On-shell constraints for perturbative and nonperturbative quark masses in QCD

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It is shown that the current, dynamical, and constituent quark masses, when anchored to the pole positions, are gauge-parameter independent both at the pole and in the deep-Euclidean region. The implications of theoretical and phenomenological quark-mass scales for chiral-symmetry breakdown and for the bridge between the relativistic theory of QCD and the nonrelativistic SU(6) quark model are discussed.

I. INTRODUCTION

It is becoming increasingly clear in the relativistic renormalizable theory of quantum chromodynamics (QCD) that there are two physically meaningful concepts of mass: the perturbative or current quark mass $m_{\rm curr}$ which occurs in the bare Lagrangian (presumably due to weak interactions) and the nonperturbative or dynamically generated quark mass $m_{\rm dyn}$ which dominates low-energy hadron physics.[1–3] Furthermore, it has long been expected that the sum of $m_{\rm curr}$ and $m_{\rm dyn}$ corresponds to the constituent quark mass $m_{\rm con}$, the latter setting the scale for the nonrelativistic quark model.[4,5] While it is now understood that the flavor-dependent current quark mass is gauge-parameter independent,[6,7] it is thought that the flavor-independent dynamically generated quark mass is gauge *dependent*.[7,8] If this were true, then our ideas about dynamical breakdown of chiral symmetry and its link[9–12] to $m_{\rm dyn}$ would be obscured, and the connection between the relativistic and nonrelativistic quark models[10–12] would be severed.

In this paper we show that the dynamically generated quark mass not only is gauge independent at the pole position, but can be defined to remain gauge independent as it runs with the aid of the renormalization-group equation.[13] Moreover, we point out that the gauge-parameter-dependent factors $3+a$ and a (where $a=0$ in Landau gauge), which occur in the self-energy components of $m_{\rm curr}$, also appear in the same self-energy components of $m_{\rm dyn}$. Thus, the sum of $m_{\rm curr}$ and $m_{\rm dyn}$ is also gauge-parameter independent at the constituent mass pole position. Away from $m_{\rm con}$, the running mass $\overline{m}(p^2)=\overline{m}_{\rm curr}(p^2)+\overline{m}_{\rm dyn}(p^2)$ also satisfies the renormalization-group equation in a gauge-independent manner.

An explanation at this point is perhaps in order concerning our use of the word "pole position" of the quark propagator. Even though the quark is never free in the confined hadronic phase so that the singular pole of the quark propagator is never achieved, it is valid to identify the running quark mass $m_{\rm dyn}(p^2)$ at the value $p^2=m_{\rm dyn}{}^2$ as the self-consistent pole position of the propagator:

$$m_{\rm dyn}=m_{\rm dyn}(p^2=m_{\rm dyn}{}^2)\ .$$

Similarly, while the free-particle Dirac equation is not applicable, it is meaningful to replace $\not p$ by $m_{\rm dyn}$ when examining this "pole position" in the context of off-shell quark Ward identities.

We begin in Sec. II by reviewing how the gauge-dependent factors $3+a$ and a enter the self-energy components of $m_{\rm curr}$ in one-loop order,[6] yet do not enter $m_{\rm curr}(p^2)$ as defined at and away from the pole position. Then in Sec. III we borrow the self-energy components of $m_{\rm dyn}$ as calculated in one-loop order[8] and again show that $m_{\rm dyn}(p^2)$ as defined at or away from the pole position is gauge independent. We also demonstrate that the gauge-dependent effective mass

$$m_{\rm eff}(p^2)=-C/D\ ,$$

where C and D are the off-shell inverse propagator components, appears in the induced pseudoscalar part of the axial-vector vertex,[14,15] thus manifesting the Nambu[16]-Goldstone[17] theorem only at the pole position, where $m_{\rm eff}=m_{\rm dyn}$ is gauge independent. In Sec. IV we incorporate the anomalous dimension and

$$d=12(33-2n_f)^{-1}$$

into the expression for the dynamically generated quark mass as determined[2] by the operator-product expansion (OPE), but now for $\overline{m}_{\rm dyn}(p^2)$ defined in a gauge-independent manner. This running mass $\overline{m}_{\rm dyn}(p^2)$ can also be fed into a manifestly gauge-independent quark loop in order to calculate the Politzer scale[2] for the quark condensate $\langle\bar{q}q\rangle_0$. The theoretical relation between $m_{\rm dyn}$ and Λ and the phenomenology associated with the scales of the chiral-symmetry-breakdown order parameters $m_{\rm dyn}$, $\langle\bar{q}q\rangle_0$, and f_π is then considered. In Sec. V the constituent quark mass $m_{\rm con}$ is shown to be gauge-parameter independent, but only at the pole position of the constituent quark mass. The bridge between the relativistic theory of QCD and the SU(6) nonrelativistic quark model is then discussed. Finally, in Sec. VI we summarize our results and parallel the OPE, momentum-loop, and axial-vector Ward-identity approaches to the dynamically generated quark mass with similar attacks on the $\pi^0\to2\gamma$ amplitude in order to stress the importance of the concepts of $m_{\rm dyn}$ and $m_{\rm con}$. The Appendix is devoted to recovering the possible running momentum structure of the dynamical quark mass in the context of the renormalization group.

