_{\mu\nu} \ , \tag{2.16c}$$

$$a_{\mu\nu} \equiv \tfrac{1}{120}R_{;\mu\nu} - \tfrac{1}{40}\Box R_{\mu\nu} - \tfrac{1}{30}R_\mu{}^\lambda R_{\lambda\nu} + \tfrac{1}{60}R^\kappa{}_\mu{}^\lambda{}_\nu R_{\kappa\lambda} + \tfrac{1}{60}R^{\lambda\rho\kappa}{}_\mu R_{\lambda\rho\kappa\mu} \ . \tag{2.17}$$

Note that in the coefficients β_μ and $\gamma^2{}_{\mu\nu}$ multiplying y^μ and $y^\mu y^\nu$ there are now terms involving the first and second derivatives, respectively, of the background field $\hat{\phi}$, in addition to those of the scalar curvature R. The latter are the only terms included in similar previous treatments. $a_{\mu\nu}$ is, of course, the tensor, whose trace gives the a_2 coefficient related to the trace anomaly.[23,24] We can simplify notation by working in Euclideanized parameters. Performing a Wick rotation $t \to -i\tau$ and defining $\overline{G}_E$ by

$$\overline{G}(t,\vec{x},t',\vec{x}\,') = \overline{G}_E(-i\tau,\vec{x},-i\tau',\vec{x}\,') \ , \tag{2.18}$$

the Euclideanized Green's function $\overline{G}_E$ satisfies the equation

$$(\partial_\tau{}^2 + \partial_x{}^2 + \alpha^2 + \hat{\beta}_\mu y^\mu + \tfrac{1}{4}\hat{\gamma}^2{}_{\mu\nu}y^\mu y^\nu)\overline{G}_E = \delta(\tau,\vec{x},\tau',\vec{x}\,'), \tag{2.19}$$

where

$$\hat{\beta}_0 = i\beta_0 \ , \quad \hat{\beta}_i = \beta_i \quad (i=1,2,3) \ ,$$
$$\hat{\gamma}^2{}_{00} = -\gamma^2{}_{00} \ , \quad \hat{\gamma}^2{}_{0i} = -i\gamma^2{}_{0i} \ , \tag{2.20}$$
$$\hat{\gamma}^2{}_{ij} = \gamma^2{}_{ij} \ .$$

With this, one can introduce a momentum space in a coordinate patch centered at point x and define the Fourier transform $\overline{G}_E(p)$ by

$$\overline{G}_E(y) = \frac{1}{(2\pi)^n}\int d^np\, e^{ipy}\overline{G}_E(p) \ , \tag{2.21}$$

where $py \equiv p_\mu y^\mu = \eta^{\mu\nu}p_\mu y_\nu$. This momentum-space representation is well-defined only in a local neighborhood of $y=0$ and does not carry the global properties of $G(x,x)$. However, for the study of ultraviolet divergences in two-point functions (like the Feynman propagator) or bivector quantities (like the energy-momentum tensor) when coincidence limits are taken, or for the study of systems involving low-order quasilocal variations of the background field, results based on the use of the momentum-space representation will be valid. Thus, the method adopted here will be useful for deriving the form of an effective "quasipotential" taking into account small changes in the classical Higgs field $\hat{\phi}$ (the so-called "kinetic terms") as an improvement over the Coleman-Weinberg potential as used in the new-inflationary-universe models. It would probably not be sufficient for the consideration of processes involving rapid changes of the background metric, as in cosmological particle production, or processes bearing on the global behavior of spacetime, e.g., topological effects on phase transitions.

The momentum-space Green's function $\overline{G}_E(p)$ satisfies

$$(p^2 + \alpha^2 - i\hat{\beta}_\mu\partial^\mu - \tfrac{1}{4}\gamma^2{}_{\mu\nu}\partial^\mu\partial^\nu)\overline{G}_E(p) = 1 \ , \tag{2.22}$$

where $\partial^\mu\overline{G}_E(p) \equiv \partial\overline{G}_E/\partial p_\mu$. Hereafter we drop the subscript E and the overbar on $\overline{G}_E$ without undue confusion. For a constant background field, $\hat{\beta} = \hat{\gamma} = 0$, we recover the flat-space result,

$$G(p) = (p^2+\alpha^2)^{-1} = \int_0^\infty ds\, e^{-\alpha^2 s}e^{-p^2 s} \ , \tag{2.23}$$

where in the second equality it is written in the proper-time (s) integral representation. In seeking a solution to (2.22) we generalize to curved space the procedure introduced by Brown and Duff[26] for the derivation of an exact effective Lagrangian with kinetic terms in flat space. In Sec. III we will show this can be achieved equivalently by using the heat-kernel technique on a spacetime-dependent anisotropic harmonic oscillator. Assume that $G(p)$ can be written in the same form as (2.23), where p^2 is replaced by a general quadratic polynomial in p^μ, i.e.,

$$G(p) = \int_0^\infty ds\, e^{-\alpha^2 s}\exp[-p^\mu A_{\mu\nu}(s)p^\nu + iB_\mu(s)p^\mu + C(s)], \tag{2.24}$$

where A, B, and C are functions of $\hat{\beta}$ and $\hat{\gamma}$ with constraints that they reduce to the zero-field results,

$$A_{\mu\nu}(s) \to \delta_{\mu\nu}s \ , \quad B_\mu(s) \to 0 \ , \quad C(s) \to 0 \ .$$

The configuration space (Euclideanized) Green's function $G(x,x')$ assumes a similar form (from definition (2.21)]:

$$G(x,x') = \frac{1}{(2\pi)^n}\int_0^\infty ds\, e^{-\alpha^2 s + C} \times \int d^np\, e^{ip\cdot y}e^{-(p\cdot A\cdot p - iB\cdot p)} \ . \tag{2.25}$$

Under the change of variable $\overline{p} = \tfrac{1}{2}A^{-1}B$ and y set to zero, the p integral becomes a Gaussian and we obtain

$$G(x,x) = \int_0^\infty \frac{ds}{(4\pi s)^{n/2}}\exp[-\alpha^2 s + C - \tfrac{1}{4}BA^{-1}B - \tfrac{1}{2}\operatorname{tr}\ln(As^{-1})] \ . \tag{2.26}$$

Following Ref. 26, it can be shown that A, B, and C are simple trigonometric functions of $\hat{\beta}$ and $\hat{\gamma}$:

$$A = \hat{\gamma}^{-1}\tanh\hat{\gamma}s \ , \tag{2.27a}$$

$$B = 2\hat{\gamma}^{-2}(1-\operatorname{sech}\hat{\gamma}s)\hat{\beta} \ , \tag{2.27b}$$

$$C = -\tfrac{1}{2}\operatorname{tr}\ln(\cosh\hat{\gamma}s) - \hat{\beta}\hat{\gamma}^{-3}(\tanh\hat{\gamma}s - \hat{\gamma}s)\hat{\beta} \ . \tag{2.27c}$$

