An extended scalar-tensor theory of gravitation

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A link between several scalar-tensor theories of gravitation and Einstein's theory is proposed. It is based on a Lagrangian in which the part depending on the scalar field vanishes under the usual conditions, but under special conditions new aspects of gravitation can be forecast.

I. INTRODUCTION

This paper tries to make a link between several scalar-tensor theories of gravitation and Einstein's theory. Many sound arguments have been presented which modify Einstein's equation of gravitation, from Jordan[1] and Brans and Dicke[2] up to some more recent works,[3–6] by adding a supplementary scalar field. Using these arguments, one can add a term of the form

$$L_S = \epsilon f_{,k} f^{,k} \tag{1.1}$$

in the action principle leading to Einstein's equation. The action principle then reads

$$\delta \int dx\, g^{1/2}(R + L_M + L_S) = 0 \tag{1.2}$$

(R is the scalar curvature of space-time and L_M is the matter Lagrangian density).

However, the gravitation experiments performed hitherto have strongly supported Einstein's equation.[7] We shall denote shortly by E the coordinate frame and the conditions under which the tests of the Einstein theory have been carried out. Thus, under the usual conditions E the factor ϵ must tend to 0. Now, in order to keep the term (1.1) in the general case, it is assumed that ϵ does not vanish under other conditions than E, in other domains of space. Therefore, the physical basis of the theory is the combination of the arguments of the existing scalar-tensor theories with the experimental data supporting Einstein's theory.

II. THE EXPRESSION FOR ϵ

One can see now that ϵ has the physical significance of being a spatial magnitude characterizing the contribution of the scalar field to gravitation. Consequently, ϵ has to be linked to a characteristic feature of space. Therefore, one may assume that the metric g_{ij} and its variation must enter the expression for ϵ. The Christoffel symbol is suitable; the simplest way to express the scalar ϵ with its help is by

$$\epsilon = -i(V^r_{,r} + \Gamma^r_{sr} V^s) , \tag{2.1}$$

where V^r is an imaginary vector (constant in E). This vector is not a measurable field, but a calculus magnitude which ensures the scalar character of ϵ. Therefore it has no physical role in the theory. The theory is based on ϵ, which is in equivalence with Γ^r_{sr} (V^r being just the link).

Following the above considerations, ϵ vanishes under conditions E:

$$(\epsilon)_0 = -i[V^r_{sr} + (\Gamma^r_{sr})_0 V^s] \cong 0 . \tag{2.2}$$

We have denoted by $(\Gamma^r_{sr})_0$ the value of Γ^r_{sr} under the conditions E. Under different conditions one may have

$$\Gamma^r_{sr} > (\Gamma^r_{sr})_0 \tag{2.3}$$

and $\epsilon \neq 0$.

III. THE FIELD EQUATIONS

The variation of Eq. (1.2) with respect to g^{ij} yields the equation

$$R_{ij} - g_{ij}(R/2) + \epsilon f_{,i} f_{,j} + \left\{ -\epsilon g_{ij} f_{,k} f^{,k}/2 + \left[\frac{\partial \epsilon}{\partial g^{ij}} - (g^{mn}/2) g_{mn,l} \frac{\partial \epsilon}{\partial g^{ij}{}_{,l}} - \left[\frac{\partial \epsilon}{\partial g^{ij}{}_{,l}} \right]_{,l} \right] f_{,k} f^{,k} - \frac{\partial \epsilon}{\partial g^{ij}{}_{,l}} (f_{,k} f^{,k})_{,l} \right\} = k_G T_{ij} , \tag{3.1}$$

where T_{ij} is the matter tensor. In obvious shorthand notation we can write Eq. (3.1) as

$$R_{ij} - g_{ij}(R/2) = k_G T_{ij} + S_{ij} . \tag{3.2}$$

In the frame and under the conditions E, Eq. (2.2) is valid and Eq. (3.2) reduces to the Einstein equation. But under different conditions, when we have Eq. (2.3), then Eq. (3.2) holds. This can be true in regions of the universe where the variation of the metric is very rapid, maybe with pulsars, black holes, gravitational collapse, etc. Of course, Eq. (2.3) is not a covariant condition, but if it is valid in one coordinate frame, ϵ, being a scalar, remains the same in every coordinate frame. The variation (1.2) with respect to f yields the equation for the scalar field

$$\Box f + f^{,j}(\epsilon_{,j}/\epsilon)=0 , \tag{3.3}$$

where $\Box$ is the covariant D'Alembertian.