II. GAUGE INDEPENDENCE OF CURRENT QUARK MASSES

By way of review, we consider the quark self-energy graph of Fig. 1, where the bare fermion mass is presumably generated by the electroweak Lagrangian. In a general covariant gauge, the gluon propagator

$$D_{\mu\nu}(k)=(-i/k^2)[g_{\mu\nu}-(1-a)k_\mu k_\nu/k^2] \quad (1)$$

combined with the QCD coupling $g\bar{q}(\lambda^i/2)\gamma_\mu q V^{i,\mu}$ leads to the second-order inverse propagator (in the absence of dynamical mass)

$$\Gamma^{(2)}(p)=\not{p}-m-\Sigma(p) \quad (2)$$

with renormalized self-energy

$$\Sigma(p)=[a\not{p}-(3+a)m]\left[\frac{4}{3}\frac{g^2}{16\pi^2}\right]\ln\left[\frac{-p^2}{\mu^2}\right], \quad (3)$$

where m is now the renormalized Lagrangian mass and μ is a reference mass. If we partition Σ into the mass-shift (Σ_1) and wave-function-renormalization (Σ_2) pieces

$$\Sigma(p)=\Sigma_1(p^2)+\Sigma_2(p^2)(\not{p}-m), \quad (4a)$$

then (3) can be rewritten as

$$\Sigma_1(p^2)=-m\left[\frac{g^2}{4\pi^2}\ln\left[\frac{-p^2}{\mu^2}\right]\right], \quad (4b)$$

$$\Sigma_2(p^2)=a\left[\frac{g^2}{12\pi^2}\ln\left[\frac{-p^2}{\mu^2}\right]\right]. \quad (4c)$$

The field-theory mass is the zero of the inverse propagator in (2), which occurs to order g^2 when

$$\not{p}=m+\Sigma_1(m^2)\equiv m_{\text{ren}} \quad (5a)$$

$$=m-m\left[\frac{g^2}{4\pi^2}\ln\left[\frac{-m^2}{\mu^2}\right]\right] \quad (5b)$$

which is then *independent*[6] of the gauge parameter a. The pole mass has also been shown to be gauge-parameter independent to two-loop order.[7] As expected, quantum field theory conspires to keep this unphysical gauge parameter from percolating into what the theory believes to be observable physics. Rescalings of g, m, a, and the renormalization subtraction point also leave the physics unaltered according to the renormalization-group equation[1,13]

$$0=\left[\mu\frac{\partial}{\partial\mu}+\beta(g)\frac{\partial}{\partial g}+m\gamma_m(g)\frac{\partial}{\partial m}+\delta(a,g)\frac{\partial}{\partial a}+\gamma_{\text{WF}}(ag)\right]\Gamma^{(2)}(p), \quad (6)$$

where the β function and the anomalous mass and wave-function dimensions γ_m and γ_{WF} are defined in the usual way.[1,13] Then explicit substitution of $\Gamma^{(2)}(p)$ from (2)–(4) to $O(g^2)$ yields

$$\left[\mu\frac{\partial}{\partial\mu}\Sigma_2(p^2)-\gamma_{\text{WF}}\right]\not{p}=0, \quad (7a)$$

$$-\mu\frac{\partial}{\partial\mu}\Sigma_1(p^2)+\left[\mu\frac{\partial}{\partial\mu}\Sigma_2(p^2)-\gamma_m-\gamma_{\text{WF}}\right]m=0, \quad (7b)$$

which in turn leads to the anomalous dimensions

$$\gamma_m=-g^2/2\pi^2, \quad (8a)$$

$$\gamma_{\text{WF}}=-g^2a/6\pi^2. \quad (8b)$$

Gauge-parameter independence of Σ_1, the non-wave-function-renormalization part of the self-energy, implies gauge-parameter independence of the anomalous mass dimension (8a), a result dictated by the gauge independence of the pole mass in (5).

Finally, the running current quark of QCD retains the deep Euclidean form[1]

$$\bar{m}_{\text{curr}}(p^2)=\bar{m}_{\text{curr}}(\mu)\left[\frac{\ln(\mu^2/\Lambda^2)}{\ln(p^2/\Lambda^2)}\right]^d, \quad (9a)$$

where Λ is the QCD energy-scale parameter, and, with

$$\beta(g)=-(33-2n_f)g^3/48\pi^2,$$

where d is obtained in the usual way:

$$d=\frac{g\gamma_m(g)}{2\beta(g)}=12(33-2n_f)^{-1} \quad (9b)$$

for n_f quark flavors. To one-loop order, d is also gauge-parameter independent, reflecting gauge independence of the anomalous dimension γ_m to this order.[13] We suspect this result to hold true to all orders; β has been argued to be gauge independent to all orders in the dimensional-regularization [i.e., modified minimal-subtraction ($\overline{\text{MS}}$)] scheme,[14] while γ_m is gauge and scheme independent through two-loop order.[7] It should be noted that for interactions that are not purely Lorentz vector, d can be gauge-parameter dependent. Such is the case for the weak interactions. Nevertheless, this gauge dependence of weak contributions to fermion masses can be shown to cancel explicitly in the ratios of running fermion masses.[6]

III. GAUGE INDEPENDENCE OF THE DYNAMICALLY GENERATED POLE MASS

While the current quark mass of Fig. 1 is a perturbative quantity, the nonperturbative mass m_{dyn} of Fig. 2 corresponds to summing over all possible gluon exchanges in

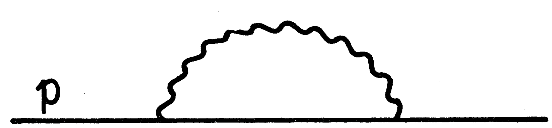

FIG. 1. Second-order gluon radiative corrections to the quark self-energy.