The one-loop contribution $L^{(1)}$ to the Euclideanized effective Lagrangian L_{eff} [which differs in sign from L_{eff} of (2.5)] which is related to the Green's function by

$$\frac{\partial L^{(1)}}{\partial\alpha^2} = \frac{\hbar}{2}G(x,x) \ , \tag{2.28}$$

can be obtained by integrating $G(x,x,)$ with respect to α^2. [The constant of integration is accounted for by the renormalization of the parameters in the gravitational action. See. Eq. (4.18).] We finally obtain the effective Lagrangian

$$L^{(1)} = -\frac{\hbar}{2(4\pi)^{n/2}}\int_0^\infty \frac{ds}{s^{1+n/2}}e^{-\alpha^2(\hat{\phi})s}e^{-f(s)} \ , \tag{2.29}$$

where

$$f(s)=-C+\tfrac{1}{4}B\cdot A^{-1}\cdot B+\tfrac{1}{2}\,\mathrm{tr}\ln(As^{-1})$$
$$=\tfrac{1}{2}\,\mathrm{tr}\ln[(\hat{\gamma}s)^{-1}\sinh(\hat{\gamma}s)]$$
$$+\hat{\beta}\cdot\hat{\gamma}^{-3}[2\tanh(\tfrac{1}{2}\hat{\gamma}s)-\hat{\gamma}s]\cdot\hat{\beta} \tag{2.30}$$

with α, β, γ, given by (2.16). Except for the Euclideanized variables, its form is identical to the flat-space result of Ref. 26, as it should, since it is based on a local (momentum-space) technique. The quasilocal expansion (2.10) of the fields and the Riemann normal coordinate expansion (2.12) of the metric result in expressions of the same form as the Taylor expansion of fields in flat space. The appearance of spacetime curvature and background-field variations are manifested exclusively through the expansion coefficients α, $\hat{\beta}$, $\hat{\gamma}$ in the Taylor series.

III. HEAT KERNEL OF ANISOTROPIC HARMONIC OSCILLATORS

Before continuing our discussion of the removal of the ultraviolet divergences in the theory, we show in this section an alternative way of deriving the exact effective Lagrangian (2.29) by means of heat-kernel techniques in Euclidean formulation. This method makes use of well-known properties of the harmonic oscillator and thus is more easily adaptable to problems requiring analytical solution. It also allows for perturbative treatments of more complicated situations, e.g., when higher-derivative terms in the background field are included.

The Green's function G associated with a differential operator H satisfies symbolically the equation

$$HG=1\ . \tag{3.1}$$

Following Schwinger[22] one can introduce a fictitious Hilbert space with "proper-time" parameters s and states $|x\rangle$ labeled by the spacetime coordinates x^μ. G has the integral representation

$$G=\int_0^\infty ds\, e^{-Hs}\ , \tag{3.2}$$

where the integrand $K=e^{-Hs}$ is the kernel of a heat equation

$$(\partial/\partial s+H)K=0\ . \tag{3.3}$$

In this space the heat kernel is realized as a two-point function with proper-time dependence

$$K(x,y,s)=\langle x\,|\,e^{-Hs}\,|\,y\rangle\ . \tag{3.4}$$

Let $|\phi_k\rangle$ be a complete set of eigenstates of H with eigenvalues λ_k; i.e., $H\,|\,\phi_k\rangle=\lambda_k\,|\,\phi_k\rangle$, then $K(x,y,s)$ can be expressed as

$$K(x,y,s)=\sum_{k=0}^{\infty} e^{-\lambda_k s}\phi_k^*(y)\phi_k(x)\ . \tag{3.5}$$

The one-loop effective action given by (2.8) now becomes

$$\Gamma^{(1)}=-\frac{\hbar}{2}\ln \mathrm{Det}(G)=-\frac{\hbar}{2}\int_0^\infty \frac{ds}{s}\int d^4x K(x,x,s)\ .$$

Thus, knowledge of $K(x,x,s)$ is sufficient to obtain G and the effective Lagrangian as

$$L^{(1)}=-\frac{\hbar}{2}\int_0^\infty \frac{ds}{s}K(x,x,s)\ . \tag{3.6}$$

As an example, consider the one-dimensional harmonic oscillator with time-dependent natural frequency ω. The Hamiltonian H_{IHO} has the general form

$$H_{\mathrm{IHO}}=\tfrac{1}{2}(p^2+\omega^2x^2)\ , \tag{3.7}$$

where x and p are the canonical coordinates and momenta. The eigenfunction ϕ_k belonging to the kth eigenvalue $\lambda_k=(k+\frac{1}{2})\omega$, $k=0,1,2,\ldots,\infty$ is given by

$$\phi_k=\frac{1}{(2^k k!)^{1/2}}\left[\frac{\omega}{\pi}\right]^{1/4} e^{-\omega x^2/2}H_k(\omega^{1/2}x)$$
$$=\frac{(\omega/\pi)^{1/4}}{(k!)^{1/2}}D_k((2\omega)^{1/2}x)\ , \tag{3.8}$$

where H_k and D_k are the kth-order Hermite polynomial and parabolic cylinder functions, respectively. With this, the heat kernel in (3.5) becomes

$$K(x,y,s)=\left[\frac{\omega}{\pi}\right]^{1/2}\sum_{k=0}^{\infty}\frac{1}{k!}e^{-(k+1/2)\omega s}$$
$$\times D_k^*((2\omega)^{1/2}y)D_k((2\omega)^{1/2}x)\ . \tag{3.9}$$

Making use of the integral representation of the Hermite polynomials,

$$H_k(x)=\frac{2^k}{\sqrt{\pi}}\int_{-\infty}^{\infty}dz(x+iz)^k e^{-z^2}\ , \tag{3.10}$$

one gets, after some manipulation, the standard result for a one-dimensional harmonic oscillator.

$$K(x,y,s)=\left[\frac{\omega}{2\pi\sinh\omega s}\right]^{1/2}$$
$$\times\exp\left[\frac{\omega}{2\sinh\omega s}[(x^2+y^2)\cosh\omega s-2xy]\right]. \tag{3.11}$$

In our present problem, the operator H is given by (2.14a)

$$H=g^{-1/4}\Box g^{-1/4}+\tfrac{1}{6}(1-\xi)R+m^2+\tfrac{1}{2}\lambda\hat{\phi}^2\ . \tag{3.12}$$

