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Canonical analysis of the fermion sector in higher-derivative supergravity

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The degree of freedom and helicity count of the fermionic sector in generic superconformal plus supergravity models is exhibited by explicit elimination of all constraints in the 3+1 decomposition of the action, which is given entirely in terms of gauge-invariant variables. In the purely superconformal, massless, case, the third-derivative action describes three helicity-$\frac{3}{2}$ excitations and one of helicity $\frac{1}{2}$. When mass is introduced by adding a Rarita-Schwinger term, a new helicity-$\frac{1}{2}$ excitation appears, and the total spectrum now consists of two massive spin-$\frac{3}{2}$ modes plus a massless one. In both cases, the excitations unavoidably emerge in sets with both signs of the Hilbert-space metric; some implications of this ghost behavior are discussed.

I. INTRODUCTION

The degree-of-freedom content of higher-derivative gauge theories, particularly gravity, has received considerable study.[1–6] There, the general fourth-order action is the sum of squares of the Weyl tensor and of the scalar curvature, to which may be added an Einstein term. The purely conformal theory (with or without the Einstein part) is of particular interest, and has been completely analyzed canonically.[5] Because of the ghost problem common to higher-derivative theories, the generic theory is expected to be unstable in that its energy, unlike that of Einstein's gravity, is not positive; in the purely conformal case the energy vanishes.[7] Since the Einstein theory and the conformal theory both have supersymmetric extensions, it is of interest to display the excitation content of the fermion sector of these theories in explicit canonical form as well. The fermionic spectrum must, of course, contain the partners of the boson excitations, with ghost properties corresponding to those occurring in the gravity sector. Further, this indefinite-Hilbert-space structure should explain how the theories can simultaneously possess the fundamental supersymmetric property that energy is a formal square of supercharge, yet violate positivity. In this paper, we carry out the explicit canonical analysis of the (flat space) kinematics of the fermionic sector for a general combination of conformal and Einstein supergravity, leaving the relation with energy to another work.[8] We shall see that (in agreement with earlier results[2,4]) the purely conformal fermion sector consists of eight massless degrees of freedom: three excitations of helicity $\frac{3}{2}$ and one of helicity $\frac{1}{2}$. These excitations fall into two sets with opposite Hilbert-space signatures. In the generic superconformal plus supergravity combination where the inverse Einstein constant provides a mass scale, some of these excitations acquire mass and a new massive helicity-$\frac{1}{2}$ excitation appears. The degree-of-freedom count jumps to ten: two massive and one massless spin $\frac{3}{2}$, again with unavoidable ghosts. These results are in accord with the corresponding bosonic counts, in which Weyl gravity represents six degrees of freedom [massless helicity (2,2,1)], while Weyl plus Einstein gravity has seven (one massless and one massive spin 2). The additional bosonic degrees of freedom needed for supersymmetry are provided in each case by the (massive or massless) vector field of the conformal supergravity sector. Our results are automatically gauge independent since there are no constraints, and therefore no gauge functions, left in the action, which is described by gauge-invariant fields. In this respect, our work differs from the earlier approaches.[2,4]

II. FERMIONIC ACTION

We shall consider the free fermionic actions obtained by taking the flat-space limit of (higher-derivative) conformal supergravity alone or together with the Rarita-Schwinger action of standard supergravity. There is a relative constant of dimension m^2 between these two actions which we fix by giving ψ_μ dimension $\frac{1}{2}$. The purely conformal theory is then obtained by taking $m=0$. Our action is of the form

$$I=I_C+I_{RS} ,$$

where[3]

$$I_C=\int dx\, \overline{W}^{\mu\nu}(\partial_\mu\phi_\nu-\partial_\nu\phi_\mu) \tag{1}$$

and

$$I_{RS}=\frac{-im^2}{4}\int dx\, \overline{\psi}_\mu\epsilon^{\mu\nu\lambda\sigma}\gamma^5\gamma_\nu(\partial_\lambda\psi_\sigma-\partial_\sigma\psi_\lambda) ,$$

with

$$W^{\mu\nu}=\partial^\mu\psi^\nu-\partial^\nu\psi^\mu-i(\gamma^\mu\phi^\nu-\gamma^\nu\phi^\mu) . \tag{2}$$