For

$$\epsilon=\text{const}=\epsilon_c \tag{3.4}$$

Eq. (3.1) reduces to

$$R_{ij}-g_{ij}(R/2)+\epsilon_c f_{,i}f_{,j}-\epsilon_c g_{ij}f_{,k}f^{,k}/2=k_G T_{ij} , \tag{3.5}$$

which corresponds to the equations from many known scalar-tensor theories of gravitation. Thus, Eq. (3.5) is identical with the equations of Tupper[4] and Lindström[8] and, setting $\epsilon_c=\frac{3}{2}$, $\epsilon_c=-1$, with those of Sen and Vanstone[9] and of Yilmaz.[10] The substitution

$$f=\ln\lambda \tag{3.6}$$

in Eq. (3.5) yields the equation of Dicke in the second formulation,[11] which by a transformation of units leads to the equation of Brans and Dicke.[2] *In vacuo*, the same substitution (3.6) yields the equation of Dunn[3] and Van den Bergh.[12]

With the condition (3.4), Eq. (3.3) becomes

$$\Box f=0 . \tag{3.7}$$

This is identical with the equations for the scalar field *in vacuo* given by the theories referred to in Refs. 2—4,8,10, and 11.

IV. EQUATION OF MOTION

The Einstein tensor being divergenceless, the right-hand side of Eq. (3.2) is also divergenceless

$$D_j(k_G T^{ij}+S^{ij})=0 . \tag{4.1}$$

In general,

$$D_j S^{ij}\neq 0 , \tag{4.2}$$

therefore also

$$D_j T^{ij}\neq 0 , \tag{4.3}$$

and the test particle which is moving only in the gravitational field does not in general follow geodesics.

The equation of motion can be deduced from Eq. (4.1). Following a standard method[13] one obtains

$$(d^2x^j/d\tau^2)+\Gamma^j_{kl}(dx^k/d\tau)(dx^l/d\tau)+\mathscr{S}^j=0 , \tag{4.4}$$

where

$$\mathscr{S}^j=(mck_G)^{-1}\frac{d}{d\tau}\int dV\sqrt{g}\,S^{4j} + (u^4/mck_G)\int dV\sqrt{g}\,\Gamma^j_{kl}S^{kl} \quad (u^4=dx^4/d\tau) , \tag{4.5}$$

or, substituting for S^{kl},

$$\mathscr{S}^j=(imck_G)^{-1}\frac{d}{d\tau}\int dV\sqrt{g}\,\{-(\nabla_r V^r)f^{,4}f^{,j}+\tfrac{1}{2}g^{4j}(\nabla_r V^r)f_{,n}f^{,n}+\tfrac{1}{2}g^{4j}\nabla_r[V^r(f_{,n}f^{,n})]\}$$
$$+(iu^4/mck_G)\int dV\sqrt{g}\,\Gamma^j_{kl}\{(\nabla_r V^r)f^{,k}f^{,l}-\tfrac{1}{2}g^{kl}(\nabla_r V^r)f_{,n}f^{,n}-\tfrac{1}{2}g^{kl}\nabla_r[V^r(f_{,n}f^{,n})]\} , \tag{4.6}$$

where ∇_r is the covariant derivative. Thus, we can consider that it is a supplementary gravitational force which deviates the test body from the geodesics. Of course under conditions E this force vanishes.

V. NEWTON'S LAW

The existence of the supplementary gravitational force leads, in the weak-field approximation, to the modification of Newton's law. Following the method of Landau,[14] one can write Eq. (3.2) in the form

$$R^k_j=k_G(T^k_j-\delta^k_j T/2)+S^k_j-\delta^k_j S/2 , \tag{5.1}$$

where

$$T=T^j_j ,\qquad S=S^j_j . \tag{5.2}$$

For slow motion, the four-vector velocity reduces to the component u^4, and the tensor $T_i^{\ j}$ reduces to

$$T_4^{\ 4}=-c^2\rho , \tag{5.3}$$

where ρ is the mass density. Then Eq. (5.1) reduces to

$$R_4^{\ 4}=(-k_G c^2\rho/2)+(S_4^{\ 4}/2) . \tag{5.4}$$

In the weak-field approximation the products of the Christoffel symbols are of the second order of magnitude. Furthermore, the terms containing the derivatives with respect to ct are also negligible. Thus, R_{44} reduces to

$$R_{44}=\frac{\partial\Gamma^a_{44}}{\partial x^a}\quad (a=1,2,3) . \tag{5.5}$$

The only component of g_{ij} differing from δ_{ij} is

$$g_{44}=(g^{44})^{-1}=-1-(2U/c^2) , \tag{5.6}$$

where U is the Newtonian potential. With $U\ll c^2$ we can write[14]

$$R_4^{\ 4}=g^{44}R_{44}\cong -R_{44} . \tag{5.7}$$

The calculation of Γ^a_{44} yields

$$\Gamma^a_{44}=\tfrac{1}{2}g^{aj}\left[\frac{\partial g_{j4}}{\partial x^4}+\frac{\partial g_{j4}}{\partial x^4}-\frac{\partial g_{44}}{\partial x^j}\right]$$

$$\cong-\tfrac{1}{2}g^{aj}\frac{\partial g_{44}}{\partial x^j}=-\tfrac{1}{2}g^{aa}\frac{\partial g^{44}}{\partial x^a} \tag{5.8}$$

(no summation over a).