(We need not explicitly display in this section the subscript B in ξ, m, and λ, knowing that before renormalization they are all bare quantities.) After introducing a Riemann normal coordinate expansion, one can write H in momentum-space representation as (see Sec. II for the conditions under which this is valid, and Sec. V for discussion of generalizations to homogeneous background spacetimes)

$$H=p_\mu p^\mu+V(y)\ . \tag{3.13}$$

From (2.15),

$$V(y)=\alpha^2+\hat{\beta}_\mu y^\mu+\tfrac{1}{4}\hat{\gamma}^2{}_{\mu\nu}y^\mu y^\nu\ , \tag{3.14}$$

where the coefficients α, $\hat{\beta}$, $\hat{\gamma}$ are c-number functions given by (2.20) and (2.16). Introduce a new variable

$$Q^\mu \equiv y^\mu + 2(\hat\gamma^{-2})^{\mu\sigma}\hat\beta_\sigma , \tag{3.15}$$

where $\hat\gamma^{-2}$ is the inverse matrix of $\hat\gamma^2$ and denote

$$\delta^2 = \alpha^2 - \hat\beta_\mu(\hat\gamma^{-2})^{\mu\nu}\hat\beta_\nu , \tag{3.16}$$

so $V(Q)$ has the quadratic form

$$V(Q) = \delta^2 + \tfrac{1}{4}Q^\mu \hat\gamma^2{}_{\mu\nu} Q^\nu , \tag{3.17}$$

and

$$K = e^{-\delta^2 s} e^{-\tilde H s} , \tag{3.18}$$

where

$$\tilde H = p_\mu p^\mu + \tfrac{1}{4} Q^\mu \hat\gamma^2{}_{\mu\nu} Q^\nu . \tag{3.19}$$

$\tilde H$ has the form of the Hamiltonian of an anisotropic harmonic oscillator with natural frequency $\hat\gamma^2{}_{\mu\nu}$ ($\mu,\nu = 0,1,\ldots,n-1$, where n is the dimension of the spacetime). Diagonalizing $(\hat\gamma^2)_{\mu\nu}$ with the orthogonal matrix S,

$$(\Gamma^2)_{ab} = S_a{}^\mu \hat\gamma^2_{\mu\nu} S^\nu{}_b = \gamma_a{}^2 \delta_{ab} , \tag{3.20}$$

where $\gamma_a{}^2$ are the eigenvalues of Γ^2 ($a = 1,\ldots,n$), one can decompose $\tilde H$ as a sum of $\tilde H_a$'s,

$$\tilde H = \sum_{a=1}^{n} \tilde H_a , \tag{3.21}$$

$\tilde H_a$ being the Hamiltonian operator for the subspace spanned by the eigenvectors ψ_{ak} belonging to the eigenvalue λ_{ak}

$$\tilde H_a \psi_{ak} = \lambda_{ak}\psi_k , \tag{3.22}$$

$$\lambda_{ak} = (k_a + \tfrac{1}{2})\gamma_{ak} , \tag{3.23}$$

$$\psi_{ak} = \frac{1}{(2^k k!)^{1/2}} \left[\frac{\gamma_a}{2\pi}\right]^{1/4} \times \exp[-\gamma_a (Q^a)^2] H_k \left[\left[\frac{\gamma_a}{2}\right]^{1/2} Q^a\right] \tag{3.24}$$

$$= \frac{1}{(k!)^{1/2}} \left[\frac{\gamma_a}{2\pi}\right]^{1/4} D_k(\gamma_a{}^{1/2} Q^a) , \tag{3.25}$$

where $Q^a = S^a{}_\mu Q^\mu$ and D_k are the parabolic cylinder functions. The heat kernel of this system is then given by [cf. Eq. (3.9)]

$$K(x+y, x+y', s) = e^{-\delta^2 s} \prod_{a=1}^{n} \left[\frac{\gamma_a}{2\pi}\right]^{1/2} \sum_{k_a k_a'} \frac{e^{-(k_a+\frac{1}{2})\gamma_a s}}{(k_a! k_a'!)^{1/2}} D_{k_a}(\gamma_a{}^{1/2} Q^a) D_{k_a'}(\gamma_a{}^{1/2} Q'^a) . \tag{3.26}$$

For the evaluation of the effective Lagrangian, we are interested in the coincidence limit $y,y' \to 0$ of K. Denoting $Q^a(y=0)$ by Q_0^a, we get

$$K(x,x,s) = e^{-\delta^2 s} \prod_{a=1}^{n} \left[\frac{\gamma_a}{2\pi}\right]^{1/2} \sum_{k_a=0}^{\infty} \frac{e^{-(k_a+1/2)\gamma_a s}}{k_a!} D_{k_a}(\gamma_a{}^{1/2} Q_0^a) D_{k_a}(\gamma_a{}^{1/2} Q_0^a) . \tag{3.27}$$

Substituting this into (3.6) and performing the s integration, we obtain an expression of $L^{(1)}$ in terms of the harmonic-oscillator eigenfunctions

$$L^{(1)} = -\frac{\hbar (\mathrm{Det}\hat\gamma)^{1/2}}{2(2\pi)^{n/2}} \sum_{k_a} \frac{1}{k_a!} \ln \left[\sum_a (k_a + \tfrac{1}{2})\gamma_a + \delta^2\right] D_{k_a}{}^2(\gamma_a{}^{1/2} Q_0^a) . \tag{3.28}$$

To show that this is identical to (2.29), we first make use of the closed form for K derived in (3.11), generalize it to the anisotropic oscillator and take the coincidence limit to get

$$K(x,x,s) = e^{-\delta^2 s} \prod_{a=1}^{n} \left[\frac{\gamma_a}{4\pi \sinh \gamma_a s}\right]^{1/2} \exp\left[-\frac{2\gamma_a[(\gamma^{-2}\cdot\hat\beta)^a]^2}{\sinh(\gamma_a s)}[\cosh(\gamma_a s) - 1]\right] . \tag{3.29}$$

Then we perform the s integration in (3.6) to get $L^{(1)}$. Noting that

$$\sum_a 2\beta_a{}^2 \gamma_a{}^{-3} \tanh(\gamma_a s/2) = 2\hat\beta\cdot\hat\gamma^{-3}\tanh(\hat\gamma s/2)\cdot\hat\beta , \tag{3.30}$$

the results (2.29), and (2.30) following immediately.

In giving this alternative derivation of the exact effective Lagrangian $L^{(1)}$, we have reduced the present problem of a varying background field propagating in a (n-dimensional) dynamic spacetime to the equivalent problem of an (n-dimensional) anisotropic harmonic oscillator. The form of $L^{(1)}$ given by (3.28) in terms of the harmonic-oscillator eigenfunction is particularly useful for studying symmetry-breaking processes due to the dynamics of spacetime. The infrared behavior of the system is determined by the contribution of the lowest eigenmodes of operator H [Eq. (3.12)] of the fluctuation field. Since the eigenfunctions of the harmonic oscillator are well known, one can derive asymptotic expressions for the effective action $L^{(1)}$ of (3.28) in the regime where a phase transition is likely to occur and analyze its critical behavior. This problem, set in the realistic context of the inflationary universe, is currently under investigation. Another advantage of drawing the analogy with the harmonic-oscillator problem is that one can extend the present method to treat cases where the background field varies more rapidly, so that derivative terms of orders higher than the second enter. This would correspond to the system of anisotropic anharmonic oscillators. Techniques already developed for treating these problems can be applied to the study of quantum processes involving background fields which vary more rapidly, such as those happening beyond the GUT time closer to the Planck time. Additional remarks on this point will be postponed to Sec. V. Let us now turn to the renormalization problem.

