Kaluza-Klein cosmologies and inflation

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We study the cosmology of Kaluza-Klein models and discuss the possibility of obtaining large amounts of inflation. The exponent that occurs in the inflation factor is simply related to the number of "extra" dimensions. The number of dimensions should therefore be large (~ 40). Quantum-gravity effects are important if the inflation is substantial. However, the appearance of inflation follows simply from thermodynamic considerations.

I. INTRODUCTION

There has been considerable interest lately in the early cosmology of Kaluza-Klein models.[1] A recent paper by Sahdev[2] has explored the possibility of inflationary cosmologies arising in a simple fashion in such models. The overall aim of such inflationary cosmologies is to solve the well-known horizon, flatness, oldness, and monopole problems.

Our analysis is of the same class of models as discussed in Ref. 2. There are three main points which we wish to emphasize in this paper concerning such Kaluza-Klein scenarios. First, to achieve a satisfactory degree of inflation[3] without unacceptably large dimensionless parameters appearing in the initial conditions, a large number of compact "extra" dimensions is required. Indeed, we will argue that to achieve the desired inflation factor of order $10^{88/3}$ one needs a number of dimensions of order $\log(10^{88/3})$. This result may be regarded either as a grave drawback or as a rather pretty link between the exponents in the large numbers of Dirac[4] and the number of extra Kaluza-Klein dimensions. The second point is that, whatever the number of dimensions, large inflation implies that much of the inflation occurs in a regime when quantum gravity is undoubtedly relevant so that simple classical calculations must be considered as only suggestive. The final point is that, regardless of the second point but under the conditions of the first point, one can argue on thermodynamic grounds that a sufficient inflation can almost certainly be achieved even in those cases where the details of inflation are not classically calculable.

In Sec. II we give a description of the model being considered. We then sketch some general arguments that imply the existence of an inflationary phase. In Sec. III we proceed with an analysis of the system using the classical (Einstein's) equations. As noted in the second point, this analysis does not apply to cases involving large inflation. Nevertheless, we believe it is instructive and promotes some confidence in the general arguments we advance. Also it is necessary to study the classical description to see where it breaks down. In Sec. IV we conclude by summarizing our results.

II. THE MODEL

The system that we study is an $(n+1)$-dimensional universe where there are $d=3$ ordinary spacelike dimensions and D extra compact spacelike dimensions, $n=d+D$. At the present time, and indeed since times of the order of the Planck time, one envisions the operation of some unknown effect that balls up the extra dimensions with a radius presumably comparable to the Planck length. This is the assumption one must make in any Kaluza-Klein model. This unknown mechanism (which may be some quantum-gravity effect) must give a cosmological constant of order 1 (in Planck-mass units[5]) in the extra dimensions but *none* in the observed four. However, in the earlier epoch we are considering, which precedes the time at which the ordinary cosmological evolution commences, one must assume that there is a zero (or negligible) cosmological constant in all dimensions.

For simplicity let us assume that in this primeval epoch the universe is radiation dominated so that $p=\rho/n$, where p is the pressure and ρ the energy density. This radiation may be excitations of the metric itself or of explicit matter or gauge fields. Let us for the moment assume that a classical description is valid. The geometry is specified by a line element

$$ds^2=-dt^2+r^2(t)g_{ij}dx^i dx^j+R^2(t)g_{ab}dX^a dX^b \;. \qquad (1)$$

The r and R are the scale factors of the three ordinary and D extra dimensions, respectively. This is a generalization of the Friedmann-Robertson-Walker form. Einstein's equations then resolve into three equations for $r(t)$ and $R(t)$ (not all independent due to the Bianchi identity and conservation of energy-momentum), with corresponding constant curvature k_d and k_D,

$$d\frac{\ddot r}{r}+D\frac{\ddot R}{R}=-8\pi\bar G\rho \;, \qquad (2a)$$

$$\frac{k_d}{r^2}+\frac{d}{dt}\left[\frac{\dot r}{r}\right]+\left[d\frac{\dot r}{r}+D\frac{\dot R}{R}\right]\left[\frac{\dot r}{r}\right]=\frac{8\pi\bar G}{n}\rho \;, \qquad (2b)$$

and

$$\frac{k_D}{R^2}+\frac{d}{dt}\left[\frac{\dot R}{R}\right]+\left[d\frac{\dot r}{r}+D\frac{\dot R}{R}\right]\left[\frac{\dot R}{R}\right]=\frac{8\pi\bar G}{n}\rho \;. \qquad (2c)$$

Here $\bar G$ is the $(n+1)$-dimensional gravitational constant related to Newton's constant G_N by[5]

$$\bar G=V_D G_N=V_D \;, \qquad (3)$$

where V_D is the volume of the compact D-dimensional manifold. If this volume is a sphere, it has physical radius $\bar{R}_{KK}$ and

$$V_D = \left[\frac{2\pi^{(D+1)/2}}{\Gamma\left[\frac{D+1}{2}\right]} \right] \bar{R}_{KK}^D . \quad (4)$$

The parameter k_D we scale to $+1$. [Thus R is not the physical radius but, for a sphere, is $R = \bar{R}/(D-1)^{1/2}$.] The parameter k_d we assume to be negative, zero, or a sufficiently small positive number. What we mean by sufficiently small we will discuss later (see Sec. III).