The field ϕ^μ is determined in terms of ψ_μ by the constraint that $W^{\mu\nu}$ be chirally self-dual and σ traceless,

$$W^{\mu\nu}=\tfrac{1}{2}\epsilon^{\mu\nu\lambda\sigma}\gamma^5 W_{\lambda\sigma},\ \ \gamma^\mu\gamma^\nu W_{\mu\nu}=0 . \tag{3}$$

This action is gauge invariant under $\delta\psi_\mu=\partial_\mu\alpha(x)$ and the $m=0$ part has a further ("conformal") local invariance under $\delta\psi_\mu=\gamma_\mu\beta(x)$; all spinors are Majorana. We use a metric with signature $(-,+,+,+)$, $\epsilon^{0123}=+1$, and imaginary (Majorana) representation γ's:

$$\{\gamma^\mu,\gamma^\nu\}=-2\eta^{\mu\nu},\ \ \gamma^5=-\gamma^0\gamma^1\gamma^2\gamma^3=-\gamma^{5\dagger} . \tag{4}$$

In order to discuss the canonical decomposition of the theory, it is more convenient to use the $(\beta,\vec{\sigma})$ matrices,

$$\gamma^0\equiv\beta=\beta^\dagger=-\beta^*,\ \ \vec{\gamma}=-i\beta\gamma^5\vec{\sigma} ,$$
$$\gamma^5=-\gamma^{5\dagger}=\gamma^{5*},\ \ \vec{\sigma}=\vec{\sigma}^\dagger=-\vec{\sigma}^* . \tag{5}$$

We also define electric and magnetic components of the ψ field strength,

$$\vec{\chi}^{\,0}\equiv\partial^0\vec{\psi}-\vec{\nabla}\psi^0,\ \ \vec{\chi}\equiv\vec{\nabla}\times\vec{\psi} , \tag{6}$$

and similarly introduce

$$(\vec{W}^0)^i\equiv W^{0i},\ \ (\vec{W})^i\equiv\tfrac{1}{2}\epsilon^{ijk}W_{jk} . \tag{7}$$

The algebraic spin-$\frac{3}{2}$ projector is defined by

$$\tilde{P}=\tilde{1}-\vec{\sigma}\tfrac{1}{3}\vec{\sigma}=\frac{(2\tilde{1}+i\vec{\sigma}\times)}{3} , \tag{8}$$

where $\vec{\sigma}\vec{\sigma}=\tilde{1}-i\vec{\sigma}\times$.

The definition (2) and constraint (3) for $W^{\mu\nu}$ uniquely determine $W^{\mu\nu}$ and ϕ^ν in terms of $\vec{\chi}^{\,0}$ and $\vec{\chi}$:

$$\vec{W}^0=\gamma^5\vec{W}=\tfrac{1}{2}\tilde{P}\cdot(\vec{\chi}^{\,0}+\gamma^5\vec{\chi}) ,$$
$$\phi^0=-\frac{\beta}{3}(\gamma^5\vec{\sigma}\cdot\vec{\chi}^{\,0}+\tfrac{1}{2}\vec{\sigma}\cdot\vec{\chi}) ,$$

and

$$\vec{\phi}=-\frac{i\beta}{2}[\tilde{P}\cdot(\vec{\chi}-\gamma^5\vec{\chi})+\tfrac{1}{3}\gamma^5\vec{\sigma}\vec{\sigma}\cdot\vec{\chi}] . \tag{9}$$

The action may be written as

$$I=2\int dx[\vec{W}^0\beta\cdot(\partial_0\vec{\phi}+\vec{\nabla}\phi^0-\gamma^5\vec{\nabla}\times\vec{\phi}) + \tfrac{1}{4}m^2(\vec{\psi}\cdot\vec{\sigma}\times\vec{\chi}^{\,0}+\psi^0\vec{\sigma}\cdot\vec{\chi}+i\vec{\psi}\cdot\gamma^5\vec{\chi})] . \tag{10}$$