IV. RENORMALIZATION

The (Euclideanized) effective Lagrangian up to one loop is given by the sum of the Euclideanized $L_E^{(0)}$ [$=-L^{(0)}$ from Eq. (2.7)] and $L^{(1)}$ [Eq. (2.29) already in Euclideanized form]:

$$L=L_E^{(0)}+L^{(1)} . \tag{4.1}$$

The expression for $L^{(1)}$ given above is formal and divergent. To identify the ultraviolet divergences, we shall examine the behavior of $L^{(1)}$ at small-proper-time parameter s. Expanding Eq. (2.30) out in a series sum in powers of s, we get

$$e^{-f(s)}=1-\frac{\hat{\gamma}^2}{12}s^2+\frac{\hat{\beta}^2}{12}s^3+\frac{\hat{\gamma}^4}{160}s^4+\cdots \equiv\sum_{l=0}^{\infty} b_l s^l , \tag{4.2}$$

where $\hat{\gamma}^2\equiv\mathrm{tr}\hat{\gamma}^2{}_{\mu\gamma}$. Separating α^2 into a constant m^2 term and a term $U=\frac{1}{2}\lambda\hat{\phi}^2-\frac{1}{6}\xi R$ containing the varying field and curvature as in Eq. (2.16a), $L^{(1)}$ can be approximated by,

$$L^{(1)}\simeq-\frac{\hbar}{2(4\pi)^{n/2}}\int_0^\infty \frac{ds\, e^{-m^2 s}}{s^{1+n/2}}(1-Us+U^2s^2/2-U^3s^3/3!+U^4s^4/4!-\hat{\gamma}^2s^2/12+\hat{\beta}^2s^3/12+\hat{\gamma}^4s^4/160+\cdots) . \tag{4.3}$$

The quantity within the parentheses is a small-proper-time expansion of the heat kernel $\bar{K}(x,x,s)$ defined by [cf. Eqs. (2.29) and (3.6)]

$$\bar{K}(x,x,s)\equiv e^{-Us-f(s)}=\sum_{l=0}^{\infty} a_l s^l . \tag{4.4}$$

The coefficients a_l are the Minakshisundaram-DeWitt coefficients[23,28,29] corresponding to the operator [Eq. (2.14)]

$$H=(-g)^{-1/4}\Box(-g)^{-1/4}+m^2+\tfrac{1}{2}\lambda\hat{\phi}^2(x)+(1-\xi)R/6$$

of an inhomogeneous scalar field $\hat{\phi}(x)$ in a background spacetime with curvature R. By examining (4.3), we obtain the following results:

$$a_0=1 , \tag{4.5a}$$

$$a_1=-U , \tag{4.5b}$$

$$a_2=\frac{U^2}{2}-\frac{\hat{\gamma}^2}{12} . \tag{4.5c}$$

Recall the definition of $\hat{\gamma}^2{}_{\mu\nu}$ from (2.16c) and (2.17). We see that

$$\tfrac{1}{4}\hat{\gamma}^2=\tfrac{1}{2}\Box U+a^\lambda{}_\lambda \tag{4.6}$$

and

$$a^\lambda{}_\lambda=-\tfrac{1}{60}(\Box R-R^{\alpha\beta}R_{\alpha\beta}+R^{\alpha\beta\gamma\delta}R_{\alpha\beta\gamma\delta}) , \tag{4.7a}$$

where R, $R_{\alpha\beta}$, and $R_{\alpha\beta\gamma\delta}$ are the scalar, Ricci, and Riemann curvature tensors, respectively. Equivalently, $a^\lambda{}_\lambda$ can be expressed in terms of the two covariant quantities F and G in the form

$$a^\lambda{}_\lambda=-\tfrac{1}{60}[\Box R-\tfrac{1}{2}(G-3F)] , \tag{4.7b}$$

where

$$F=R^{\alpha\beta\gamma\delta}R_{\alpha\beta\gamma\delta}-2R^{\alpha\beta}R_{\alpha\beta}+\tfrac{1}{3}R^2 , \tag{4.8a}$$

which reduces to the Weyl curvature tensor squared $C^{\alpha\beta\gamma\delta}C_{\alpha\beta\gamma\delta}$ in four-dimensions, and

$$G=R^{\alpha\beta\gamma\delta}R_{\alpha\beta\gamma\delta}-4R^{\alpha\beta}R_{\alpha\beta}+R^2 \tag{4.8b}$$

is the Gauss-Bonnet density, a total divergence in four-dimensions. This generalized form of a_l given in (4.5) has been derived before by Gilkey[30] using differential geometric methods. Our derivation by means of the quasilocal expansion follows the method of DeWitt.[23] It is well known that the a_2 coefficient is related to the trace anomaly T in the standard way:

$$T=\langle T^\mu{}_\mu\rangle_{\mathrm{ren}}=-\frac{a_2(x)}{16\pi^2} . \tag{4.9}$$

Our results (4.5c) and (4.6) show that the trace anomaly of an inhomogeneous scalar field in curved spacetime acquires an additional nonlocal contribution from the second variation of the scalar field. (Note the absence of first derivative terms $\propto\beta$, as expected.) For models involving time-dependent Higgs fields (e.g., symmetry breaking in GU theories, induced gravity theories, or higher-order gravitational theories), their contribution to the trace anomaly (analogous to the $\Box R$ term) may have interesting theoretical and cosmological consequences.

The one-loop effective Lagrangian is given by

$$L^{(1)}=-\frac{\hbar}{2(4\pi)^{n/2}}\sum_{l=0}^{\infty} a_l m^{n-2l}\Gamma(l-n/2) , \tag{4.10}$$

where

$$\Gamma(x)\equiv\int_0^\infty ds\, s^{x-1}e^{-s}$$

is the Γ function. In order to keep this in the same dimension as $L^{(1)}$, it is customary to introduce a renormalization mass parameter such that

$$L^{(1)}=-\frac{\hbar}{2(4\pi)^{n/2}}\left[\frac{m}{\mu_D}\right]^{n-4}\sum_{l=0}^{\infty}a_l m^{4-2l}\Gamma(l-n/2)\ . \qquad (4.11)$$

In the spirit of dimensional regularization, we expand the first three terms in the series near $n=4$. Making use of the well-known expansion relations for $(m/\mu_D)^{n-4}$ and $\Gamma(l-n/2)$, we get

$$L^{(1)}=\frac{\hbar m^4}{32\pi^2}\left\{\frac{1}{n-4}+\frac{1}{2}\left[\ln\left[\frac{m^2}{4\pi\mu_D{}^2}\right]+\gamma-\tfrac{3}{2}\right]\right\}-\frac{\hbar a_1 m^2}{16\pi^2}\left\{\frac{1}{n-4}+\frac{1}{2}\left[\ln\left[\frac{m^2}{4\pi\mu_D{}^2}\right]+\gamma-\tfrac{1}{2}\right]\right\}$$
$$+\frac{\hbar a_2}{16\pi^2}\left\{\frac{1}{n-4}+\frac{1}{2}\left[\ln\left[\frac{m^2}{4\pi\mu_D{}^2}\right]+\gamma\right]\right\}-\frac{\hbar}{2(4\pi)^2}\sum_{l=3}^{\infty}a_l m^{4-2l}\Gamma(l-2)\ . \qquad (4.12)$$

Here and only here $\gamma=0.5772$ is the Euler constant.