We further assume that, for the period we are interested in, interactions occur at a rate adequate to maintain thermal equilibrium. In that case entropy is conserved. In particular, the entropy S in a volume comoving with the expansion (contraction) of the universe is constant:

$$S \sim r^d R^D T^n = \text{const} . \quad (5)$$

For future reference we write that

$$\rho = N_{\text{pol}} a_n T^{n+1} \equiv C (r^d R^D)^{-(n+1)/n} . \quad (6)$$

Here N_{pol} is the number of polarization states of the radiation and a_n is the $(n+1)$-dimension Stefan Boltzmann constant given by

$$a_n = \frac{n\Gamma\left[\frac{n+1}{2}\right]\zeta(n+1)}{\pi^{(n+1)/2}} . \quad (7)$$

C is just a (n-dependent) constant defined by Eq. (6).

Now there are two ways to look at the model we have presented. One may examine solutions to the classical equations [Eqs. (2)] or one may make thermodynamic arguments. The former strategy has the advantage of permitting a more detailed treatment. However, we shall see in Sec. III that the classical equations are not to be trusted in the most interesting cases. The thermodynamic argument is more general and powerful, and we present it first.

The general behavior[2] of $r(t)$ and $R(t)$ is as depicted in Fig. 1. We wish to have all of the dimensions start off with a "big bang." Perhaps this should be called the "first big bang." From this point both r and R expand. If k_d is negative, zero, or positive but sufficiently small, then $r(t)$ behaves in an "open" manner as we shall see in Sec. III. $R(t)$, however, reaches a maximum and begins to recollapse. As we shall argue, this collapse of $R(t)$ drives a rapid inflation of $r(t)$. Ultimately, R will reach some minimum value which we call the Kaluza-Klein radius, R_{KK}. At that point quantum-gravity effects are assumed to stabilize R. From that time until the present $R = R_{KK}$. After that time, which we call the collapse time t_c, the ordinary dimensions enter a Robertson-Walker phase and expand as $t^{1/2}$. Since we aim to achieve inflation from the collapse of R, we assume that no other significant inflations, coming from phase transitions or other mechanisms, disturb the Robertson-Walker expansion after R has collapsed.

Now what are the goals of inflationary cosmologies? Essentially they are three: (1) to achieve a huge ($\sim 10^{88/3}$) inflation of the scale factor $r(t)$ in a short period of time so as to solve the horizon problem, (2) to generate the huge entropy we now see (in the form of the 10^{88} blackbody photons within our horizon) within a causal volume at some early time, and (3) to do all of this without having to tune fundamental dimensionless parameters appearing in the initial conditions to be either very large or very small (e.g., 10^{+88} or 10^{-88}). We might call these the inflation, the entropy, and the fine-tuning aspects of the problem.

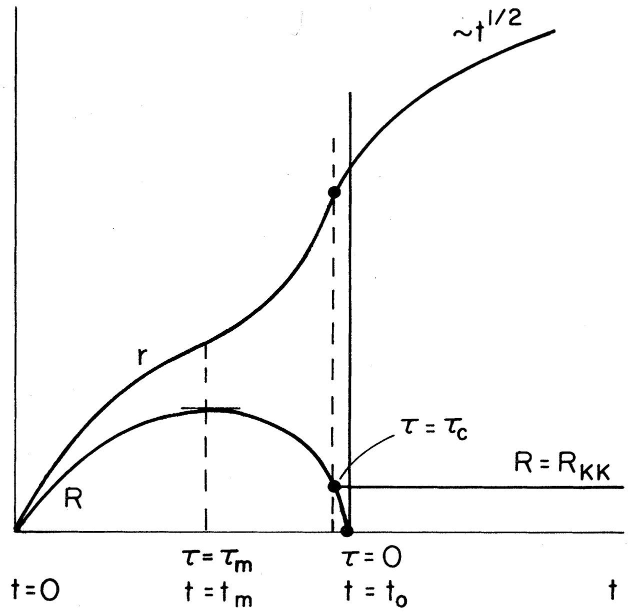

FIG. 1. The behavior of the scale factors of the ordinary (r) and extra (R) dimensions as a function of time. As $t \to t_0$, R goes to zero and r blows up. At $t = t_c < t_0 (\tau = \tau_c)$ some (quantum?) effect stabilizes R at R_{KK} and henceforth, r expands in the usual FRW way.

As we shall see in Sec. III, the rapid inflation of r begins at about the time that R achieves its maximum (which we call t_m.) The rapid inflation stops at the collapse time t_c. It is convenient then to specify "initial" conditions at t_m. Goal (2) above tells us that at t_m we want to have an entropy of order 10^{88} (or larger) within a causal volume. This causal volume is, at that time, an n-*dimensional* volume with radius of order t_m. It is not hard to see from Einstein's equations (which are valid at t_m), and we will show it in Sec. III [see Eqs. (17)], that

$$R_m \equiv R(t_m) \sim r(t_m) \sim t_m . \quad (8a)$$

Thus the entropy in a causal volume is

$$S \sim (t_m)^n T^n \sim (RT)^n \big|_{t=t_m} . \quad (8b)$$

This must be at least of order 10^{88}. (Remember we have assumed that entropy is conserved.) Thus, there are two possibilities. The first is that n is small and RT is huge initially. For example, with $n=11$ we need $R_m T_m \sim 10^8$, or with $n=6$, $R_m T_m \sim 10^{15}$. But this does not satisfy goal (3) which was to avoid introducing huge (or tiny) dimensionless parameters into the initial conditions. Far preferable is to have D (and n) relatively large. Then $(R_m T_m) \sim 10^{88/n}$ can be of reasonable scale. With n of order 40 we need not introduce any numbers greater than about 100 into the initial conditions.

A further desirable requirement is that $T_m \lesssim 1$. Otherwise we would expect to have gravity coupling very strongly to thermal energy and the metric fluctuating wildly. On the other hand, we do not want T_m to be extremely small for the usual reason of avoiding unnaturally fine adjustments. (For confirmation that this situation can, in fact, be arranged see the discussion below.)