Introducing Γ^a_{44} in Eq. (5.5) we obtain

$$R_{44}=\frac{\partial}{\partial x^a}\left[-\tfrac{1}{2}g^{aa}\frac{\partial g_{44}}{\partial x^a}\right]\cong\Delta U/c^2 . \tag{5.9}$$

Thus, in view of Eqs. (5.7) and (5.9), Eq. (5.4) becomes

$$\Delta U=(k_G c^4\rho/2)-(c^2 S_4{}^4/2) , \tag{5.10}$$

where $S_4{}^4$ has the form

$$S_4{}^4=-i[(\nabla_r V^r)f_{,a}f^{,a}+(V^r/2)(f_{,s}f^{,s})] . \tag{5.11}$$

Computing $\Gamma^r_{sr}V^s$ we obtain

$$\Gamma^r_{sr}V^s=(g^{rn}/2)\frac{\partial g_{rn}}{\partial x^s}V^s=(g^{rr}/2)\frac{\partial g_{rr}}{\partial x^a}V^a$$

$$=(1/c^2)\frac{\partial U}{\partial x^a}V^a , \tag{5.12}$$

and Eq. (5.10) takes the form

$$\Delta U-\xi^a\frac{\partial U}{\partial x^a}=\phi , \tag{5.13}$$

where

$$\xi^a=iV^a f_{,b}f^{,b}/2 ,$$

$$\phi=(k_G c^4\rho/2)+(ic^2/2)[V^s_{,s}f_{,a}f^{,a}+V^s(f_{,n}f^{,n})_{,s}/2] \tag{5.14}$$

$$=(k_G c^4\rho/2)+\varphi .$$

A solution of Eq. (5.13) for a point particle of mass m can be put into the form[15]

$$U=-(k_G c^4/8\pi)(m/r)\exp(\lambda)-\bar{U} , \tag{5.15}$$

with the shorthand notations

$$\lambda=\zeta-\eta ,$$

$$\zeta=if_{,a}f^{,a}\vec{V}\cdot\vec{r}/4 ,$$

$$\eta=if_{,a}f^{,a}(V^b V_b)^{1/2}r/4 , \tag{5.16}$$

$$\bar{U}=\exp(\zeta)\int(d\vec{r})(\varphi/r)\exp(-\eta)+\text{small term} .$$

The gravitational force on a point particle of mass m' is then

$$F=-m'\frac{\partial U}{\partial r}=-(k_G c^4/8\pi)(m'm/r^2)\exp(\lambda)$$

$$+(k_G c^4/8\pi)(m'm/r)\frac{\partial\lambda}{\partial r}\exp(\lambda)$$

$$+m'\frac{\partial\bar{U}}{\partial r} . \tag{5.17}$$

One can absorb $\exp(\lambda)$ and $(\partial\lambda/\partial r)\exp(\lambda)$ into the gravitational "constant"

$$G=(k_G c^4/8\pi)\left[1+r\frac{\partial\lambda}{\partial r}\right]\exp(\lambda) , \tag{5.18}$$

and the force takes the form

$$F=-Gm'm/r^2+m'\frac{\partial\bar{U}}{\partial r} . \tag{5.19}$$

However, it is instructive to write

$$F=-Gm'm/r^2$$

$$+(Gm'm/r^2)\left[1-\left[1-r\frac{\partial\lambda}{\partial r}\right]\exp(\lambda)\right]+m'\frac{\partial\bar{U}}{\partial r} , \tag{5.20}$$

with the constant

$$G=k_G c^4/8\pi . \tag{5.21}$$

Thus, the force is of the form

$$F=-Gm'm/r^2+F_S . \tag{5.22}$$

Of course, under the conditions E, the part F_S vanishes, but Eq. (5.22) gives rise to an interesting possibility. In view of Eqs. (5.20), (5.14), and (5.16), if $\bar{U}$ and $\partial\lambda/\partial r$ are positive, then $\partial\bar{U}/\partial r$ and

$$(Gm'm/r)\left[1-\left[1-r\frac{\partial\lambda}{\partial r}\right]\exp(\lambda)\right]$$

can be positive, and therefore F_S is positive. Now, we have not excluded the possibility that under special conditions

$$|F_S|>|Gm'm/r^2| \tag{5.23}$$

and then a sui generis antigravitation could appear.

VI. CONCLUSIONS

The present proposal for a theory of gravitation exhibits some features which we consider worthy of attention. The theory is in full agreement with the experiments performed so far. It reduces to Einstein's theory of gravitation under the conditions in which the theory has been tested.

The theory comprises several scalar-tensor theories of gravitation in the sense that the equations of these theories can be obtained in particular cases. Furthermore, under special conditions, the present proposal forecasts interesting new aspects of the gravitational phenomena.

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