In a regularization scheme the divergent terms $[\sim(n-4)^{-1}]$ can be removed by the introduction of counterterms L_c in the effective Lagrangian, which in their most general form contain linear combinations of F and G [defined in (4.8)]. The regularized Lagrangian then consists of terms in (4.12) together with the counterterms, (see, e.g., Ref. 31). In a renormalization scheme, the divergent terms are attributed to changes in the coupling constants of the theory—the cosmological constant Λ, Newton's constant G_N, constants ϵ_1, ϵ_2, ϵ_3, and ϵ_4 of the curvature squared, and $\Box R$ terms [see (4.16) below] for the geometry and m, θ, ξ, and λ for the fields. Since this procedure is by now standard we do not intend to belabor it (see, e.g., Ref. 1, Sec. 6.2). Instead, we want to illustrate this procedure in a slightly different way, which is more suitable for interacting fields in curved spacetime.

The expansion of the integrand of $L^{(1)}$ in (2.29) for small s can be performed in different ways. Factoring out the constant (mass)^2 term as we did in (4.3) and grouping the field variables with the curvature terms is closest in spirit and form to flat-space field theory. The curvature of spacetime and the fields enter only through the a_l coefficients. However, when dealing with problems involving interacting fields in dynamic spacetimes, the mass acquires additional contributions from $\hat{\phi}$, and R, and it is the modified mass $\alpha^2=m^2+\frac{1}{2}\lambda\hat{\phi}^2-\xi R/6$ or its related quantities $\mathscr{M}^2$or$\overline{\mathscr{M}}^2$ which are the more relevant physical parameters. For example, in symmetry-breaking problems, on the classical level, the phase transition occurs near $\overline{\mathscr{M}}^2=m^2+\frac{1}{6}\lambda\hat{\phi}^2+(1-\xi)R/6=0$, while on the quantum level it occurs near $\overline{\mathscr{M}}^2+\frac{1}{2}\lambda\langle\phi^2\rangle_0=0$, where the effective mass acquires a radiative correction. In dynamic spacetimes, further contributions can arise from the kinetic terms of the operator. One should seek approximations according to the correct range of the effective mass instead of the intrinsic mass. (For a more detailed discussion of this point, see Ref. 13.) This difference shows up more distinctly in problems like symmetry breaking when one is interested in the long-range or infrared (IR) behavior of the theory. On the other hand, the identification of ultraviolet (UV) divergences (corresponding to the large-M or small-R domain) should be insensitive to the local structure of spacetime. In the same vein the UV behavior should also be identical to that obtained from exact models, as long as the spacetime is locally Minkowskian. We shall demonstrate this by comparing the result we are about to obtain with that obtained earlier for the Einstein universe.[13] The comparison will also enable us to derive general results on the renormalization-group equations with little extra effort.

Carrying out a small-s expansion of $L^{(1)}$ using α^2 as the mass parameter, we obtain the following results for the regularized one-loop effective Lagrangian [cf. (4.12) with redefined μ where $\ln\mu^2=\ln 4\pi\mu_D{}^2-\gamma-2/(n-4)$, μ now agrees with μ_H of Ref. 13]:

$$L^{(1)}=\frac{\hbar}{64\pi^2}\left[\alpha^4\left[\ln\frac{\alpha^2}{\mu^2}-\frac{3}{2}\right]-\frac{\hat{\gamma}^2}{6}\ln\frac{\alpha^2}{\mu^2}-\sum_{l=3}^{\infty}b_l\alpha^{4-2l}\Gamma(l-2)\right]\ , \qquad (4.13)$$

where

$$b_0=a_0=1\ ,\quad b_1=0\ ,\quad b_2=-\frac{\hat{\gamma}^2}{12}\ . \qquad (4.14)$$

Note the disappearance of U terms in b_l but the presence of nonlocal term $\Box U$ in b_2. The terms in (4.13) proportional to $\hat{\gamma}^2$, which contains contributions arising from the changing background field, will be the focus of attention in future discussions of dynamical processes. To compare this with our earlier result for the Einstein universe, the effective potential $V^{(1)}$ $[=L^{(1)}]$ is obtained from Eqs. (12), (15), and (16) of Ref. 13 to be

$$V^{(1)}=\frac{\hbar}{4\pi^2a^4}\left[\left[\frac{x^2}{16}\ln\frac{x}{\mu^2a^2}-\frac{3}{2}\right]\right]\quad(x\gg1)\ , \qquad (4.15)$$

which is identical to the first term in (4.13), as $\hat{\gamma}=0$ in static spacetimes. For the Einstein universe which is conformally flat, a more natural set of parameters to use is the conformally related mass $x=\alpha^2a^2$.

In anticipation of the appearance of higher-order curvature terms in the renormalization procedure, one should write the classical Lagrangian for the background spacetime in the most general form:

$$L_g^{(0)}=\Lambda+\kappa R+\tfrac{1}{2}\epsilon_1R^2+\tfrac{1}{2}\epsilon_2C^2+\epsilon_3G+\epsilon_4\Box R,\qquad (4.16)$$

where Λ is the cosmological constant, $\kappa=(16\pi G_N)^{-1}$, G_N being Newton's constant, ϵ_1 and ϵ_2 are, respectively, the coupling constants for the scalar curvature-squared R^2 and Weyl curvature-squared $C^2=C^{\alpha\beta\gamma\delta}C_{\alpha\beta\gamma\delta}$ [which is equal to F defined by (4.8a) in four-dimensions]. We have also added the total divergence terms G and $\Box R$ with coupling constants ϵ_3 and ϵ_4 for completeness, but they only enter in spacetimes with boundaries. The complete Lagrangian to one-loop is given by the sum of (4.16), (2.7), and (4.13):

$$L = L_g^{(0)} + L_\phi^{(0)} + L^{(1)} . \tag{4.17}$$

All parameters of the theory prior to this point should be regarded as bare quantities. We now proceed to define the renormalized parameters by suitable renormalization conditions [Eq. (14) of Ref. 13]:

$$m^2 = \left[\mathrm{Re}\frac{\partial^2 L}{\partial\hat\phi^2}\right]_{\hat\phi=0,\ R=0}, \quad \lambda = \left[\mathrm{Re}\frac{\partial^4 L}{\partial\hat\phi^4}\right]_{\hat\phi=\phi_0,\ R=0},$$

$$\theta = \left[\mathrm{Re}\frac{\partial L}{\partial\Box\hat\phi^2}\right]_{\hat\phi=\phi_\theta,\ R=0},$$

$$\frac{1-\xi}{6} = \left[\mathrm{Re}\frac{\partial^3 L}{\partial R\,\partial\hat\phi^2}\right]_{\hat\phi=0,\ R=R_0},$$

$$0 = (\mathrm{Re}L)_{\hat\phi=\langle\phi\rangle,\ R=0}, \quad \kappa = \left[\mathrm{Re}\frac{\partial L}{\partial R}\right]_{\hat\phi=0,\ R=0}, \tag{4.18}$$

$$\epsilon_1 = \left[\mathrm{Re}\frac{\partial^2 L}{\partial R^2}\right]_{\hat\phi=0,\ R=R_1}, \quad \epsilon_2 = 2\left[\mathrm{Re}\frac{\partial L}{\partial C^2}\right]_{\hat\phi=0,\ R=R_2},$$

$$\epsilon_3 = \left[\mathrm{Re}\frac{\partial L}{\partial G}\right]_{\hat\phi=0,\ R=R_3}, \quad \epsilon_4 = \left[\mathrm{Re}\frac{\partial L}{\partial\Box R}\right]_{\hat\phi=0,\ R=R_4}.$$