To repeat, our point of view is that the second goal of inflation—to have a huge ($>10^{88}$) entropy within a causal volume at early times—is best achieved by having a reasonable temperature T_m (and thus a reasonable entropy density) but an enormous causal volume at t_m. This enormous causal volume can be achieved by having a large number ~ 40 dimensions and a reasonable value of $R(t_m)$. In this way the third goal, that no fundamental parameters be initially huge, is also achieved. (We regard R as more fundamental than R^D.)

Now the crucial question is whether goal (1), a huge and rapid inflation of r, can be achieved in models which satisfy the above constraints. The answer is yes and, in fact, achievement of goal (1) *follows* quite naturally when the other goals are also achieved. In Sec. III we will show this to be the case assuming that quantum effects are ignorable. However, the following *thermodynamic* argument suggests that a large inflation is *generally* present in such scenarios.

Let us consider two times. At an initial time t_i, the universe is effectively n dimensional ($T_i R_i > 1$). At the final state t_f, R has reached its minimum $R_{\rm KK} \gtrsim 1$ and has decoupled ($T_f R_{\rm KK} < 1$) so that the universe is effectively three dimensional. At both times we assume thermal equilibrium. If entropy is conserved between these two times, then, setting the entropy in a comoving volume at t_i and t_f equal, we find[6]

$$(r_i{}^3 R_i{}^D T_i{}^n) N_{\rm pol}^{(3+D)} \left[\frac{4+D}{3+D}\right] a_{3+D} \left[\frac{V_D}{R_{\rm KK}^D}\right] = (r_f{}^3 T_f{}^3) N_{\rm pol}^3 (\tfrac{4}{3}) a_3 \sim 10^{88} , \tag{9a}$$

where the last step follows from assuming that goal (2) has been achieved at $t_i = t_m$. Thus, using Eq. (8), we have

$$\left[\frac{r_f}{r_i}\right] = \left[\frac{T_i}{T_f}\right] (R_i T_i)^{D/3} \left[\frac{N_{\rm pol}^{(3+D)} (V_D / R_{\rm KK}^D)[(D+3)/(D+4)] \alpha_{3+D}}{N_{\rm pol}^{(3)}(\frac{4}{3}) a_3}\right]^{1/3}$$

$$\underset{D \gg 3}{\sim} \left[\frac{T_i}{T_f}\right] \sqrt{D} \times 10^{88/3} . \tag{9b}$$

We have assumed that $T_i \le 1$ and, since we have assumed that at the final time things have cooled enough so that the universe looks three dimensional, we also know that $T_f \lesssim 1/R_{\rm KK} \lesssim 1$. Thus, within the interval of time between t_m and t_f, an inflation in r of order $\gtrsim 10^{88/3}$ must have occurred. In fact that is the amount of inflation required to solve the horizon problem. If entropy is generated during the inflationary phase, then our reasoning suggests an even larger inflation of r.

We see then that the Kaluza-Klein extra dimensions play a crucial role in two ways. First, for D large (but not exponentially large), they allow an enormous volume (and thus an enormous entropy) within a not very large radius, thus "solving" the (exponentially) large-number puzzle. Second, their contraction drives a huge inflation of r as we were able to show from purely thermodynamic arguments. In Sec. III we will put flesh on these bare bones and confirm our faith in our arguments by working out the essential features of the solutions to Einstein's equations.

III. EINSTEIN'S EQUATIONS

Our course in this section will be, first, to deduce certain general features of the solutions of Eqs. (2) without actually solving them. Second, we will perform a quantitative but approximate study of these solutions, and confirm [at least to the extent that we may trust Eqs. (2)] the validity of our foregoing analysis. Finally, we will determine where one might expect the classical treatment to break down.

Let us first assume that $k_d \le 0$ in Eq. (2b). Then we may easily demonstrate two things. (a) If r and R commence in a big bang, r will continue to increase monotonically rather than recollapsing. (b) When r recollapses toward its minimum value, r is driven to expand rapidly.

We may prove (a) by contradiction. Suppose r expanded and reached a maximum and recollapsed. At its imagined maximum we have $\dot r = 0$ and $\ddot r \le 0$. Looking at Eq. (2b) in this case we see

$$\left[\frac{k_d}{r^2} + d\frac{\ddot r}{r} = \frac{8\pi \bar G \rho}{n}\right]$$

that the left-hand side is negative while the right-hand side is positive ($k_d \le 0$, $\rho > 0$). So r never turns over. Similarly we may prove (b) from Eq. (2b). Define t_0 to be the time when R would recollapse to zero were quantum-gravity effects not to supervene. We assume that R behaves, to leading order, like a power of $(t_0 - t)$ for $t \to t_0$ from below. As $R \to 0$ near t_0 it is clear that $\dot R / R \to -\infty$. Further we showed in proving (a) above that $\dot r / r > 0$. Thus the term $D(\dot R/R)(\dot r/r)$ on the left-hand side of Eq. (2b) is driven to $-\infty$. Since $k_d \le 0$, and the right-hand side of Eq. (2b) is positive, it must be that either $(\dot r/r)^2$ or $\ddot r/r$ (or both) goes to $+\infty$ as $t \to t_0$. Thus

r also behaves in leading order as a power of (t_0-t) and it must be that r itself tends to $+\infty$, i.e., it must behave as a negative power of (t_0-t). Thus we see that it is the coupling of r and R through the term $D(\dot{r}/r)(\dot{R}/R)$ and the collapse of R that may be said to drive the inflation of r.

The above discussion assumed that $k_d \leq 0$. In fact, the solution may look like Fig. 1 for a certain range of positive values of k_d. This will be discussed later.