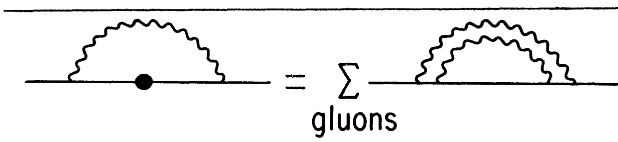

FIG. 2. Dynamically generated quark mass as nonperturbative infinite sum of radiative corrections to the quark line.

all orders. Here we work in the chiral limit with $m_{\rm curr}=0$ along with $m=0$, the latter being the bare mass in the QCD Lagrangian. The dynamically generated mass $m_{\rm dyn}$ then owes its existence to the mass-dimension-three quark condensate $\langle \bar{q}q \rangle \neq 0$. On dimensional grounds alone, we therefore expect

$$m_{\rm dyn}(p^2) \propto \langle \bar{q}q \rangle_0 / p^2 \; . \tag{10}$$

The p^{-2} dependence in (10) can be interpreted as the gluon pole in the operator-product-expansion (OPE) graph of Fig. 3, corresponding to the coordinate-space quark propagator

$$\int d^4x\, e^{ipx} \langle T\psi(x)\bar{\psi}(0) \rangle_0$$

as first considered in Ref. 2. For the inverse quark propagator, now expressed in the chiral limit as

$$S^{-1}(p) = \not{p} - \Sigma(p) \; , \tag{11}$$

it is possible to exploit again the OPE and show that (11) has the gauge-dependent self-energy part[8]

$$\Sigma(p^2) = -\left[a \frac{\not{p}}{p^2} m_{\rm dyn} - (3+a) \right] \left[\frac{4\pi\alpha_s(p^2)}{9} \right] \frac{\langle \bar{q}q \rangle_0}{p^2} \; . \tag{12}$$

We differ from Ref. 8 only in that the mass in (12) must be the self-consistent dynamically generated mass

$$m_{\rm dyn} = m_{\rm dyn}(p^2 = m_{\rm dyn}{}^2) \; , \tag{13}$$

rather then the Lagrangian mass m which vanishes in (11) and (12) in the chiral limit.

Comparing (12) with (3), we see that the gauge dependence for the current and dynamical self-energies is identical at the pole mass $\not{p} = m_{\rm dyn}$ and $p^2 = m_{\rm dyn}{}^2$ by (13). Separating off the gauge-dependent self-energy part Σ_2 as in (4), it is again obvious the mass-renormalization part Σ_1 in (12) is gauge independent at the pole:

$$\Sigma_1(p^2 = m_{\rm dyn}{}^2) \equiv m_{\rm dyn} = -\frac{4\pi}{3} \alpha_s(m_{\rm dyn}{}^2) \frac{\langle \bar{q}q \rangle_0}{m_{\rm dyn}{}^2} \; . \tag{14}$$

This result follows directly from Ref. 2 in Landau gauge with $a=0$ (up to a factor of $-\frac{1}{12}$ missed in Ref. 2). The work of Ref. 8 suggests that the gauge independence of (14) holds only in four dimensions.

There is a second dynamically generated quark mass $m_{\rm eff}$ defined directly from the inverse quark propagator $S^{-1} = C + D\not{p}$ as $m_{\rm eff} = -C/D$. It appears in the axial-vector vertex part[10,15,18]

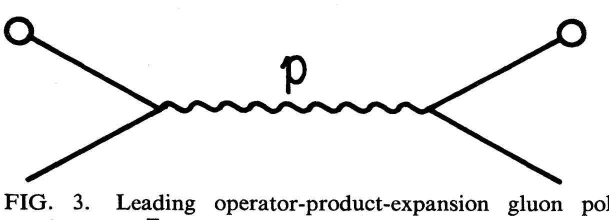

FIG. 3. Leading operator-product-expansion gluon pole graph for $T\psi(x)\bar{\psi}(0)$.

$$\Gamma_{\mu 5}(p;q) \propto \gamma_\mu \gamma_5 - \frac{2m_{\rm eff}(p^2)}{q^2} q_\mu \gamma_5 \; , \tag{15}$$

as found from the axial-vector Ward identity

$$-iq^\mu \Gamma_{\mu 5} = S^{-1}(p + \tfrac{1}{2}q)\gamma_5 + \gamma_5 S^{-1}(p - \tfrac{1}{2}q) \tag{16}$$

by inverting (16) (i.e., dividing by q_μ) and dropping the transverse gauge-dependent part. In order to render (15) gauge independent, we must go to the pole position $p^2 = m_{\rm dyn}{}^2$, at which point $m_{\rm eff} = m_{\rm dyn}$ as well, the latter being gauge independent by (14). Thus, once again it is pole mass [in (15)] which reflects the physical (Nambu[16]-Goldstone[17]) induced pseudoscalar $m_\pi^2 = 0$ pole structure of the axial-vector current. This gauge-independent result was obtained by Nambu[16] for on-mass-shell free fermions instead of the off-shell confined quarks considered here, but the conclusion is the same in either case.