The real part in the above equations should be taken, as the renormalized terms are required to be real quantities. The coupling constants λ, θ, ξ, ϵ_1, ϵ_2, ϵ_3, and ϵ_4 are defined at the energy scales corresponding to the values $\phi=\phi_0$, ϕ_θ, $R=R_0$, R_1, R_2, R_3, and R_4, respectively, which may in general be all different as the energy scales at which they are measured need not be the same. How these coupling constants behave under changes of scales is determined by their associated renormalization-group equations.

The renormalized parameters are related to the bare ones by counterterms $m^2 = m_B{}^2 + \delta m^2$, etc., where the counterterms δm^2 are obtained from (4.18). Some of the counterterms have been calculated before [Eq. (17) of Ref. 13], and we present the complete result here:

$$\delta m^2 = \tilde h\lambda m^2\left[\ln\frac{m^2}{\mu^2} - 1\right] \equiv \tilde h\lambda m^2(\tau-\tau_m), \tag{4.19a}$$

$$\delta\lambda = 3\tilde h\lambda^2\left[\ln\frac{m^2+\frac12\lambda\phi_0{}^2}{\mu^2} + \frac83 - \frac{8(m^2+\frac14\lambda\phi_0{}^2)m^2}{3(m^2+\frac12\lambda\phi_0{}^2)^2}\right]$$
$$\equiv 3\tilde h\lambda^2(\tau-\tau_\lambda), \tag{4.19b}$$

$$\delta\theta = -\frac{\tilde h\lambda}{12}\left[\ln\frac{m^2+\frac12\lambda\phi_\theta{}^2}{\mu^2}\right] \equiv -\frac{\tilde h\lambda}{12}(\tau-\tau_\theta), \tag{4.19c}$$

$$\delta\xi = \tilde h\lambda\xi\left[\ln\frac{m^2-\frac16\xi R_0}{\mu^2}\right] \equiv \tilde h\lambda\xi(\tau-\tau_0), \tag{4.19d}$$

$$\delta\Lambda = \frac{\tilde h m^4}{2}\left[\ln\frac{m^2}{\mu^2} - \frac32\right] + \delta\Lambda(\langle\phi\rangle)$$
$$\equiv \frac{\tilde h m^4}{2}(\tau-\tau_\Lambda) + \delta\Lambda(\langle\phi\rangle), \tag{4.19e}$$

$$\delta\kappa = -\tilde h m^2\frac{\xi}{6}\left[\ln\frac{m^2}{\mu^2} - 1\right] \equiv -\tilde h m^2\frac{\xi}{6}(\tau-\tau_\kappa), \tag{4.19f}$$

$$\delta\epsilon_1 = \frac{\tilde h\xi^2}{36}\left[\ln\frac{m^2-\frac16\xi R_1}{\mu^2}\right] \equiv \frac{\tilde h\xi^2}{36}(\tau-\tau_1), \tag{4.19g}$$

$$\delta\epsilon_2 = \frac{\tilde h}{60}\left[\ln\frac{m^2-\frac16\xi R_2}{\mu^2}\right] \equiv \frac{\tilde h}{60}(\tau-\tau_2), \tag{4.19h}$$

$$\delta\epsilon_3 = -\frac{\tilde h}{360}\left[\ln\frac{m^2-\frac16\xi R_3}{\mu^2}\right] \equiv -\frac{\tilde h}{360}(\tau-\tau_3), \tag{4.19i}$$

$$\delta\epsilon_4 = \frac{\tilde h}{36}(\xi+\tfrac15)\left[\ln\frac{m^2-\frac16\xi R_4}{\mu^2}\right]$$
$$\equiv \frac{\tilde h}{36}(\xi+\tfrac15)(\tau-\tau_4), \tag{4.19j}$$

where $\tau=\ln(m^2/\mu^2)$, $\tilde h = \hbar/32\pi^2$. (We have renamed the R_i and the τ_i from Ref. 13.) We include the terms $\epsilon_4\Box R$ and $\theta\Box\hat\phi^2$ in our classical actions, $L_g^{(0)}$ and $L_\phi^{(0)}$ in (4.16) and (2.7), even though these are purely surface terms, since divergent terms proportional to them occur at the one-loop level. These divergences are most satisfactorily removed by absorption into the appropriate parameters in the bare Lagrangian. The Gauss-Bonnet density G is introduced for similar reasons, its integral over the manifold is a topological invariant, $32\pi^2\chi$, χ being the Euler characteristic. We thus expect χ to be present in the action for all manifolds, with or without boundary.

From the counterterms (4.19) one can obtain the renormalization-group equations for each of the parameters:

$$\frac{dm^2}{d\tau} = \tilde h\lambda m^2, \quad \frac{d\lambda}{d\tau} = 3\tilde h\lambda^2, \quad \frac{d\theta}{d\tau} = -\frac{\tilde h\lambda}{12}, \quad \frac{d\xi}{d\tau} = \tilde h\lambda\xi,$$

$$\frac{d\Lambda}{d\tau} = \frac{\tilde h m^4}{2}, \quad \frac{d\kappa}{d\tau} = -\tilde h m^2\frac{\xi}{6}, \quad \frac{d\epsilon_1}{d\tau} = \frac{\tilde h\xi^2}{36}, \tag{4.20}$$

$$\frac{d\epsilon_2}{d\tau} = \frac{\tilde h}{60}, \quad \frac{d\epsilon_3}{d\tau} = -\frac{\tilde h}{360}, \quad \frac{d\epsilon_4}{d\tau} = \frac{\tilde h}{36}(\xi+\tfrac15).$$