Our procedure thus far has been to solve explicitly for $(W_{\mu\nu},\phi_\nu)$ as functions of ψ_μ and its time derivatives, rather than keep them as independent, "Ostrogradsky" variables. This has the advantage that all time derivatives are manifest from the start and the most convenient choice for new variables becomes reasonably transparent. At this point, the leading time-derivative term is $\vec{W}^0\cdot\beta\vec{\phi}_{,0}\sim\vec{\psi}_{,0}\cdot\vec{\psi}_{,00}$; all the other terms are of lower order. When we cast the theory into first-order canonical form,

$$I\sim\int dx\left[\frac{i}{2}\Psi S\partial_0\Psi-\mathscr{H}(\Psi)\right] , \tag{11}$$

where the matrix S is symmetric (and includes the traditional β factor of $\overline{\psi}$), the vector Ψ will include both the original ψ_μ field and the Ostrogradsky variables. To this end, we observe that the coefficient of $\vec{\psi}_{,00}$ is $-\vec{W}^0\beta$; we therefore define

$$\vec{\psi}_1=-i\beta(\vec{\chi}^{\,0}+\gamma^5\vec{\chi}) \tag{12}$$

and rewrite the action (10) in terms of $\vec{\psi}_1$ and of a Lagrange multiplier field, $\vec{\psi}_2$, which enforces (12). (The index on $\vec{\psi}_a$ defines the number of time derivatives it carries with respect to $\vec{\psi}$ itself.) Then (10) becomes

$$I=\int dx\Bigg\{-i\vec{\psi}_2\cdot(\vec{\chi}^{\,0}+\gamma^5\vec{\chi}-i\beta\vec{\psi}_1)+i(\tilde{P}\cdot\vec{\psi}_1)\cdot\Bigg[-\tfrac{1}{2}\vec{\psi}_{1,0}+\vec{\nabla}\times(\gamma^5\vec{\psi}_1+i\beta\vec{\chi})-\vec{\nabla}\left[\frac{i}{3}\gamma^5\vec{\sigma}\cdot\vec{\psi}_1-\tfrac{1}{2}\beta\vec{\sigma}\cdot\vec{\chi}\right] + \vec{\nabla}\times[\tfrac{1}{2}\gamma^5\tilde{P}\cdot\vec{\psi}_1-(\tilde{I}-\vec{\sigma}\tfrac{1}{2}\vec{\sigma})\cdot\vec{\chi}]\Bigg] + \frac{m^2}{2}(\psi^0\vec{\sigma}\cdot\vec{\chi}+i\vec{\psi}\cdot\gamma^5\vec{\chi}+i\vec{\psi}\beta\vec{\sigma}\times\vec{\psi}_1-\vec{\psi}\gamma^5\vec{\sigma}\times\vec{\chi})\Bigg\} . \tag{13}$$

The action is now of first-order form; only first-time derivatives of the field $\vec{\Psi}\equiv(\vec{\psi}_2,\vec{\psi}_1,\vec{\psi})$ appear. However, there are constraints so that not all components of $\vec{\Psi}$ may be independently specified as initial-value data. In order to exhibit the constraints, we use the identity

$$\tilde{P}\cdot\vec{\nabla}\times\equiv\tilde{P}\cdot\vec{\nabla}\times\tilde{P}+\tfrac{1}{3}\tilde{P}\cdot\vec{\nabla}i\vec{\sigma} \tag{14}$$

and the vector-spinor orthonormal decomposition,[9,10]

$$\vec{\psi}=\vec{\psi}^{T}+(\tfrac{3}{2})^{1/2}\frac{\tilde{P}\cdot\vec{\nabla}i\vec{\sigma}\cdot\vec{\nabla}}{(-\nabla^2)}\psi^L+\frac{i\vec{\sigma}}{\sqrt{3}}\psi^l\,, \tag{15}$$

where $\vec{\psi}^T$ is the doubly transverse, helicity-$\frac{3}{2}$ part,

$$\vec{\sigma}\cdot\vec{\psi}^T\equiv 0\equiv\vec{\nabla}\cdot\vec{\psi}^T\,, \tag{16}$$

while the helicity-$\frac{1}{2}$ components are ψ^l, the $\vec{\sigma}$-longitudinal part of $\vec{\psi}$,

$$\vec{\sigma}\cdot\tilde{P}\cdot\vec{\psi}=i\sqrt{3}\psi^l\,, \tag{17}$$

and ψ^L, the $\vec{\sigma}$-transverse projection of the (ordinary) longitudinal part of the vector $\vec{\psi}$,

$$\vec{\nabla}\cdot\tilde{P}\cdot\vec{\psi}=-(\tfrac{2}{3})^{1/2}i\vec{\sigma}\cdot\vec{\nabla}\psi^L\,. \tag{18}$$

