Kaon decays and a determination of the scale of chiral symmetry

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Using effective chiral Lagrangians, we consider the occurrence of quadratic terms in the $\Delta I=\frac{1}{2}$ $K\to 3\pi$ matrix element. The inclusion of higher-derivative Lagrangians leads to a significant improvement over the usual current-algebra analysis. This allows an extraction directly from experimental data of Λ_χ, a measure of the scale of chiral symmetry. We give an operational definition of the chiral scale, and find $\Lambda_\chi \approx 1$ GeV. The issue of uncertainty in the phenomenological determination of the B parameter is clarified.

I. INTRODUCTION

One of the most important features of the strong interactions is the existence of an approximate, dynamical chiral symmetry.[1] All indications are that the basic features of chiral invariance emerge from quantum chromodynamics (QCD) with the only explicit symmetry breaking being due to the (small) quark masses. This means that very-low-energy physics is described in terms of the pseudo-Goldstone bosons of the theory, pions and kaons, with interactions being strongly constrained by the chiral structure.[2] Modifications to the low-energy predictions of chiral symmetry are suppressed by factors of q^2/Λ_χ^2, where Λ_χ is a scale parameter which indicates the energy above which the predictions must be modified. We shall call Λ_χ "the scale of chiral symmetry" and, in this paper, determine it from data on nonleptonic kaon decays.

Internal to the low-energy dynamics of pions and kaons, the most important dimensional parameter is $F_\pi=0.094$ GeV. If this were also the scale Λ_χ, then the predictions of chiral symmetry would be worthless. Conversely, the success of current algebra and PCAC[3] (partial conservation of axial-vector current) indicates that the scale must be rather larger, at least greater than m_K. Indeed, it has been proposed theoretically that $\Lambda_\chi \simeq 4\pi F_\pi$ (Ref. 4).

Kaon decays can provide an *empirical* estimate of Λ_χ. This occurs because the lowest-order prediction of chiral symmetry allows constant plus linear terms in the amplitude for $K\to 3\pi$, but forbids quadratic contributions.[5] (The reader unfamiliar with the terminology will find it defined in Sec. III.) Higher-order terms can be systematically studied via the effective-Lagrangian technique and can introduce quadratic dependence in the matrix element. This quadratic dependence can be used to extract the strength of the higher-order Lagrangians, which may in turn be converted into an estimate of scale Λ_χ.

Specifically, there are three aspects of this study on which we will focus. At the most superficial level, we are updating the phenomenological analyses of the kaon system. In the late 1960's, chiral symmetry was applied to kaon decays, using an assumed linear momentum dependence for the $K\to 3\pi$ amplitudes.[6] There is a longstanding ($\sim$20%) discrepancy between the chiral-symmetry prediction for the $K\to 3\pi$ amplitude and its experimental value. In addition, newer, more precise experiments have confirmed quadratic momentum dependence in the $K\to 3\pi$ amplitude.[7] Here we demonstrate that higher-order effective chiral Lagrangians can provide an explanation of *both* of these features, allowing a description of the amplitudes good to about 10%. At a slightly deeper level, the result so obtained reveals something about chiral symmetry itself. The quadratic terms which we find are relatively small. Barring accidental cancellations, this implies that the scale parameter of chiral symmetry must be somewhat above m_K. As we argue explicitly later in the paper, there is no *unique* way to characterize this scale. However, we shall provide a definition of Λ_χ which measures the *effect* of higher-order terms in physical amplitudes. While not unambiguous, this does provide a new and independent probe of the chiral scale. Finally, a third aspect of interest which emerges from this study is an estimate of the chiral-symmetry uncertainties in the calculation of the so-called B parameter, a low-energy matrix element occurring in the $K^0\bar{K}^0$ system. As defined and discussed below, the B parameter is not directly probed by the experimental data of this study. However, its determination does rely on linear momentum dependence in kaon decay amplitudes, which is the subject of this paper. Again barring accidental cancellations, the demonstration that quadratic effects are small should plausibly apply to it also.