IV. INCLUSION OF THE ANOMALOUS DIMENSION IN THE RUNNING NONPERTURBATIVE MASS

If we were to generalize Politzer's OPE analysis[2] for running nonperturbative quark mass in an arbitrary gauge, we would find

$$\bar{m}_{\rm eff}(p^2) = -\frac{1}{3} \frac{g^2(p^2)}{p^2} \left[\frac{g^2(p^2)}{g^2(M^2)} \right]^{-d} \times \langle (\bar{q}q)_M \rangle_0 \left[\frac{3 + a(p^2)}{3} \right] , \tag{17}$$

where M is now the renormalization-point mass and the gauge parameter $a(p^2)$ also runs. Once again, if we anchor $m_{\rm eff}$ to the pole mass $p^2 = m_{\rm dyn}{}^2$, equivalent to the self-consistency equation (13), the gauge-dependent factor $a(p^2)$ disappears as in (14). Since

$$m_{\rm curr}(M^2) \langle (\bar{q}q)_M \rangle_0$$

is renormalization-point invariant, i.e.,

$$\langle (\bar{q}q)_M \rangle_0 \sim (\ln M^2)^d \; ,$$

it is clear from (17) that[10–12]

$$\bar{m}_{\rm dyn}(p^2) \propto p^{-2} (\ln p^2)^{d-1}$$

is independent of M. We may express this result in a manner analogous to (9) as

$$\bar{m}_{\rm dyn}(p^2) = \frac{M^2}{p^2} \bar{m}_{\rm dyn}(M^2) \left[\frac{\ln(M^2/\Lambda^2)}{\ln(p^2/\Lambda^2)} \right]^{1-d} \tag{18}$$

in the deep-Euclidean region. We require that this nonperturbative running mass (18) satisfies the self-consistency equation (13) and is therefore gauge-parameter independent even for $p^2 \neq m_{\rm dyn}{}^2$.

While (18) is a stronger relation than (17), the coordinate-space OPE condensate scale of (14) or (17) is lost. It can, however, be regained by direct computation of $\langle (\bar{q}q)_M \rangle_0$ via the momentum-space graph of Fig. 4 using only[10,12] the running mass (18). In particular we have for three-quark colors

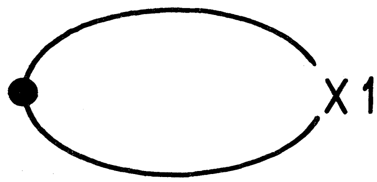

FIG. 4. Momentum-space quark loop for $\langle(\bar{q}q)_M\rangle_0$. The dark circle represents a dressed quark propagator with mass $m_{\rm dyn}(p^2)$.

$$\langle(\bar{q}q)_M\rangle_0=\frac{-i\,3\times 4}{(2\pi)^4}\int^M \frac{d^4p\,\overline{m}_{\rm dyn}(p^2)}{p^2-\overline{m}_{\rm dyn}{}^2(p^2)} \tag{19a}$$

$$=-\frac{3}{4\pi^2 d}\overline{m}_{\rm dyn}(M^2)M^2\ln(M^2/\Lambda^2)\ , \tag{19b}$$

where we take the renormalization-point deep-Euclidean prescription for M in (19a) as corresponding to the upper limit in the integral. We have also employed (18) in the deep-Euclidean region with $y=-p^2$, $d^4p=i\pi^2 y\,dy$, and $\overline{m}_{\rm dyn}(-y)=\overline{m}_{\rm dyn}(y)$ for $y>0$ in order to go from (19a) to (19b). Then using the asymptotic-freedom QCD coupling[13]

$$\alpha_s(p^2)=\pi d/\ln(p^2/\Lambda^2)\ ,$$

we may convert (19b) to

$$-\langle(\bar{q}q)_M\rangle_0=\frac{3}{4\pi}\alpha_s{}^{-1}(M^2)M^2\overline{m}_{\rm dyn}(M^2)\ , \tag{20}$$

which is then the obvious generalization of the gauge-independent pole mass relation (14). Apart from recovering the Politzer scale of (17) (for $a=0$), the momentum-space-loop graph of Fig. 4 contains no explicit gluon propagators; consequently, this figure and (19)–(20) are manifestly gauge independent.

In the Appendix, the pole mass of the fermion propagator is shown to satisfy a gauge-parameter-independent renormalization-group equation from which either the $[g^2(\mu^2)]^{-d}$ structure of (9a) or the $p^{-2}[g^2(M^2)]^d$ OPE structure of (17) are realizable.

Having discussed $m_{\rm dyn}(p^2)$ from the coordinate space OPE, the momentum space Feynman integral (19), along with the renormalization-group approach of the Appendix, we now extract the scale of $m_{\rm dyn}$ from the quark condensate determined[19] at $M\approx 1$ GeV with value

$$-\langle(\bar{q}q)_M\rangle_0\approx(249\ {\rm MeV})^3\ .$$

Since $M\sim 1$ GeV roughly corresponds to the strong-to-weak-coupling freeze-out region,[20] we can combine (10) and (13) to obtain

$$\overline{m}_{\rm dyn}(M^2)\approx m_{\rm dyn}{}^3/M^2\ . \tag{21}$$

Then, substituting (21) back into (20), we recover (14) or[12]

$$m_{\rm dyn}=\left[-\frac{4\pi}{3}\alpha_s(M^2)\langle(\bar{q}q)_M\rangle_0\right]^{1/3}\approx 319\ {\rm MeV} \tag{22}$$

for the accepted value α_s (1 GeV^2)$\approx$0.50, the latter obtained from the $\bar{s}s$ ϕ meson, the $\bar{c}c$ ψ meson,[5] nonleptonic hyperon decays[10] and the phenomenologically determined $\Lambda_{\overline{\rm MS}}(5)\approx 150$ MeV, and $\Lambda_{\overline{\rm MS}}(3)\approx 250$ MeV for five and three quark flavors, respectively[21] [recall the gauge independence of β in the $\overline{\rm MS}$ renormalization program[14]]. The fact that $m_{\rm dyn}$ in (22) is near the weak-binding value $m_{\rm dyn}\approx m_N/3\approx 313$ MeV is as expected; other estimates of $m_{\rm dyn}$ range between[12] 310 and 320 MeV.