Note that $(i\sigma\cdot\nabla)^2=(-\nabla^2)$. Orthonormality follows from these properties: any two vector-spinors have the inner product

$$\int d^3\vec{r}\,\vec{\psi}\cdot\vec{\zeta}=\int d^3\vec{r}(\vec{\psi}^T\cdot\vec{\zeta}^T+\psi^L\zeta^L+\psi^l\zeta^l)\,. \tag{19}$$

Returning to the action (13), we note that there are two constraints (the equations obtained by varying ψ^0 and ψ_1^l do not involve any time derivatives):

$$\delta\psi_1^l\Longrightarrow\psi_2^l=-\frac{m^2}{2}\psi^l\,,$$
$$\delta\psi^0\Longrightarrow i\vec{\nabla}\cdot\vec{\psi}_2+\frac{m^2}{2}\vec{\sigma}\cdot\vec{\chi}=0\,. \tag{20}$$

Using the identity, Eq. (14), we rewrite the second equation as

$$\psi_2^L=-\frac{m^2}{2}(\psi^L+2\sqrt{2}\psi^l)\,. \tag{21}$$

The above constraints may be used to eliminate ψ_2^L and ψ_2^l regardless of whether m^2 vanishes. In the purely conformal theory ($m^2=0$), the result is simply

$$\vec{\psi}_2=\vec{\psi}_2{}^T\,. \tag{22}$$

It is instructive to link these results (and the degree-of-freedom count) to the gauge invariances of the theory. First, we note that the helicity-$\frac{3}{2}$ parts, being transverse, are entirely gauge-independent and unconstrained. With regard to the helicity-$\frac{1}{2}$ fields, the original ψ_0 is the usual gauge variable associated with the $\delta\psi_\mu=\partial_\mu\alpha$ invariance, and its coefficient is the gauge constraint fixing ψ_2^L. Likewise, ψ_1^l and its constraint (fixing ψ_2^l) are the Ostrogradsky images of the former. This leaves the variables (ψ_1^L,ψ^L,ψ^l) in general. However, one more component disappears, namely the gradient part of $\vec{\psi}$, not unlike A_L in electrodynamics, and two helicity-$\frac{1}{2}$ parts remain. Finally, in the purely conformal case one of these variables disappears as well because of the local $\delta\psi_\mu=\gamma_\mu\beta$ invariance which states that $\gamma^\mu\psi_\mu$ is also absent from the action. (Since the invariance is algebraic rather than differential, there is no associated constraint.)

In our linearized action, the helicity-$\frac{3}{2}$ and helicity-$\frac{1}{2}$ components cannot mix as orthogonality forbids any combination of the form $\int d^3r\,\vec{\psi}^T\cdot\vec{\mathscr{O}}\chi$, where $\vec{\mathscr{O}}$ denotes any linear combination of $\vec{\sigma}$ and $\vec{\nabla}$ and, after some algebra, the action may be written as

$$I=I_{3/2}+I_{1/2}\,,$$

where

$$I_{3/2}=\int dx\left[\frac{i}{2}(\vec{\psi}_2^T\cdot\partial_0\vec{\psi}^T+\vec{\psi}^T\cdot\partial_0\vec{\psi}_2^T-\vec{\psi}_1^T\cdot\partial_0\vec{\psi}_1^T)+\tfrac{3}{2}\vec{\psi}_1^T\cdot\gamma^5(\vec{\sigma}\cdot\vec{\nabla})\vec{\psi}_1^T-\vec{\psi}\cdot\gamma^5(\vec{\sigma}\cdot\vec{\nabla})\vec{\psi}^T-\vec{\psi}_2^T\cdot\beta\vec{\psi}_1^T+2\vec{\psi}_1^T\cdot\beta\nabla^2\vec{\psi}^T\right.$$
$$\left.+m^2\vec{\psi}^T\cdot\gamma^5(\vec{\sigma}\cdot\vec{\nabla})\vec{\psi}^T+\frac{m^2}{2}\vec{\psi}^T\cdot\beta\vec{\psi}_1^T\right]$$

and

$$I_{1/2}=\int dx\left[\frac{i}{2}(-m^2\lambda\partial_0\lambda-\psi_1^L\partial_0\psi_1^L)+\tfrac{1}{2}\psi_1^L\gamma^5(\vec{\sigma}\cdot\vec{\nabla})\psi_1^L+\frac{m^2}{2}[\lambda\gamma^5(\vec{\sigma}\cdot\vec{\nabla})\lambda+2\lambda\beta\psi_1^L]\right]\,. \tag{23}$$