We begin our discussion by noting that the lowest-order prediction of chiral symmetry for the $\Delta I=\frac{1}{2}$, $K_L\to\pi^+\pi^-\pi^0$ amplitude is

$$A^{\text{linear}}=A_0\left[1+3\left[\frac{m_\pi}{m_K}\right]^2 Y\right]=(0.75+0.18Y)\times 10^{-6}\ , \tag{1}$$

where we use the variables

$$Y=(s_3-s_0)/m_{\pi^+}{}^2\ ,\qquad X=(s_2-s_1)/m_{\pi^+}{}^2\ , \tag{2}$$

and

$$s_i=(p_K-p_i)^2,\ \ s_0=\tfrac{1}{3}(m_K{}^2+m_1{}^2+m_2{}^2+m_3{}^2)\ .$$

A_0 is predicted in terms of the $\Delta I = 1/2$ amplitude for $K_S \to \pi^+\pi^-$ by[8]

$$A_0 = -\frac{i}{6F_\pi}\text{Amp}^{(1/2)}(K_S \to \pi^+\pi^-)\frac{m_K{}^2}{m_K{}^2 - m_\pi{}^2}$$
$$= 0.75 \times 10^{-6} . \qquad (3)$$

The above amplitude is an example of one with constant plus linear (in Y) dependence. The $\Delta I = \frac{1}{2}$ rule can be used to obtain all the other $K \to 3\pi$ amplitudes.

By means of a study of effective Lagrangians in Sec. III, we shall demonstrate that the only possible forms for higher-order terms are

$$A^{\text{quad}} = a_1[\tfrac{2}{9}m_K{}^2(m_K{}^2 - 3m_\pi{}^2) + \tfrac{1}{3}m_\pi{}^2(m_K{}^2 + 3m_\pi{}^2)Y - m_\pi{}^4 Y^2] + a_2[\tfrac{4}{9}m_K{}^2(m_K{}^2 - 3m_\pi{}^2) - \tfrac{1}{3}m_\pi{}^2(m_K{}^2 + 3m_\pi{}^2)Y + \tfrac{1}{2}m_\pi{}^4(Y^2 + X^2)] . \qquad (4)$$

These two amplitudes have quadratic dependence (in X and Y), in addition to constant and linear terms. The data indicates the presence of *each* type of term[9]

$$A^{\text{expt}} = [(0.915 \pm 0.0024) + (0.258 \pm 0.004)Y - (0.0037 \pm 0.0011)(Y^2 + X^2/3) - (0.0125 \pm 0.0012)(Y^2 - X^2/3)] \times 10^{-6} . \qquad (5)$$

The quadratic terms appear small, but in fact are quite sizable [note the attendant factors of $m_\pi{}^4$ in Eq. (4)]. If we fit a_1 and a_2 to the measured quadratic pieces in Eq. (5) (thus yielding $a_1 = 3.1 \times 10^{-6}/m_K{}^4$, $a_c = -0.95 \times 10^{-6}/m_K{}^4$), we determine

$$A^{\text{quad}} = [(0.20 \pm 0.07) + (0.13 \pm 0.01)Y - (0.0037 \pm 0.0011)(Y^2 + \tfrac{1}{3}X^2) - (0.0125 \pm 0.0012(Y^2 - \tfrac{1}{3}X^2)] \times 10^{-6} . \qquad (6)$$

We can then use this amplitude to form a definition of the chiral scale Λ_χ by considering the ratio of quadratic to linear amplitudes and, since there are two additional powers of momentum, relating this to the ratio of the overall mass scale $m_K{}^2$ and the square of the chiral scale

$$\frac{m_K{}^2}{\Lambda_\chi{}^2} \equiv \frac{A^{\text{quad}}}{A^{\text{linear}}} . \qquad (7)$$

This ratio of the quadratic amplitude $A^{\text{quad}}(X,Y)$ to the linear amplitude $A^{\text{linear}}(X,Y)$ is essentially uniform over the Dalitz plot, and yields the scale parameter

$$\Lambda_\chi = 970 \pm 140 \text{ MeV} , \qquad (8)$$

where the error is only the experimental error on the quadratic terms. A second (independent) procedure, involving a least-squares fit to the entire amplitude with an assumed 10% theoretical uncertainty, yields a good fit ($\chi^2 = 1.8$ for 2 DOF) with

$$\Lambda_\chi = 993 \pm 50 \text{ MeV} . \qquad (9)$$

Here the quoted error is due to the 10% uncertainty assumed. Here the uncertainties quoted are merely statistical and probably should be taken "cum grano solis." Since there are two independent amplitudes with opposite sign coefficients, there is clearly some sort of cancellation taking place. Nevertheless, we feel that these values of Λ_χ do provide a reasonable (though rough) estimate of the chiral scale.

In the remainder of the paper we present in detail the analysis which has been sketched above. Section II is devoted to a discussion of the effective chiral Lagrangians. In Sec. III we consider evaluation of the kaon decay rates and compare them with the data. Section IV contains further discussion and a summary.