Now we wish to make quantitative statements. An exact analytic solution of Eqs. (2) and (6) is not possible except in very special cases[2] which are not realistic. Nevertheless, we can extract useful information about the solution by a systematic approximation. The equations are simple to solve by an expansion about the points $t=0$ and $t=t_0$. The interesting physics occurs near the latter point. But we wish for completeness to study also the structure near the former point. Therefore we will expand the solutions about both points.

To see the general structure of the solutions it is useful to define the effective volume element

$$\mathscr{V}=r^d R^D , \tag{10a}$$

which satisfies the equation [Eq. (2b) times d plus (2c) times D]

$$\frac{d}{dt}\left[\frac{\dot{\mathscr{V}}}{\mathscr{V}}\right]+\left[\frac{\dot{\mathscr{V}}}{\mathscr{V}}\right]^2=8\pi\bar{G}C\mathscr{V}^{-(n+1)/n}-\frac{D}{R^2} . \tag{10b}$$

In this equation and the subsequent discussion we have taken $k_d=0$. We are interested here in the behavior of $\mathscr{V}$ in the limit $\mathscr{V}\rightarrow 0$. For the parameter values of interest[7] the final, curvature term in Eq. (10b) is always negligible in this limit. There are thus two further possibilities: (1) the term $8\pi\bar{G}\mathscr{V}^{-(n+1)/n}$ is important in this limit; (2) it is *not* important in this limit. In the former case it follows immediately from Eq. (10b) that, for $\mathscr{V}$ behaving as a power of t as $t\rightarrow 0$, it must be true that

$$\mathscr{V}^{-(n+1)/n} \underset{t\rightarrow 0}{\propto} t^{-2} \tag{11a}$$

or

$$\mathscr{V} \underset{t\rightarrow 0}{\propto} t^{2n/(n+1)} . \tag{11b}$$

In the second case, where both terms on the right-hand side of Eq. (10b) are to be ignored, it is even easier to solve. For reasons which will be clear shortly, it is useful to define the independent variable to be τ. We find immediately that

$$\mathscr{V} \underset{\tau\rightarrow 0}{\propto} \tau . \tag{12}$$

The two possible behaviors illustrated by Eqs. (11) and (12) are exactly those of interest here. The limit $t\rightarrow 0$ is identified with the approach (backwards in time) to the big bang, while $\tau=(t_0-t)\rightarrow 0$ is identified with the (forward) approach to the time when R collapses. Note the distinction that for $t\rightarrow 0$ the leading behavior includes the influence of the density term but not the curvature term while for $\tau\rightarrow 0$ neither term is important to the leading behavior. We now study each case in more detail.

A. The solution near $t=0$

Again we are treating the case $k_d=0$. We will see below how far this may be relaxed. At the first big bang we may write

$$R(t)=At^{\alpha}(1+A't^{\delta}+\cdots) \tag{13a}$$

and

$$r(t)=at^{\beta}(1+a't^{\delta}+\cdots) . \tag{13b}$$

In fact, for $k_d=0$, a is an arbitrary scale of no physical relevance. From Eqs. (10), (11), and (13) it follows that

$$\alpha D+\beta d=\frac{2n}{n+1} , \tag{14a}$$

and that

$$8\pi\bar{G}C(A^D a^d)^{-(n+1)/n}=\frac{2n(n-1)}{(n+1)^2} . \tag{14b}$$

Using these relations in Eq. (2a) yields

$$\alpha=\beta=\frac{2}{n+1} . \tag{15}$$

Thus near the first big bang we have r, $R\sim t^{2/(n+1)}$, i.e., both r and R vanish at $t=0$ explaining the identification of case (1) with the big bang. It is easy to check from Eqs. (2) and (6) that the curvature term $1/R^2$ is of higher order than the derivative terms $[(\dot{r}/r)^2$, etc.] and the density terms which both go as t^{-2}. Thus the density term is important and the curvature term is not as $t\rightarrow 0$ as is required to use the results of case (1) above [i.e., Eq. (11)]. The first subleading terms in Eq. (13) account for the effects of the curvature $(1/R^2)$ term which tends to turn R over. By substituting Eqs. (13) into Eqs. (2) and matching subleading terms one can solve for δ, A', and a' in terms of A to find, in the limit of large D,

$$\delta=2\left[\frac{n-1}{n+1}\right]\simeq 2\left[1-\frac{2}{D}\right] , \tag{16a}$$

$$A'\simeq-\frac{1}{A^2}\left[\frac{1}{12}+O\left[\frac{1}{D}\right]\right] , \tag{16b}$$

and

$$a'\simeq+\frac{1}{A^2}\left[\frac{1}{12}+O\left[\frac{1}{D}\right]\right] . \tag{16c}$$

Since we desire D to be large for the reasons given in Sec. II, we will neglect terms of order $1/D$ (except when they will later be multiplied by D). Note that the fact that both A' and a' are nonzero serves as an *a posteriori* justification for using the same subleading power (δ) in both R and r. The time t_m when R turns over is estimated simply by setting $\dot{R}=0$ in the two-term approximate expression obtained above. Thus, from Eqs. (13) and (16) we find

$$t_m\simeq A^{1+2/D}\left[\frac{12}{D}\right]^{1/2} , \tag{17a}$$

$$R_m \simeq A^{1+2/D}\left[\frac{12}{D}\right]^{1/D}, \tag{17b}$$

and

$$r_m \simeq \left[\frac{a}{A}\right] R_m . \tag{17c}$$

B. The solution near $t = t_0$

The solution near the point $t=t_0$ is more subtle. As above we define $\tau=(t_0-t)$ and expand in powers of τ. Thus, we write for τ near 0