Apart from the phenomenological determinations of $m_{\rm dyn}$, there is a recent theoretical calculation[22] based on the gap-equation approach of Nambu and Jona-Lasinio.[16] The chiral-invariant Lagrangian (with $m_0=0$) is partitioned into a massive free-particle part and a counterterm at mass $m_{\rm dyn}$. The second-order radiative corrections in Fig. 1 then refer to the "physical" pole mass in the chiral limit $m_{\rm dyn}$ which is kept independent of p^2. To order g^2, the self-energy preserves this mass $m_{\rm dyn}$ if we have to one-loop accuracy a mass shift, including finite surface terms calculated in the $\overline{\rm MS}$ renormalization scheme[22]

$$\delta m=\frac{4}{3}\times\frac{3\alpha_s}{4\pi}m_{\rm dyn}\left[\ln\frac{\mu^2}{m_{\rm dyn}{}^2}+\frac{1}{3}\right]\ , \tag{23a}$$

where the subtraction point mass μ as in (5a) now replaces the usual QED ultraviolet cutoff. When (23a) is combined with the analog of (5a) or (13), the self-consistent pole-mass equations

$$\delta m=m_{\rm dyn} \tag{23b}$$

for $m_0=0$, the nonperturbative mass $m_{\rm dyn}\neq 0$ cancels out. The renormalization-group-improved version of the gap equation (23) is then[22]

$$\left[1+\frac{\alpha_s}{\pi d}\left[\ln\frac{m_{\rm dyn}{}^2}{\mu^2}-\frac{1}{3}\right]\right]^d=0\ . \tag{24}$$

Since

$$\alpha_s(\mu^2)=\pi d/\ln(\mu^2/\Lambda^2)\ ,$$

the renormalization scale μ in (24) can be eliminated in favor of Λ, leading to

$$m_{\rm dyn}=\Lambda e^{1/6}\ . \tag{25}$$

This mass represents the summation of the "leading-logarithm" components of the self-energy graphs in Fig. 2. [Note that naive application of this approach to (23) rather than (24) could not have resulted in an expression like (25) which is independent of μ.] The exponent $\frac{1}{6}$ in (25) stems from the surface term of $\frac{1}{3}$ in (23a). While the latter is a Landau-gauge result, if one works in a general gauge, (23a) is more complicated but the end result (25) remains unchanged.[23] In Ref. 22 it is shown that (25) also holds in two-loop order.

Again we observe that (25), although first derived in Landau gauge beginning with (23a), is in fact gauge-parameter independent because the self-consistency (pole) condition (23b) ensures that the gauge parameter a enters into only the wave-function renormalization, but not into $m_{\rm dyn}$ as in (5) and (12) at $\not p=m_{\rm dyn}$ and $p^2=m_{\rm dyn}{}^2$. Thus, assuming that (25) is valid even in higher orders and remains gauge independent, it is a fundamental result relating the nonperturbative mass parameter to the energy scale Λ of QCD. In fact, (25) is the QCD analog of the BCS equation for the superconducting gap energy[24] $\Delta \approx 2\omega_D e^{-5}$, where twice the Debye energy $2\omega_D$ is the ultraviolet cutoff (of order melting energy) suppressed by the exponential factor $e^{-5} \sim 10^{-2}$. In (25) the QCD energy scale Λ acts as an infrared cutoff enhanced by the exponential factor $e^{1/6} \approx 1.18$. For $\Lambda_{\overline{MS}}(3) \approx 250$ MeV corresponding to α_s (1 GeV)≈ 0.50 and α_s (3 GeV)≈ 0.28, (25) predicts

$$m_{\rm dyn} \approx 295 \text{ MeV} , \tag{26}$$

but since $\Lambda \approx 267$ MeV increases α_s only slightly higher to α_s (1 GeV)≈ 0.52 and α_s (3 GeV)≈ 0.29, the associated value $m_{\rm dyn} \approx 315$ MeV is almost as reasonable a consequence of (25) as is (26).

One immediate application of the deep-Euclidean gauge-independent form (18) and scale (22) is in the calculation of the infinite-range integral for the pion decay constant.[9,25] In the chiral limit, the Goldberger-Treiman relation at the quark level extracted from the axial-vector current (15) is

$$f_\pi g_{\pi qq} = m_{\rm dyn} . \tag{27}$$

When this is combined with (18) and the Feynman loop for the axial-vector matrix element $\langle 0 | A_\mu | \pi \rangle$ in Fig. 5 as $q \to 0$, we obtain

$$i f_\pi = g_{\pi qq} \frac{3 \times 4}{(2\pi)^4} \int d^4 p \frac{\overline{m}_{\rm dyn}(p^2)}{[p^2 - \overline{m}_{\rm dyn}{}^2(p^2)]^2} , \tag{28a}$$

in analogous fashion to the quark-loop integral (19a), so that the nonperturbative mass scale $m_{\rm dyn} \neq 0$ cancels out as in (23). The resulting dimensionless integral for $y = -p^2$ is[11,12]

$$\xi \equiv \frac{1}{m_{\rm dyn}} \int_0^\infty \frac{dy\, y m_{\rm dyn}(-y)}{[y + m_{\rm dyn}{}^2(-y)]^2} \approx 0.92 , \tag{28b}$$

converging in the ultraviolet region due to the p^{-2} structure of (18) for the numerator mass in (28b) and in the infrared region due to the p^{-2} of (10) in the denominator mass in (28b). In the infrared region we may alternatively cut off $m_{\rm dyn}(p^2)$ at $m_{\rm dyn} \approx 315$ MeV. The dimensionless bound-state πqq coupling constant is then[11,12]

$$g_{\pi qq} = \frac{2\pi}{\sqrt{3\xi}} \approx 3.78 , \tag{29a}$$

quite close to the ratio determined directly from the Goldberger-Treiman relation (27) using (22) along with the chiral-limiting value[10,26] $f_\pi \approx 90$ MeV:

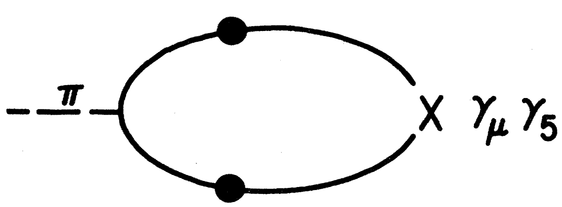

FIG. 5. Pion matrix element of the axial-vector current.

$$g_{\pi qq} = \frac{m_{\rm dyn}}{f_\pi} \approx \frac{319 \text{ MeV}}{90 \text{ MeV}} \approx 3.54 . \tag{29b}$$

The gauge-independent nonperturbative running quark mass thus plays a key role in (20), (22), and (29), and as such, unifies the quark nonperturbative "order parameters" of QCD: $\langle \bar{q} q \rangle_0$, $m_{\rm dyn}$, f_π. The superconductivity-type relation (25) then relates all of these nonperturbative energy scales back to the QCD energy-scale Λ in a gauge-independent manner.

V. GAUGE INDEPENDENCE OF CONSTITUENT QUARK MASSES

Away from the chiral limit, the quark masses in the quark propagator and in the renormalization-group analysis are the sum of the flavor-dependent perturbative or Lagrangian masses and the flavor-independent nonperturbative quark mass[2,3] all of which run:

$$m_i(p^2) = \overline{m}_{i,\rm curr}(p^2) + m_{\rm dyn}(p^2) , \tag{30}$$

where $i = u,d,s$ are the SU(3) quark flavors. Since the gauge-dependent structure of the self-energy parts in (3) and (12) are identical at the constituent pole position, now at

$$m_{i,\rm con} = m_i(p^2 = m_{i,\rm con}{}^2) , \tag{31}$$

the gauge-dependent coefficients a cancel at

$$\Sigma(p = m_{i,\rm con},\ p^2 = m_{i,\rm con}{}^2)$$

both in (3) and in (12). This occurs self-consistently for m in (3) now corresponding to $m_{i,\rm con}$ with radiative corrections computed from Fig. 1 and in (12) with $m_{\rm dyn}$ replaced by $m_{i,\rm con}$ which are the masses in the quark condensates $\langle \bar{u} u \rangle_0$, $\langle \bar{d} d \rangle_0$, $\langle \bar{s} s \rangle_0$ away from the chiral limit.

As a further check on the consistency of (30) and (31), we note that $m_{\rm curr}(p^2)$ and $m_{\rm dyn}(p^2)$ are separately gauge-parameter-independent solutions of the pole-mass (i.e., constituent-mass) renormalization-group equation developed in the Appendix when the self-consistency requirement (31) is imposed. Consequently, the sum of these two masses in (30) is also a solution.

Given that the constituent quark masses determined by (30) and (31) are gauge independent and run in a manner consistent with the renormalization-group equations, we may express (30) in each flavor sector according to[10-12]

$$\hat{m}(p^2) = \hat{m}_{\rm curr}(p^2) + m_{\rm dyn}(p^2) \tag{32a}$$

for average nonstrange mass $\hat{m} \equiv (m_u + m_d)/2$ and

$$m_s(p^2) = m_{s,\rm curr}(p^2) + m_{\rm dyn}(p^2) . \tag{32b}$$

The nonrelativistic SU(6) quark model and, specifically, the observed baryon magnetic moments require[5]

$$\hat{m}_{\rm con} \approx 340 \text{ MeV}, \quad m_{s,\rm con} \approx 510 \text{ MeV} . \tag{33}$$

Evaluating (32a) at $p^2 = \hat{m}_{\rm con}{}^2$, (32b) at $p^2 = m_{s,\rm con}{}^2$, employing (31), and the flavor-independent dimensionful relation (21) below freeze-out (at $M \sim 1$ GeV), the ratio of (32b) to (32a) leads to the unique current quark mass ratio

$$\left[\frac{m_s}{\hat{m}}\right]_{\text{curr}} = \left[\frac{m_{s,\text{con}}{}^3 - m_{\text{dyn}}{}^3}{\hat{m}_{\text{con}}{}^3 - m_{\text{dyn}}{}^3}\right]\left[\frac{\hat{m}_{\text{con}}}{m_{s,\text{con}}}\right]^2$$
$$\approx 3.5 \text{ to } 6.8 \qquad (34)$$

for $m_{\text{dyn}} \approx 295-320$ MeV and constituent quark masses (33).

Apart from the debate between strong- and neutral-PCAC (partial conservation of axial-vector current) current quark mass ratios,[10,27] we stress that (34) is (i) freeze-out and running-mass independent since the p^2's in (32) are below ~ 1 GeV^2, and (ii) completely gauge-independent and therefore believable if $m_{\text{dyn}} \approx 310 \pm 10$ MeV. Only if $m_{\text{dyn}} \approx \hat{m}_{\text{con}}$ will the ratio in (34) approach the strong-PCAC ratio ~ 25. But then (32b) implies $m_{s,\text{curr}}$ (1 GeV^2)$\sim$370 MeV, still too high to satisfy strong-PCAC advocates. However, (34) as it stands is perfectly compatible with neutral PCAC. Whatever the outcome, we are hopeful that (32)—(34) buoyed up by the gauge independence of the quark masses will provide a tight link between relativistic QCD and the nonrelativistic quark model.