II. EFFECTIVE CHIRAL LAGRANGIANS

One of the most efficient ways to obtain the predictions of chiral symmetry is to use nonlinear effective Lagrangians. For example, in the chiral limit the strong interactions of pions and kaons can be described by the Lagrangian[10]

$$L_{\text{strong}} = \frac{F_\pi{}^2}{4}\text{Tr}(\partial_\mu M \partial^\mu M^\dagger) , \qquad (10)$$

where

$$M = \exp(i\vec{\lambda}\cdot\vec{\phi}/F_\pi) \qquad (11)$$

with $F_\pi = 94$ MeV and $\text{Tr}\lambda^A\lambda^B = 2\delta^{AB}$. Under left-handed and right-handed SU(3) chiral transformations M transforms as

$$M \to U_L M U_R{}^{-1} , \qquad (12)$$

where U_L (U_R) is a 3×3 unitary matrix describing the left (right) transformation. Expansion of the exponentials to order ϕ^4 yields the usual kinetic energy piece plus Weinberg's results for $\pi\pi$ scattering [and its SU(3) generalizations].[11] The masses of the pseudoscalars can be incorporated by an explicit chiral symmetry breaking term

$$L_{\text{mass}} = \text{Tr}(mM + M^\dagger m) \qquad (13)$$

with $m = a + b\lambda_3 + c\lambda_8$.

In principle, there may also be chirally invariant terms with more derivative factors, for example,

$$L'_{\text{strong}} = \left[\frac{F_\pi}{2\Lambda}\right]^2 \text{Tr}([\partial_\mu M, \partial^\nu M^\dagger][\partial_\mu M, \partial^\nu M^\dagger]) . \qquad (14)$$

However, at low enough energies ($q^2/\Lambda^2 \ll 1$) these terms are always unimportant. The success of the Weinberg predictions for $\pi\pi$ scattering requires that Λ be much larger than m_π. Note that these higher-derivative terms are required in the (still speculative) soliton or Skyrme models of baryons.[12] One explicit example which has been worked out[13] uses the above Lagrangian with $\Lambda \simeq 1$ GeV.

The weak interactions in the standard model involve only left-handed fields and hence are a singlet under right-handed transformations. For the dominant $\Delta I = \frac{1}{2}$ piece, the left-handed transformation property must be that of an octet, i.e., (8,1) under (L,R) transformations.

The lowest-order Lagrangian with this feature was written down long ago by Cronin:[14]

$$L_0 = g\,\mathrm{Tr}(\lambda_6 \partial_\mu M \partial^\mu M^\dagger) \ . \tag{15}$$

Any possible $(3,\bar{3})$ terms, induced by explicit breaking from the quark masses, can be removed entirely by a diagonalization of the mass terms in the Lagrangian.[15] This Lagrangian incorporates the unique form of the $K \to \pi$, $K \to 2\pi$, and $K \to 3\pi$ amplitudes which results when one expands them to first order in bilinear momentum factors (i.e., $P_i \cdot P_j$), demands consistency in all soft-pion limits, and imposes the theorem which requires the $K \to 2\pi$ amplitudes to vanish in the SU(3) limit.[16] It has been used successfully to relate $K \to 2\pi$ and $K \to 3\pi$ amplitudes.[17]

Again, other effective Lagrangians with more derivative factors, but respecting the (8,1) transformation property, can be constructed. For example, one of these is

$$L_w' = \frac{g}{\Lambda^2}\,\mathrm{Tr}(\lambda_6 \partial_\mu M \partial^\mu M^\dagger \partial_\nu M \partial^\nu M^\dagger) \ . \tag{16}$$

Here Λ must be large enough not to destroy the relation between $K \to 3\pi$ and $K \to 2\pi$. (Note that, having four derivatives, L_w' contributes to $K \to 3\pi$ but vanishes for $K \to 2\pi$ and therefore cannot reproduce the same results as L_0.) There are, in addition, many other possible forms of these higher-order terms. The remainder of this section is devoted to classifying these possibilities and reducing them to a manageable number by the use of various identities.

It is most convenient to work with objects which are automatically singlets under right-handed transformations. For example, the current

$$X_\mu = (\partial_\mu M) M^\dagger \tag{17}$$

transforms as

$$X_\mu \to U_L X_\mu U_L^{-1} \tag{18}$$

under left and right transformations. Note that since $MM^\dagger = 1$,

$$X_\mu^\dagger = -X_\mu \ . \tag{19}$$

Higher-order Lagrangians can be obtained by stringing together four factors of X_μ, but they can also involve objects containing more derivatives, such as

$$Y_{\mu\nu} = (\partial_\mu \partial_\nu M) M^\dagger \tag{20}$$

and

$$Z_{\mu\nu\lambda} = (\partial_\mu \partial_\nu \partial_\lambda M) M^\dagger \ . \tag{21}$$

In this case, $M^\dagger M = 1$ implies that

$$Y_{\mu\nu} + Y_{\mu\nu}^\dagger = -(X_\mu X_\nu^\dagger + X_\nu X_\mu^\dagger) \ . \tag{22}$$