$$R(\tau)=B\tau^{\gamma}(1+B'\tau^{\epsilon}+\cdots), \tag{18a}$$

$$r(\tau)=b\tau^{\eta}(1+b'\tau^{\epsilon}+\cdots). \tag{18b}$$

Thus, from Eqs. (2a), (12), and (18) it follows that

$$D\gamma+d\eta=1 \tag{19a}$$

and

$$D\gamma(\gamma-1)+d\eta(\eta-1)=0, \tag{19b}$$

so that

$$\gamma=\frac{1+[(d/D)(n-1)]^{1/2}}{n} \underset{D\gg 1}{\sim} \frac{1+\sqrt{d}}{D}=\frac{1+\sqrt{3}}{D}, \tag{19c}$$

$$\eta=\frac{1-[(D/d)(n-1)]^{1/2}}{n} \underset{D\gg 1}{\sim} -\frac{1}{\sqrt{d}}=-\frac{1}{\sqrt{3}}. \tag{19d}$$

Again we can easily check that the derivative terms in Eq. (2) go as τ^{-2} to leading order. Thus both $1/R^2$ and ρ go as less negative powers of τ. Thus, to leading order in $1/\tau$, one may solve Eqs. (2) keeping only derivative terms as we did in case (2) to find Eq. (12). The important feature of these solutions is that $\gamma>0$ while $\eta<0$ so that for $\tau\to 0$, $R\to 0$ while $r\to\infty$. This explains the identification of this situation with the collapse scenario and the identification with the corresponding solutions [$R(t)$ and $r(t)$] near $t=0$ in Sec. III A.

The subtlety of this regime arises in determining the *next-to-leading* terms. Whether ρ or $1/R^2$ is next most important after the derivative terms depends on the number of dimensions. For large D the ρ term is more important[7] than $1/R^2$. This is the case of interest to us. Returning to Eq. (10) to match the subleading power in $\mathscr{Y}(\sim\tau^{\epsilon-2})$ to the behavior of the density term $(\sim\tau^{-(n+1)/n})$, we find

$$\epsilon=\frac{n-1}{n}. \tag{20}$$

Then, using Eqs. (2a) and (10) (but ignoring the curvature term), one may solve for B' and b' in terms of the quantity

$$\kappa\equiv\frac{8\pi\bar{G}C}{\epsilon(\epsilon+1)(B^D b^d)^{(n+1)/n}} =\frac{2n^3}{(n+1)^2(2n-1)}\left[\left[\frac{A}{B}\right]^D\left[\frac{a}{b}\right]^d\right]^{(n+1)/n}, \tag{21}$$

where Eq. (14b) was used in the final step. We find

$$B'=-\frac{1}{D}\frac{\epsilon+\eta}{\gamma-\eta}\kappa=\left[\frac{1}{n}+\frac{d}{n-1}(\eta-\gamma)\right]\kappa, \tag{22a}$$

$$b'=\frac{1}{d}\frac{\epsilon+\gamma}{\gamma-\eta}\kappa=\left[\frac{1}{n}+\frac{D}{n-1}(\gamma-\eta)\right]\kappa. \tag{22b}$$

Again the fact that b' and B' are both nonzero justifies the use of identical subleading powers for both R and r in Eq. (18). Note the remarkable feature that B' is negative similarly to A' in the case near $t=0$ except that in the present approximate solution the effect of the curvature term has not been included.

To explicitly connect the solutions [Eq. (18)] for τ near 0 to those appropriate to t near 0 [Eq. (13)] is nontrivial at the present level of approximation. One might attempt, for example, to simply match the two sets of solutions at some intermediate point such as t_m [defined by $dR/dt=0$ as in Eq. (17)] taken to correspond to τ_m (defined by $dR/d\tau=0$). The difficulty with this procedure is simply that for τ near τ_m it is not true that the curvature term can be ignored (e.g., the equations for R do not allow $\dot{R}=0$ if $k_D=0$). However, it is true that our two-term *approximate* solution does exhibit a zero derivative at a finite τ which offers at least a crude approximation to the actual point of maximum.

While factors of order 2 are surely introduced in such an approximation, they will be irrelevant in what follows. Thus we define τ_m as the point where the two-term approximation has zero derivative and find, in the limit of large D,

$$\tau_m\sim\frac{\sqrt{d}+1}{\sqrt{d}-1}\kappa^{-(1+1/D)}. \tag{23}$$

Identifying this point with $\tau_0-\tau_m$ and matching the two solutions yields, again in the limit of large D,

$$B\sim A, \tag{24a}$$

$$b\sim aA^{1/\sqrt{d}}\left[\frac{12}{D}\right]^{1/2\sqrt{d}}\left[\frac{\sqrt{d}(\sqrt{d}-1)}{d+1}\right]^{(2+\sqrt{d})/2} \times\left[\frac{\sqrt{d}+1}{\sqrt{d}-1}\right]^{1/2\sqrt{d}}, \tag{24b}$$

and thus, from Eq. (21), we have

$$\kappa\sim\left[\frac{1}{A}\left[\frac{D}{12}\right]^{1/2}\left[\frac{\sqrt{d}+1}{\sqrt{d}-1}\right]^{1/2} \times\left[\frac{d+1}{\sqrt{d}(\sqrt{d}-1)}\right]^{d/2}\right]^{1+1/D} \tag{24c}$$