Here we have introduced the field

$$\lambda\equiv\psi^L+\sqrt{2}\psi^l\,.$$

The (unconstrained, gauge-independent) helicity-$\frac{3}{2}$ action may be written as

$$I_{3/2}=\int dx\left[\frac{i}{2}\vec{\Psi}\cdot S\partial_0\vec{\Psi}-\tfrac{1}{2}\vec{\Psi}M\vec{\Psi}\right]\,, \tag{24}$$

where

$$\vec{\Psi}=\begin{bmatrix}\vec{\psi}_2\\ \vec{\psi}_1\\ \vec{\psi}\end{bmatrix}\,,$$

$$S=\begin{bmatrix}0&0&1\\0&-1&0\\1&0&0\end{bmatrix}\,,$$

and

$$M=\beta\begin{bmatrix} 0 & 1 & -\vec{\gamma}\cdot\vec{p} \\ 1 & 3\vec{\gamma}\cdot\vec{p} & \left[2p^2-\frac{m^2}{2}\right] \\ -\vec{\gamma}\cdot\vec{p} & \left[2p^2-\frac{m^2}{2}\right] & (2m^2\vec{\gamma}\cdot\vec{p}) \end{bmatrix} .$$

Here $\vec{p}$ is $(1/i)\vec{\nabla}$, $p^2\equiv\vec{p}\cdot\vec{p}$, and we have reexpressed the $\vec{\sigma}$ matrices in terms of $\vec{\gamma}$. The Fourier-transformed field equation then reads

$$\mathscr{D}\Psi=[\omega S-M(p)]\Psi=0 , \qquad (25)$$

where

$$\mathscr{D}=\beta\begin{bmatrix} 0 & -1 & \beta\omega+\vec{\gamma}\cdot\vec{p} \\ -1 & -(\beta\omega+3\vec{\gamma}\cdot\vec{p}) & -\left[2p^2-\frac{m^2}{2}\right] \\ \beta\omega+\vec{\gamma}\cdot\vec{p} & -\left[2p^2-\frac{m^2}{2}\right] & -2m^2\vec{\gamma}\cdot\vec{p} \end{bmatrix} .$$

This equation may be solved to yield

$$\vec{\psi}_1^T=(\vec{\gamma}\cdot\vec{p}-\gamma^0p_0)\vec{\psi}^T ,$$
$$\vec{\psi}_2^T=-\left[(3\vec{\gamma}\cdot\vec{p}+\gamma^0p_0)(\vec{\gamma}\cdot\vec{p}-\gamma_0p^0)+2p^2-\frac{m^2}{2}\right]\vec{\psi}^T , \qquad (26)$$

and

$$(\gamma^\mu p_\mu)(m^2+p^\mu p_\mu)\vec{\psi}^T=0 .$$

Note that the first two equations are necessarily noncovariant; ψ_1 and $\vec{\psi}_2$ are essentially $\vec{W}^0$ and $\partial^0\vec{W}^0$, so they must have the complicated Lorentz transformation properties which are implicit in these relations. The final equation is the covariant wave equation from which it is apparent that the excitations all satisfy

$$p^\mu p_\mu=-m^2 \qquad (27)$$

or

$$p^\mu p_\mu=0 .$$

Without having to perform an explicit diagonalization of (25), we can immediately conclude that, in the helicity-$(\pm\frac{3}{2})$ sector, there are two massive and one massless degrees of freedom (times 2 for $\pm$ helicity). Clearly, the number of degrees of freedom does not change if $m\equiv0$.

The helicity-$\frac{1}{2}$ modes in (23) must be analyzed separately in the zero-mass and nonzero-mass cases. In the $m=0$ case λ does not appear; only the ψ_1^L degree of freedom remains in the action, which has the form

$$I_{1/2}^{m=0}=\int dx\,\tfrac{1}{2}(\beta\psi_1^L)\beta\left[\gamma^\mu\frac{1}{i}\partial_\mu\right](\beta\psi_1^L) \qquad (28)$$

and exhibits a standard spin-$\frac{1}{2}$ zero-mass Majorana particle with negative metric. The full mode count thus consists of three helicity-$\frac{3}{2}$ plus a helicity-$\frac{1}{2}$ degree of freedom.