Solutions to these equations can be obtained with the appropriate boundary conditions which preserve Eq. (4.19) at the specific renormalization point:

$$m^2(\tau) = m^2(\tau_m)\left[\frac{L(\tau_m)}{L(\tau)}\right], \tag{4.21a}$$

$$\lambda(\tau) = \frac{\lambda(\tau_\lambda)}{L^3(\tau)}, \tag{4.21b}$$

$$\theta(\tau) = \theta(\tau_\theta) - \frac{1}{12}\left[\ln\frac{L(\tau_\theta)}{L(\tau)}\right], \tag{4.21c}$$

$$\xi(\tau)=\xi(\tau_0)\left[\frac{L(\tau_0)}{L(\tau)}\right], \tag{4.21d}$$

$$\Lambda(\tau)=\Lambda(\tau_\Lambda)-\frac{m^4}{2\lambda(\tau_\lambda)}L^2(\tau_m)[L(\tau)-L(\tau_\Lambda)], \tag{4.21e}$$

$$\kappa(\tau)=\kappa(\tau_\kappa)+\frac{\xi(\tau_0)m^2(\tau_m)}{6\lambda(\tau_\lambda)}L(\tau_m)L(\tau_0)[L(\tau)-L(\tau_\kappa)], \tag{4.21f}$$

$$\epsilon_1(\tau)=\epsilon_1(\tau_1)+\frac{\xi^2(\tau_0)}{36\lambda(\tau_\lambda)}L^2(\tau_0)[L(\tau_1)-L(\tau)], \tag{4.21g}$$

$$\epsilon_2(\tau)=\epsilon_2(\tau_2)+\frac{\tilde{h}}{60}(\tau-\tau_2), \tag{4.21h}$$

$$\epsilon_3(\tau)=\epsilon_3(\tau_3)-\frac{\tilde{h}}{360}(\tau-\tau_3), \tag{4.21i}$$

$$\epsilon_4(\tau)=\epsilon_4(\tau_4)+\frac{\xi(\tau_0)L(\tau_0)}{72\lambda(\tau_\lambda)}[L^2(\tau_4)-L^2(\tau)]+\frac{\tilde{h}}{180}(\tau-\tau_4), \tag{4.21j}$$

where $L(\tau)\equiv[1-3\tilde{h}\lambda(\tau_\lambda)(\tau-\tau_\lambda)]^{1/3}$ and τ_m to τ_4 are defined in Eq. (4.19). Henceforth, whenever the explicit functional dependence of the parameters on τ is not displayed they should be regarded as assuming their values at their respective renormalization points.

The renormalized one-loop effective Lagrangian is given by

$$\begin{aligned}L_{\rm eff}=&\Lambda+\kappa R+\frac{\epsilon_1}{2}R^2+\frac{\epsilon_2}{2}C^2+\epsilon_3 G+\epsilon_4\Box R+\tfrac{1}{12}(1-\xi)R\hat{\phi}^2+\theta\Box\hat{\phi}^2+L_{\rm CW}\\
&+\frac{\hbar}{64\pi^2}\left[-\frac{\lambda}{6}\Box\hat{\phi}^2\ln\left|\frac{M^2}{m^2+\frac{1}{2}\lambda\phi_\theta^2}\right|-\frac{\xi}{6}\lambda\hat{\phi}^2R\left[\ln\left|\frac{M^2}{M^2-\xi R_0/6}\right|-\frac{3}{2}\right]-\tfrac{1}{3}m^2\xi R\left[\ln\left|\frac{M^2}{m^2}\right|-\frac{1}{2}\right]\right.\\
&+\frac{\xi^2}{36}R^2\left[\ln\left|\frac{M^2}{m^2-\xi R_1/6}\right|-\frac{3}{2}\right]+\tfrac{1}{60}C^2\ln\left|\frac{M^2}{m^2-\xi R_2/6}\right|-\tfrac{1}{180}G\ln\left|\frac{M^2}{m^2-\xi R_3/6}\right|\\
&\left.+\tfrac{1}{18}(\xi+\tfrac{1}{5})\Box R\ln\left|\frac{M^2}{m^2-\xi R_4/6}\right|\right]\\
&+\delta\Lambda(\langle\phi\rangle)+\frac{\hbar}{64\pi^2}\left[\alpha^4-\frac{\hat{\gamma}^2}{6}\right]\ln\frac{\alpha^2}{M^2}-\frac{\hbar}{32\pi^2}\int_0^\infty\frac{ds}{s^3}\left[e^{-\alpha^2 s-f(s)}-1+\frac{\hat{\gamma}^2s^2}{12}\right],\end{aligned} \tag{4.22}$$

where $\delta\Lambda$ is determined by the vacuum energy of flat space. Here $L_{\rm CW}$ denotes the Coleman-Weinberg effective Lagrangian for $\lambda\phi^4$ theory in flat space given by

$$\begin{aligned}L_{\rm CW}=&\tfrac{1}{2}\hat{\phi}\Box\hat{\phi}+\tfrac{1}{2}m^2\hat{\phi}^2+\frac{\lambda}{4!}\hat{\phi}^4+\frac{\hbar}{64\pi^2}\left[M^4\ln\left|\frac{M^2}{m^2}\right|-\frac{m^2}{2}\lambda\hat{\phi}^2\right.\\
&\left.+\frac{\lambda\hat{\phi}^4}{4}\left[\ln\left|\frac{m^2}{m^2+\frac{1}{2}\lambda\phi_0^2}\right|-\frac{25}{6}+\frac{8}{3}\frac{m^2(m^2+\frac{1}{4}\lambda\phi_0^2)}{(m^2+\frac{1}{2}\lambda\phi_0^2)^2}\right]\right].\end{aligned} \tag{4.23}$$

The integral expression in (4.22) contains terms which were finite to start with. The analytic form of $f(s)$ is given by (2.30).

Combining the results obtained above for the effective action with the results of integrating the renormalization-group equations (4.21), we obtain the renormalization-group-improved effective Lagrangian

$$\begin{aligned}L_{\rm eff}=&\Lambda(\tau)+\kappa(\tau)R+\tfrac{1}{2}\epsilon_1(\tau)R^2+\tfrac{1}{2}\epsilon_2(\tau)C^2+\epsilon_3(\tau)G+\epsilon_4(\tau)\Box R+\tfrac{1}{2}\hat{\phi}\Box\hat{\phi}+\theta(\tau)\Box\hat{\phi}^2+\tfrac{1}{2}m^2(\tau)\hat{\phi}^2\\
&+\tfrac{1}{12}[1-\xi(\tau)]R\hat{\phi}^2+\frac{\lambda(\tau)}{4!}\hat{\phi}^4-\frac{\hbar}{32\pi^2}\int_0^\infty\frac{ds}{s^3}\left[e^{-\alpha^2s-f(s)}-1+\frac{\hat{\gamma}^2s^2}{12}\right].\end{aligned} \tag{4.24}$$

In the above expression $\tau=\ln(\alpha^2/m^2)$ and the parameters are as in Eq. (4.21).

As a reminder, all expressions in Secs. III and IV up to this point are in Euclidean form. To rewrite expressions in Lorentzian form the transformations introduced in Sec. II must be undone. By using Eq. (2.20) $\hat{\beta}$ and $\hat{\gamma}$ can be converted back to β and γ by $\tau\to it$ and $\sqrt{-g_E}L_E\to-\sqrt{-g}L$.