C. Inflation

To examine the inflation of r let us look at Fig. 2, where $\ln r$ is plotted versus $\ln t$. Near the first big bang this line has slope $2/(n+1)$ as indicated in Eq. (15). Near $t \simeq t_m (\tau \simeq \tau_m)$ the scale factor r begins to inflate rapidly. This is shown as an almost vertical segment on the ln-ln plot. When $t = t_c$ $(\tau = \tau_c = t_0 - t_c)$, R reaches its final and minimum value and the inflation of r ends (see later discussion). If, by that time, $T < 1/R_{\rm KK}$ the modes in the extra dimensions will have de-excited and a normal Robertson-Walker expansion will commence with $r \sim t^{1/2}$. Thus the curve in Fig. 2 will have slope $\frac{1}{2}$ (changing to $\frac{2}{3}$ when matter later begins to dominate). This picture is somewhat modified if de-excitation occurs *after* the collapse time t_c as is discussed later. As noted earlier, if $k_d = 0$ the overall scale of r is arbitrary. We have chosen to normalize it in Fig. 2 so that it is presently equal to the size of our observable universe. Thus the line of unit slope in Fig. 2 (which is just a plot of $\ln t$) intersects $\ln r(t)$ when t equals the present age of the universe.

In order to consider the horizon problem, to which we now turn, the crucial quantity is $r(t)/t$ at the time when inflation ceases and normal FRW expansion commences. As indicated in the figure, if we simply calculate backward from the present time (ignoring for simplicity the distinction between the matter and radiation-dominated regimes), we find, in our units[5]

$$\frac{r(t)}{t} \sim T(t) \times 10^{94/3} \tag{25}$$

assuming $(rT)^3 \sim 10^{88}$. Hence, if the end of inflation is marked, for example, by temperatures of order 10^{-2}, we must require $r(t)/t$ to be of order $10^{88/3}$ at this time (a factor we recall from our earlier discussion). Thus the horizon problem is treated satisfactorily, in the present scheme, if the inflation of r (due to the collapse of R) carries us rapidly from a time t_1 when $r(t_1)/t_1 \sim 1$ (i.e., r describes an initially causal volume) to a time t_2 when $r(t_2)/t_2 \sim 10^{88/3}$ and $T(t_2) \sim 10^{-2}$. With our present approximation scheme the quantity $r(t)/t$ is difficult to study directly in the region $\tau \to 0$ because we cannot evaluate t_0 quantitatively, and thus cannot specify the relationship between t and τ quantitatively. However, since r is varying rapidly compared to t in this region (as $\tau \to 0$ $r \sim \tau^\eta$ while $t \to t_0 - \tau \sim t_0$) and since we are concerned with factors of order 10^{40} not of order 2 (or even of order 10), we can set $t_2 \sim t_1 \sim r(t_1)$ [but *not,* in the same approximation, $r(t_2) \sim r(t_1)$ and *not* $\tau_2 \sim \tau_1$]. Then we can identify t_1 (i.e., τ_1) with the τ_m discussed above and t_2 (i.e., τ_2) with τ_c. We expect all these operations to be good to factors of 2. Thus we focus on what we call the inflation factor, $\mathscr{I}$, defined as

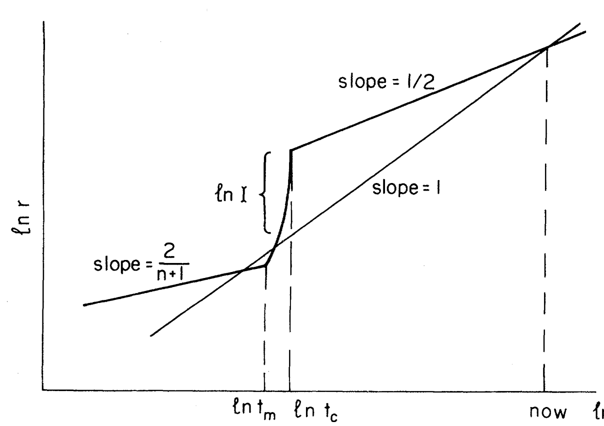

FIG. 2. A plot of $\ln r$ versus $\ln t$. Before $t = t_m$, we have $r \sim t^{2/(n+1)}$. Between t_m and t_c, r increases rapidly. After t_c, $r \sim t^{1/2}$ (until matter dominates). The inflation factor is $r(t_c)/t_c$, denoted $\mathscr{I}$.

$$\mathscr{I} \equiv \frac{r(\tau_c)}{r(\tau_m)} \ . \tag{26}$$

From Eq. (18), τ_c is (well) determined (for small τ_c) by our approximate solution to be

$$\tau_c \simeq \left[\frac{R_{\rm KK}}{B}\right]^{1/\gamma} \underset{D \gg d}{\sim} \left[\frac{R_{\rm KK}}{B}\right]^{D/(1+\sqrt{d})} , \tag{27a}$$

which is much less than unity for $B > R_{\rm KK}$. In the same approximation we have

$$r(\tau_c) \simeq b \left[\frac{R_{\rm KK}}{B}\right]^{\eta/\gamma} \sim b \left[\frac{R_{\rm KK}}{B}\right]^{-D/(d+\sqrt{d})} . \tag{27b}$$

For the larger τ_m we use the same notation but keep the subleading terms [and use Eq. (23)] to find

$$\tau_m \simeq \left[\frac{R_m}{B}\right]^{1/\gamma} (1+B'\tau_m^\epsilon)^{-1/\gamma}$$
$$\sim \left[\frac{R_m}{B}\right]^{D/(1+\sqrt{d})} \left[1-\frac{\sqrt{d}+1}{D}\right]^{-D/(\sqrt{d}+1)} \tag{28a}$$

and

$$r(\tau_m) \simeq b \left[\frac{R_m}{B}\right]^{\eta/\gamma} (1+B'\tau_m^\epsilon)^{-\eta/\gamma}(1+b'\tau_m^\epsilon)$$
$$\sim b \left[\frac{R_m}{B}\right]^{-D/(d+\sqrt{d})} \left[1-\frac{\sqrt{d}+1}{D}\right]^{D/(d+\sqrt{d})}$$
$$\times \left[\frac{d+1}{d-\sqrt{d}}\right] . \tag{28b}$$