In the $m\neq0$ case, the rescaled field $\rho\equiv m\lambda$ also appears and the resultant equations are

$$\left[(\gamma^\mu p_\mu)\begin{bmatrix}1&0\\0&1\end{bmatrix}-m\begin{bmatrix}0&1\\1&0\end{bmatrix}\right]\begin{bmatrix}\rho\\ \psi_1^L\end{bmatrix}=0 \qquad (29)$$

so that each field obeys the covariant equation

$$(p^\mu p_\mu+m^2)(\rho,\psi_1^L)=0 . \qquad (30)$$

Diagonalized equations may be defined from (29) using combinations (involving β) of (ρ,ψ_1^L). One may choose

$$\frac{\beta}{\sqrt{2}}(\rho\pm\psi_1^L) \qquad (31)$$

as the negative-metric diagonal fields.

The norm of the Hilbert space in the spin-$\frac{3}{2}$ sector is determined by the signature of S^{-1}. With the overall sign of our action corresponding to the conventional sign for the bosonic sector, the equal-time anticommutation relations are given by

$$\{\vec{\Psi}^T(\vec{r},t),\vec{\Psi}^T(\vec{r}\,',t)\}=S^{-1}\langle\vec{r}\,|\,\tilde{P}_{3/2}\,|\,\vec{r}\,'\rangle . \qquad (32)$$

Hence, the form (24) of S immediately tells us that there is one linear combination $(\vec{\psi}_2+\vec{\psi})$ with positive norm and two combinations $(\vec{\psi}_2-\vec{\psi},\ \vec{\psi}_1)$ with negative norm.

To find the explicit values of the norms of the physical helicity-$\frac{3}{2}$ eigenstates, we include a source, $\vec{\eta}$, in the action:

$$I_\eta=I_{3/2}+\int d^3x\,\vec{\eta}\cdot\vec{\Psi}^T . \qquad (33)$$

The field equation is then

$$\mathscr{D}\Psi=\eta , \qquad (34)$$

which implies

$$\vec{\psi}_1=\beta(\vec{\alpha}\cdot\vec{p}+\omega)\vec{\psi}-\beta\vec{\eta}_2 ,$$
$$\vec{\psi}_2=\left[p^2+2\omega\vec{\alpha}\cdot\vec{p}-\omega^2+\frac{m^2}{2}\right]\vec{\psi}-\beta\vec{\eta}_1-(3\vec{\alpha}\cdot\vec{p}-\omega)\vec{\eta}_2 ,$$

and

$$\vec{\psi}=-\left[\frac{\vec{\alpha}\cdot\vec{p}+\omega}{(m^2+p^2-\omega^2)(p^2-\omega^2)}\right]\times\left[\left[\frac{m^2}{2}+p^2+2\omega\vec{\alpha}\cdot\vec{p}-\omega^2\right]\vec{\eta}_2+(\vec{\alpha}\cdot\vec{p}+\omega)\beta\vec{\eta}_1+\vec{\eta}\right] . \qquad (35)$$

The norms of the states given by the poles at $\omega^2=(p^2+m^2,p^2)$ are governed by the signs of the residues at the poles. The pole terms in $\mathscr{G}=\mathscr{D}^{-1}$ are

$$\begin{bmatrix}A\\B\\1\end{bmatrix}\frac{\vec{\alpha}\cdot\vec{p}+\omega}{(m^2+p^2-\omega^2)(p^2-\omega^2)}(A^\dagger,B^\dagger,1) , \qquad (36)$$

where

$$A \equiv \frac{m^2}{2} + p^2 + 2\omega \vec{\alpha} \cdot \vec{p} - \omega^2 \quad \text{and} \quad B \equiv \beta(\vec{\alpha} \cdot \vec{p} + \omega) .$$