V. DISCUSSION

The result obtained here generalizes the Coleman-Weinberg (CW) potential to curved spacetime with slowly varying background fields. It provides a better framework than the existing flat-space field theory for the analysis of quantum processes in the early universe where both the background spacetime and background fields are dynamic. The one-loop effective Lagrangian (4.22) is in the form of a quasi-potential, in that it includes up to the second-order derivatives in the background field, but is exact to all orders of the coupling constant λ. As an improvement over perturbative (in λ) methods in curved-spacetime interacting field theory, the exact form we obtained is useful for the study of gravitational and dynamic field effects on symmetry breaking. When generalized to gauge theories, it should replace the flat-space CW potential usually assumed in most discussions of the (new) inflationary universe.[15,16] One can use it to study the effect of the so-called kinetic term in this quasipotential. The effect of spacetime curvature on symmetry breaking was the subject of our earlier paper.[13] Similar analysis carried out for the present problem shows that the effective mass $\mathscr{M}$ in the theory depends on the second derivative of the background field, suggesting a form of dynamical symmetry breaking due to the changing background spacetime alone, without invoking particle interactions. Derivation of this result and the discussion of its cosmological implication will appear in a companion paper.

There are a number of ways in which the present work can be extended for the treatment of more complicated situations:

(1) Generalization to non-Abelian gauge fields would be useful for the analysis of more realistic particle models in grand unification theories such as SU(5). The basic framework for treating time-dependent background gauge fields in flat space was discussed in Ref. 26. The extension to curved spacetime will be similar to that treated here. The behavior of the coupling constants in an asymptotically free theory was discussed recently by Parker and Toms.[5] Their method, which was based on the renormalization-group (RG) equations, cannot be extended to regions where there is symmetry breaking. This is because the RG equations derived from the counterterms scale only the local, ultraviolet behavior, while nonlocal infrared behavior of the theory would be important during phase transitions. The method presented here, especially the heat-kernel technique (discussed in Sec. III) would allow one to analyze to a limited extent the behavior of the system close to the critical point.

(2) The present method can be applied to the consideration of finite-temperature quantum fields in curved space. The conceptual problem of defining thermal equilibrium in a dynamic background and the technical problem of setting up such a theory in curved space have been discussed in earlier work by one of us.[25] There we pointed out the importance of the existence of a global conformal Killing vector in the spacetime and the conformal invariant properties of fields as conditions for the maintenance of thermal equilibrium. We introduced the quasi-adiabatic n-particle states, and used an adiabatic expansion to calculate the finite-temperature energy density for free fields as well as the finite-temperature effective potential for interacting fields. To the extent that the method adopted in the present work for zero-temperature theory is similar to the adiabatic expansion method (see discussion in Sec. II), one can apply it to finite-temperature theory in curved space in parallel with what we have done, subject to the same physical conditions as we have described. Technically, finite-temperature theory in curved space is only well defined for fields which are nearly conformally invariant and spacetimes which are nearly conformally static. Under those conditions one can define thermal Green's functions by imposing quasi-periodic conditions on the imaginary time and adopting a Riemann normal-coordinate expansion for the background spacetime to include the effect of spacetime curvature on the thermal properties of the system. This approach was adopted by Drummond[32] and Critchley *et al.*[33] in the derivation of the thermal Green's function for Robertson-Walker spacetimes. Combining the results from the present calculation, one can in principle derive an exact finite-temperature effective (quasi-) potential including variations of the background field up to the second order. It would be useful for studying finite-temperature effects in early-universe quantum processes. In practice, the quasilocal expansion technique used in this paper, like the adiabatic expansion used in our earlier work, is only applicable for high-temperature regimes. Moreover, in symmetry-breaking problems, as far as the influence of background fields in the critical temperature is concerned, calculations up to the second adiabatic order are not expected to yield results different from that of the flat space. The effect arising from nonadiabatic variations of the background field and spacetime only begin to appear in the fourth adiabatic order.[34]

(3) Extension to include higher-order variations of the background field will enable one to analyze situations where the spacetime changes more rapidly, as in the early universe near the Planck time or in black-hole collapses. The quasilocal expansion for homogeneous fields is related to the adiabatic expansion. Although exact forms of the effective Lagrangian, including higher-order nonlocal terms, do not always exist, one can nevertheless carry out perturbation expansions (in a parameter similar to the nonadiabaticity parameter $\overline{\Omega}$ in Ref. 25) using the second-order exact Lagrangian as a starting point. The corresponding model is that of an anisotropic anharmonic oscillator. When the nonlocality of the background field is viewed as possessing spatial inhomogeneity $\hat{\phi}(x^i)$ rather than being dynamic $\hat{\phi}(t)$, the present problem is analogous to assuming next-next-neighbor spin interactions in a Ginsburg-Laudau theory. It is also similar to the multipole expansion in QCD for the description of soft-gluon processes, where large-distance (or low-momentum) behavior becomes more important. Of course, higher-order nonlocal approximations will break down beyond a certain point. When tackling problems where the spacetime changes so rapidly that particle production becomes important, or when global properties of the spacetime (e.g., topology or boundary) are involved, nonperturbative methods will have to be invoked.

(4) One can generalize the momentum-space technique to global spacetimes with symmetry. The momentum-space technique is local in nature as it relies on the Riemann normal-coordinate expansion in a local patch. It allows one to transcribe Feynman propagator rules from flat-space field theory to curved space and greatly facilitates the treatment of ultraviolet divergences in quantum field theory. However, since it is a local technique, it would not be sensitive to the global properties of a spacetime. As remarked earlier in Sec. II, the result obtained from it is useful for the study of local curvature and dynamic field effects on quantum processes such as dynamical symmetry breaking and cosmological particle production (not involving event horizons), but insufficient for, say, symmetry breaking due to changes in topology or boundary. To improve on this while still adhering to the momentum-space approach, at least for quantum fields in spacetimes possessing a certain degree of symmetry (as in homogeneous cosmology), one can try to construct viable quantum field theories with techniques from harmonic analysis on group manifolds. One example previously studied is the SO(3) spatially homogeneous Bianchi type-IX universe. In the diagonal mixmaster metric[35]

$$ds^2 = -dt^2 + \sum_{a=1}^{3} l_a{}^2(t)(\sigma^a)^2 ,$$

where ${}^{(3)}g_{ab} = l_a{}^2\delta_{ab}$ is the metric of the hypersurface and the invariant one-forms σ^a obey the structure relation $d\sigma^a = \frac{1}{2}\epsilon^a{}_{bc}\sigma^b \Lambda \sigma^c$, the Hamiltonian operator for a scalar field[36] has a kinetic term of the form $H = {}^{(3)}g^{ab}e_a e_b$, where e_a are the invariant vectors of the rotation group, $\sigma^a[e_b] = \delta^a{}_b$. Here e_a and σ^a play the roles of canonical "momenta" and "coordinates" on a group manifold. With them one can construct a "momentum representation" of field propagators in such spaces. Nonlocal Riemann expansion techniques on homogeneous background spaces (rather than Minkowski space) and group-theoretical methods have been developed earlier for the study of wave equations in homogeneous cosmologies.[37] More recently they have been used for analyzing the stability of instanton solutions in quantum gravity and Kaluza-Klein theories.[38] These more sophisticated methods are direct extensions of the quasilocal techniques for near-flat-space theories. They will make possible the analysis of effects due to the global properties of spacetime on quantum processes such as symmetry breaking and particle production. We hope to report on our investigations of the above-mentioned problems in the future.

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