Combining these results we find for $D \gg d$,

$$\mathscr{I} \sim \left[\frac{R_m}{R_{\rm KK}}\right]^{D/(d+\sqrt{d})} e^{1/\sqrt{d}} \left[\frac{d-\sqrt{d}}{d+1}\right]$$
$$\sim \left[\frac{R_m}{R_{\rm KK}}\right]^{D/(d+\sqrt{d})} , \tag{29}$$

which is required to be of order $10^{88/3}$. Thus this Kaluza-Klein scenario offers an explanation of the horizon problem as long as $R_m/R_{\rm KK}$, which is essentially just R_m measured in units of the Planck length, is of order 10^2. This is just the constraint (for $T_m \sim 1/R_{\rm KK} \lesssim 1$) of Eq. (8) which was required in order to obtain the desired entropy and thus yields the required inflation as already argued on thermodynamics grounds in Eq. (9). The large inflation is again seen to arise from the large volume in the extra dimensions at early times.

Now we are in a position to see why the classical equations cannot be used when there is a great deal of inflation. As is apparent from Eqs. (27), (28), and (29) the huge inflation stems primarily from the small size of τ_c. In fact, we can write

$$\mathscr{I} \sim \left[\frac{\tau_c}{\tau_m}\right]^{-1/\sqrt{d}} \tag{30}$$

and then use the order-of-magnitude (or two) relations $\tau_m \sim R_m \sim R_{\mathrm{KK}}\mathscr{I}^{(d+\sqrt{d})/D}$ to find, in the usual units,[5]

$$\tau_c \sim R_{\mathrm{KK}}\mathscr{I}^{(d+\sqrt{d})/D}\mathscr{I}^{-\sqrt{d}} \sim \mathscr{I}^{-\sqrt{d}} \le 10^{-88/\sqrt{3}} . \tag{31}$$

Remember that this is in units of the Planck time. At those times $(\dot{r}/r)^2 \sim (\dot{R}/R)^2 \sim 1/\tau^2$. With frequency components of order $1/\tau_c > 10^{88/\sqrt{3}}$ being excited it is reasonable to expect that quantum-gravity effects ought to be important. We believe one should not trust Einstein's equations for τ less than 1. By that time there has still been relatively little inflation. In fact, almost all of the inflation occurs when quantum-gravity effects can be expected to be important, so that we cannot calculate reliably. Notwithstanding this difficulty, let us pursue the classical description further.

One aspect that we have glossed over is the de-excitation of the modes in the extra dimensions. This occurs roughly when T falls below $1/R$. After this happens the equation of state drastically changes. The pressure in the extra dimensions drops from ρ/n to nearly zero. The pressure in the ordinary dimensions goes to $\rho/3$ (assuming equilibrium). We should here emphasize that even if de-excitation occurs before t_c, while R is still collapsing, nevertheless inflation will continue until t_c. The point is that for $\tau \approx 0$ the curvature and density terms in Einstein's equations are not as important as the derivative terms. Even if the equation of state changes late in the collapse of R, the leading behavior of r and R, as given in Eqs. (18) and (19), will be unaltered until R stabilizes around R_{KK} and $\dot{R}$ becomes negligible.

In principle there are two cases to consider in which the de-excitation time t_d is greater or less than the collapse time t_c. In practice, however, de-excitation will presumably occur over some finite span of time rather than instantaneously and, since quantum effects can be expected to be important here, we will not discuss the two cases in great detail. Note also that the assumption that thermodynamic equilibrium is maintained is unlikely to be valid when $\mathscr{I}$ is large.

Case I ($t_d \le t_c$). Figure 3(a) exhibits the idealized behavior of T and $1/R$ versus t for this case. Assuming equilibrium we see that the temperature peaks at t_d. From Eqs. (6) and (10a), T behaves as $\mathscr{V}^{-1/n}$, so from Eq. (12) it follows that, for $\tau \to 0$, T behaves as $\tau^{-1/n}$. Once de-excitation has occurred the radiation becomes effectively three dimensional and red-shifts rapidly while r continues to inflate. Ordinary evolution proceeds after τ_c.

Case II ($t_d > t_c$). Idealized behavior for this case is illustrated in Fig. 3(b). The radiation is still effectively n dimensional when R stabilizes at R_{KK} and the inflation of r ceases. Thereafter from Eq. (6) $T \sim (r^d R^d)^{-1/n} \sim r^{-d/n}$. Hence T starts to decrease after t_c. In this case $RT > 1$ at $t = t_c$ and decreases subsequently. By the time RT is of order unity de-excitation has occurred and ordinary cosmological evolution proceeds.