The residue at the zero-mass pole is ($\omega = |p|$)

$$\sum_\lambda \begin{bmatrix} \frac{m^2}{2} + 2p^2 \\ 2\beta |p| \\ 1 \end{bmatrix} \frac{\mu_\lambda^{(0)} \mu_\lambda^{(0)\dagger}}{m^2} \left[\frac{m^2}{2} + 2p^2, \; 2|p|\beta, \; 1 \right] \geq 0 ,$$

while the residue at the $\omega^2 = p^2 + m^2$ pole is $[\omega = +(p^2+m^2)^{1/2}]$

$$-\frac{1}{2m^2} \sum_\lambda \begin{bmatrix} A \\ B \\ 1 \end{bmatrix} (\mu_\lambda^{(m)} \mu_\lambda^{(m)\dagger} + \mu_\lambda^{(-m)} \mu_\lambda^{(-m)\dagger})(A, B^\dagger, 1) \leq 0 \tag{37}$$

with

$$m\beta + \vec{\alpha} \cdot \vec{p} + \omega = \sum_\lambda \mu_\lambda^{(m)} \mu_\lambda^{(m)\dagger} .$$

These two massive helicity-$\frac{3}{2}$ ghosts combine with the helicity-$\frac{1}{2}$ ghosts of (30,31) to make two massive spin-$\frac{3}{2}$ excitations when $m \neq 0$. For $m = 0$, they are still ghosts as is the single helicity-$\frac{1}{2}$ particle of (28).

We remark that our final actions (both for $m = 0$ and for $m \neq 0$) are, of course, not manifestly Lorentz invariant since we have eliminated the redundant variables. Indeed, we have the form which normal (massive or massless) spin-$\frac{3}{2}$ actions take when reduced to canonical form.[9]

III. COMMENTS

We have exhibited the canonical form of both the purely conformal fermionic theory and the generic conformal plus Rarita-Schwinger combination. In the former we found eight massless excitations, namely, those associated with the three transverse traceless helicity-$\frac{3}{2}$ fields $(\vec{\psi}_2, \vec{\psi}_1, \vec{\psi})^T$ and the helicity-$\frac{1}{2}$ field ψ_1^L. These eight degrees of freedom pair with the six massless helicity (2,2,1) metric degrees of freedom of conformal gravity plus the spin-1 axial photon required for superconformal invariance. When the Rarita-Schwinger term is added to yield a supersymmetric but not conformally invariant theory, a second helicity-$\frac{1}{2}$ degree of freedom appears which, along with the original helicity-$\frac{1}{2}$ field, combines with two of the helicity-$\frac{3}{2}$ degrees of freedom to form two massive spin-$\frac{3}{2}$ degrees of freedom while one helicity-$\frac{3}{2}$ field remains massless. The corresponding massive bosonic sector has massive spins 2 and 1 (including the axial vector) plus a massless spin 2. The pairing yields a massless spin $(2, \frac{3}{2})$ representation of the supersymmetry and a massive spin $(2, \frac{3}{2}, \frac{3}{2}, 1)$. It should be noted that, as usual with gauge theories, the $m \to 0$ limit is not the same as the $m = 0$ theory since, although all modes become massless, the lower helicity modes need not disappear. (This is even true for the massive Rarita-Schwinger theory.[9])

As one would expect, the massive multiplets all have negative Hilbert-space norm (or, for the bosons, negative energy). The zero-mass excitations have both positive energy and positive norm. Although these specific assignments depend upon the signs chosen for the two parts of the Lagrangian, the conclusion that there must be negative-norm excitations does not: We may change either the overall sign of the fermionic terms, the sign of the mass term or both. If we change the overall sign, we have positive-norm massive excitations and negative-norm massless excitations (of course, supersymmetry now no longer obtains unless the sign of the gravitational action is also changed). If we change the sign of m^2, the excitations become tachyonic but still have the same norm.

The fact that the fundamental dynamical variables do not have positive-definite commutation relations immediately undermines the usual argument that the energy in a supersymmetric theory must be positive semidefinite. In any such theory the generator of time translations, P^0, may indeed still be expressed as the anticommutator, $P^0 = \frac{1}{4} \operatorname{tr}\{Q, Q\}$, but this is manifestly positive semidefinite only if the norm of the Hilbert space is positive.[11] Since the supercharges Q^α are odd in the (indefinite norm) fermionic variables, there is no argument that P^0 is non-negative and no contradiction with the classical bosonic results. The detailed consequences of this will be presented elsewhere.[8]

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