VI. CONCLUSION

In this paper, we have shown that when the current, dynamical, and constituent quark masses are anchored to their respective pole positions, then their running values remain gauge-parameter independent for covariant gauges throughout the deep-Euclidean region. This result is not surprising in that QCD parallels QED because the vector nature of the massless bosons in both theories blocks the gauge parameters from percolating into the "physical" quantities.

Apart from the gauge independence of the quark masses, we have focused on the nonperturbative mass scale of m_{dyn} as seen from three independent formal viewpoints: (i) Operator-product expansion of $\psi(x)\bar{\psi}(0)$ in coordinate space, (ii) momentum-space quark-loop graph for $\langle(\bar{q}q)_M\rangle_0$, and (iii) axial-vector Ward identity and the PCAC induced pseudoscalar pole term m_{dyn}/q^2. These three steps have their analogs in the $\pi^0 \to 2\gamma$ problem: (i) Point splitting of the axial-vector current $\bar{\psi}(x+\epsilon)\gamma_\mu\gamma_5\psi(x-\epsilon)$ in coordinate space leading to the anomalous divergence term,[28] (ii) momentum-space (quark-) loop graph for $\pi^0 \to 2\gamma$ using $g_{\pi qq}\bar{q}\gamma_5 q\pi$ leading to[29,10] $F_{\pi\gamma\gamma} = -(\alpha/\pi f_\pi)$, and (iii) axial-vector Ward identity with the anomalous divergence term and PCAC leading to[30] $F_{\pi\gamma\gamma} = -(\alpha/\pi f_\pi)$, in complete agreement with (ii) and experiment.

The parallel between m_{dyn} and $\pi^0 \to 2\gamma$ convinces us that one cannot dismiss the theoretical or phenomenological consequences of this nonperturbative flavor- (and gauge-) independent quark mass m_{dyn}. It is related to the other dynamical chiral-breakdown order parameters $\langle\bar{q}q\rangle_0$, f_π and to the QCD energy scale Λ. Its flavor-dependent partners, the nonstrange and strange constituent quark masses then provide a strong link between relativistic QCD and the nonrelativistic SU(6) quark model.

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APPENDIX: GAUGE-PARAMETER-INDEPENDENT RENORMALIZATION-GROUP EQUATION FOR THE NONPERTURBATIVE POLE MASS

Following the methodology of Ref. 15, we begin by introducing a renormalized fermion mass parameter m_R that is nonvanishing even in the limit that the bare Lagrangian mass m_0 is zero, the chiral limit. Formally, this construction is possible through the relation

$$m_R = Z_m{}^{-1}(\Lambda/\mu, g)\, m_0(\Lambda) \qquad (A1)$$

provided m_0 and Z_m approach zero at the same rate as $\Lambda \to \infty$.[31] The parameter m_R is not to be identified with the physical fermion mass m_{pole}, which is defined to be the pole of the renormalized fermion propagator $S(p)$. In order to develop a renormalization-group equation for m_{pole}, we note that the unrenormalized fermion propagator is subtraction-point independent:

$$0 = \mu\frac{d}{d\mu}[Z_2{}^{-1}(\Lambda/\mu, g_{un}, a_{un})S^{-1}(p,\mu,m_R,g_R,a_R)] .$$

One can obtain the renormalization-group equations given in Ref. 14 for the dimensionless coefficients F and G defined by

$$S^{-1} = \not{p}F(p,\mu,m_R,g_R,a_R) - m_R G(p,\mu,m_R,g_R,a_R) ,$$

provided rescalings of the momentum are consistent with the absence of canonical mass dimensions in F and G. In other words, we demand that

$$A(\kappa p,\mu,m_R,g_R,a_R) = A(\kappa p/\mu, m_R/\mu, g_R, a_R)$$

for $A = F, G$, in order to find that

$$\mu\frac{\partial}{\partial\mu}A(\kappa p,\dots) = -\kappa\frac{\partial}{\partial\kappa}A(\kappa p,\dots) - m_R\frac{\partial}{\partial m_R}A(\kappa p,\dots) . \qquad (A2)$$

Our concern, however, is with the propagator pole, which is $m_R G(p)F^{-1}(p)$ evaluated when p^2 is equal to $(m_{\text{pole}})^2$. It is precisely this self-consistency condition that ensures gauge-parameter independence of the pole mass in Sec. III. Define a quantity

$$m_{\text{pole}}(\kappa p, m_R, g_R, a_R, \mu) = m_R G(\kappa p,\dots)F^{-1}(\kappa p,\dots) = m_R\xi(\kappa p,\dots) . \qquad (A3)$$