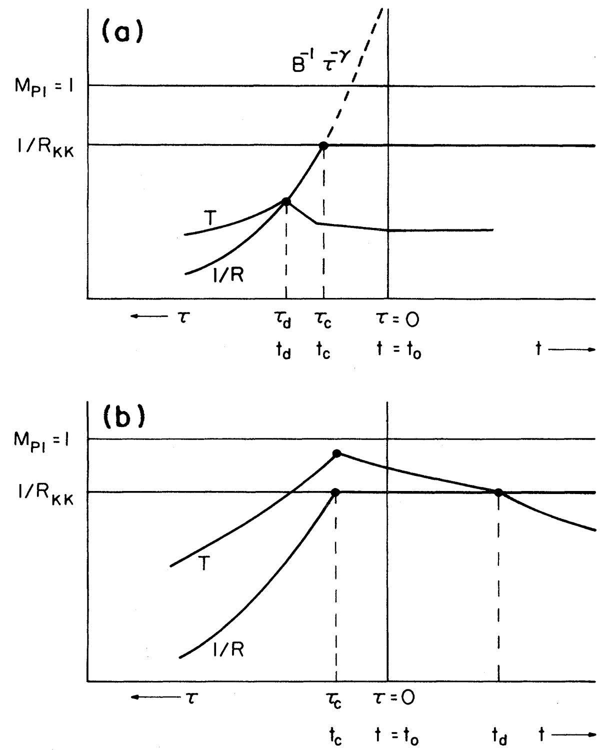

FIG. 3. A plot of $1/R$ and T versus t. (a) Decoupling before the collapse time t_c. The temperature peaks at the decoupling time. (b) Decoupling after the collapse time t_c. The temperature peaks at the collapse time.

There is one more point to be discussed and that is the size of k_d. Let us assume $k_d > 0$. How small must k_d be? Or, equivalently, if k_d is normalized to unity how big must r be, where r now has physical significance, i.e., the "size" of the three-dimensional universe? If r is not to recollapse with R, we must have, at least, that k_d/r^2 is unimportant compared to $8\pi\bar{G}\rho/n$ at $t = t_m$. Then [see Eq. (14b)] we require

$$\frac{k_d}{r^2(t_m)} < 8\pi\bar{G}\rho/n \simeq \frac{2n(n-1)}{(n+1)^2}\frac{1}{{t_m}^2} \tag{32}$$

or

$$r(t_m) > t_m . \tag{33}$$

But at t_m we have $R_m \sim t_m$ [see Eqs. (17)]. Thus the necessary condition is that (r/R) be greater than unity at t_m. This seems not unreasonable. The bound in Eq. (33) just says that the "physical size" of the ordinary three-dimensional universe is larger than a horizon at t_m. From Fig. 2 this can be seen to imply the statement that the size of the universe is larger than the horizon *now*. This is a fact we already know of course. So there is no interesting constraint on k_d.

IV. CONCLUSIONS

We have argued that, because of the presence of the extra Kaluza-Klein dimensions, it is possible to achieve a large (10^{88}) entropy at early times ($t = t_m$) within a causal volume without the introduction of absurdly large (10^{88}) numbers as fundamental parameters as long as the number of extra dimensions is ~ 40. In this case the large entropy is present at $t = t_m$ (when the extra dimensions have maximum size) due to the huge volume of the compact space which is characterized by $(R_m)^D$. Thus with $D \sim 40$ the required large volume $\sim 10^{88}$ requires only a $R_m \sim 10^2$. In this way we explain the mysterious large numbers of Dirac by relating their exponents to the number of extra dimensions. Then from thermodynamics alone (the conservation of entropy) we argued that the inflation of the scale size of ordinary space is just such as to ensure that the universe inside the present horizon was inside a causal volume also at an earlier time. Thus a number of extra dimensions of ~ 40 offers an explanation of the entropy, inflation, and fine-tuning problems mentioned earlier. Detailed analytic (but approximate) calculations using Einstein's equations support this conclusion, though we expect the thermodynamic argument to be generally valid and Einstein's equations not to be valid in the cases of real interest.

We should emphasize that there are serious uncertainties arising from the fact that in these models there is an epoch where quantum-gravity effects are important. For instance, this situation makes it difficult to say anything definite about expansion rates during the period in which most of the inflation occurs. This in turn raises questions concerning the details of the matter spectrum and the return to equilibrium after inflation ends. Furthermore, we can say little about the maintenance of homogeneity during inflation (since quantum fluctuations are presumably large). This is important as a prime goal of inflation is to explain the large-scale homogeneity of the universe.

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[1]For Kaluza-Klein theories, see A. Salam and J. Strathdee, Ann. Phys. (N.Y.) **141**, 316 (1982), and references therein. For Kaluza-Klein cosmology, see P. G. O. Freund, Nucl. Phys. **B209**, 146 (1982); T. Appelquist and A. Chodos, Phys. Rev. Lett. **50**, 141 (1983); E. Alvarez and M. Belen Gavela, *ibid.* **51**, 931 (1983); also LPTHE Orsay Report No. 83/30 (unpublished); S. Barr and L. S. Brown, Phys. Rev. D **29**, 2779 (1984); Q. Shafi and C. Wetterich, CERN Report No. TH.3613, 1983 (unpublished); S. Randjbar-Daemi, A. Salam, and J. Strathdee, Phys. Lett. **135B**, 388 (1984); E. W. Kolb and R. Slansky, *ibid.* **135B**, 378 (1984); D. Sahdev, *ibid.* **137B**, 155 (1984); M. Böhm, J. L. Lucio M., and A. Rosado, report, 1983 (unpublished).

[2]Sahdev (Ref. 1).

[3]See M. Turner, in *Fourth Workshop on Grand Unification,* edited by H. A. Weldon, P. Langacker, and P. J. Steinhardt (Birkhauser, Boston, 1983), for a review of inflation.

[4]See, for example, John D. Barrow, Q. J. R. Astron. Soc. **22**, 388 (1981).

[5]We use throughout units in which $G_N = M_{\rm Pl}^{-2} = 1$ (and $c = 1$). All lengths are in units of the Planck length and energies are in units of the Planck mass.

[6]Barr and Brown (Ref. 1).

[7]For $d \geq 1$, ρ dominates over $1/R^2$ if $D \geq 3$.