The dimensionless quantity $\xi(\kappa p,\dots)$ satisfies the renormalization-group equation

$$\left[-\kappa\frac{\partial}{\partial\kappa}+\beta(g)\frac{\partial}{\partial g_R}-[1-\gamma_m(g_R)]m_R\frac{\partial}{\partial m_R}+\gamma_m\right]$$
$$\times\xi(\kappa p,\dots)=-\delta(a_R,g_R)\frac{\partial}{\partial a_R}\xi(\kappa p,\dots)\ , \qquad (A4)$$

where

$$\beta(g_R)=\mu\frac{\partial}{\partial\mu}g_R(\Lambda/\mu,g_{\rm un})\,|_{\Lambda,g_{\rm un}}$$

is negative, and where γ_m, defined by

$$m_R\gamma_m(g_R)=\mu\frac{\partial}{\partial\mu}m_R(\Lambda,\mu)$$
$$=m_R(\Lambda,\mu)\Lambda\frac{\partial}{\partial\Lambda}\ln[Z_m(\Lambda/\mu)]\ ,$$

is also negative $[\gamma_m(g)=-g^2/2\pi^2]$. Gauge-parameter-dependent wave-function anomalous dimensions

$$\gamma(g)\equiv\mu Z_2{}^{-1}(\Lambda/\mu)(\partial Z_2/\partial\mu)$$

appearing in renormalization-group equations for F and G do not enter into the equation for ξ. From Sec. III, we know that the explicit gauge-parameter dependence of $\xi(\kappa p,\dots,a_R)$ will vanish provided $\kappa=m_{\rm pole}/p$. Hence, if this self-consistency requirement is met, the right-hand side of (A4) will vanish. Formally, $m_{\rm pole}(m_{\rm pole})=m_R\xi(\kappa)$, where $\xi(\kappa)$ is the solution to the gauge-parameter-independent renormalization-group equation

$$\left[-\kappa\frac{\partial}{\partial\kappa}+\beta(g)\frac{\partial}{\partial g_R}-[1-\gamma_m(g_R)]m_R\frac{\partial}{\partial m_R}+\gamma_m\right]$$
$$\times\xi(\kappa,\dots)=0\ , \qquad (A5)$$

with solution

$$\xi(\kappa,g_R,m_R,\mu)=\xi(1,g(\kappa,g_R),m(\kappa,m_R))$$
$$\times\exp\left[\int_{g_R}^{g(\kappa,g_R)}\frac{\gamma_m(g')dg'}{\beta(g')}\right]. \qquad (A6)$$

Note that all p dependence of ξ must run through κ such that $\kappa\sim p^{-1}$. Running values of $g(\kappa,g_R)$, $m(\kappa,m_R)$ are solutions to the homogeneous renormalization-group equation

$$\left[-\kappa\frac{\partial}{\partial\kappa}+\beta(g_R)\frac{\partial}{\partial g_R}-[1-\gamma_m(g_R)]m_R\frac{\partial}{\partial m_R}\right]$$
$$\times\begin{bmatrix}g(\kappa,g_R)\\ m(\kappa,m_R)\end{bmatrix}=0\ , \qquad (A7)$$

such that $g(1,g_R)=g_R$, $m(1,m_R)=m_R$. These solution properties are satisfied provided

$$\frac{d\kappa'}{\kappa'}=\frac{dg(\kappa',g_R)}{\beta[g(\kappa',g_R)]}=\frac{dg'}{\beta(g')}\ , \qquad (A8)$$

$$m(\kappa,m_R)=\frac{m_R}{\kappa}\exp\left[\int_1^{\kappa}\gamma_m(g(\kappa',g_R))\frac{d\kappa'}{\kappa'}\right]$$
$$=\frac{m_R}{\kappa}\left[\frac{g^{-2}(\kappa,g_R)}{g_R{}^2}\right]^d. \qquad (A9)$$

We note that the solutions (A6) contain an arbitrary function of $g(\kappa,g_R)$ and $m(\kappa,m_R)$. Such solutions are *not,* however, arbitrary functions of p when p dependence is determined by the $\kappa p=m_{\rm pole}$ self-consistency condition. For example, a solution for the pole mass that has no dependence on powers of p is obtained from (A6) by defining

$$\xi(1,g(\kappa,g_R),m(\kappa,m_R))=N\ ,$$

where N is some arbitrary constant. We see that no powers of κ are present in the solution for $\xi(\kappa)$, leading to the result

$$m_{\rm pole}(\kappa p=m_{\rm pole})=Nm_R\left[\frac{g^2(\kappa,g_R)}{g_R{}^2}\right]^d. \qquad (A10)$$

This solution corresponds to the current quark mass. Of course, $g(\kappa,g_R)$ can enter any solution of $\xi(\kappa)$ to arbitrary powers, as $g(\kappa,g_R)$ is a solution of (A7), the homogeneous renormalization-group equation. The true significance of (A9) is that the power of $g_R{}^2$, the unrescaled coupling, is not arbitrary; $g_R{}^{-2d}$ dependence follows from insisting that $\xi(\kappa)$ not depend on powers of κ, which are inverse powers of p^{-1}. Indeed, this $g_R{}^{-2d}$ dependence corresponds to the $[g(\mu)]^{-2d}$ dependence occurring within (9a), as $g(\mu)$ is the unrescaled coupling in (9a).

A solution for $m_{\rm pole}(\kappa p=m_{\rm pole})$ that goes like p^{-2} (hence, like κ^2) is obtained by having

$$\xi(1,g(\kappa,g_R),m(\kappa,m_R))\sim[m(\kappa,m_R)]^{-2}\ .$$

Then we see from (A6) and (A9) that

$$\xi(\kappa)\sim\kappa^2[g^2(\kappa,g_R)/g_R{}^2]^{-d}\ , \qquad (A11)$$

a solution which correlates p^{-2} dependence of the dynamical pole mass with $g_R{}^{2d}$ dependence of the unrescaled coupling. Such a correlation is manifest in the $[g(M)]^{2d}$ dependence of the unrescaled coupling occurring in (17).

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