It is advantageous to use the CP properties of the currents in order to simplify matters. Under the CP operation,

$$X_\mu \to -(X_\mu)^T \ , \quad Y_{\mu\nu} \to (Y_{\mu\nu}^\dagger)^T \ . \tag{23}$$

Let us construct a combination of Y and $Y^\dagger$ which has *definite CP* properties. To this end, define

$$\tilde{Y}_{\mu\nu} = Y_{\mu\nu} - Y_{\mu\nu}^\dagger \ , \quad \tilde{Y}_{\mu\nu}^\dagger = -\tilde{Y}_{\mu\nu} \ , \tag{24}$$

which transforms under CP as

$$\tilde{Y}_{\mu\nu} \to -\tilde{Y}_{\mu\nu}^T \ . \tag{25}$$

The combination which is the sum of Y and $Y^\dagger$ and can be written in terms of X_μ, so we need not consider it separately. We will use the shorthand notation

$$\tilde{Y} \equiv g^{\mu\nu} \tilde{Y}_{\mu\nu} \ . \tag{26}$$

Quantities involving $Z_{\mu\nu\lambda}$, such as

$$\mathrm{Tr}(\lambda_6 Z_{\mu\nu\lambda} X^{\dagger\mu} g^{\nu\lambda}) \ , \tag{27}$$

can always be rewritten in terms of X and $\tilde{Y}$ by integration by parts to shift one factor of the derivative from Z to elsewhere in the expression.

There is a useful partition of the Lagrangian into classes containing two, one, or no factors of $\tilde{Y}_{\mu\nu}$. The physics of each class is distinct. Those with two factors of $\tilde{Y}$ can contribute to all of $K \to \pi$, $K \to 2\pi$, and $K \to 3\pi$. Those with one factor of $\tilde{Y}$, and hence two of X_μ, have no $K \to \pi$ matrix element, but do contribute to $K \to 2\pi$ and $K \to 3\pi$. Finally, those with no factors of $\tilde{Y}$, and hence four factors of X_μ, contribute only to $K \to 3\pi$.

To demonstrate how the analysis proceeds, let us consider a set of Lagrangians from the first class, e.g.,

$$\begin{aligned} L_1 &= \mathrm{Tr}(\lambda_6 \tilde{Y}\tilde{Y}) \ , \\ L_1' &= \mathrm{Tr}(\lambda_6 \tilde{Y}_{\mu\nu} \tilde{Y}^{\mu\nu}) \ , \\ L_1'' &= \mathrm{Tr}(\lambda_6 \tilde{Y}) \mathrm{Tr}(\tilde{Y}) \ , \\ L_1''' &= \mathrm{Tr}(\lambda_6 \tilde{Y}_{\mu\nu}) \mathrm{Tr}(\tilde{Y}^{\mu\nu}) \ . \end{aligned} \tag{28}$$

However, the latter two can be shown to vanish because

$$\mathrm{Tr}\,\tilde{Y}^{\mu\nu} = 0 \ . \tag{29}$$

Also, by integrating twice by parts, L_1' can be transformed into L_1. Other terms with additional factors of λ matrices [e.g., $\mathrm{Tr}(\lambda_6 \lambda^A \tilde{Y} \lambda^A \tilde{Y})$] can be reduced to the above set by use of identities such as

$$\lambda_{ij}^A \lambda_{kl}^A = -\tfrac{2}{3}\delta_{ij}\delta_{kl} + 2\delta_{il}\delta_{kj} \ . \tag{30}$$

Thus the form L_1 uniquely characterizes this class.

In class two, we have Lagrangians such as

$$\begin{aligned} L_2 &= \mathrm{Tr}([\lambda_6, \tilde{Y}] X_\mu X^\mu) \ , \\ L_2' &= \mathrm{Tr}(\{\tilde{Y}, \lambda_6\} Y_\mu X^\mu) \ , \\ L_2'' &= \mathrm{Tr}(\lambda_6 X_\mu \tilde{Y} X^\mu) \ , \\ L_2''' &= \mathrm{Tr}(\lambda_6 \tilde{Y}) \mathrm{Tr}(X_\mu X^\mu) \ , \\ L_2'''' &= \mathrm{Tr}(\lambda_6 X_\mu X^\mu) \mathrm{Tr}\,\tilde{Y} \ , \\ L_2''''' &= \mathrm{Tr}(\lambda_6 X_\mu \tilde{Y}) \mathrm{Tr} X^\mu \ , \end{aligned}$$

and (31)

$$L_3=\mathrm{Tr}([\lambda_6,\widetilde{Y}_{\mu\nu}]X^\mu X^\nu)\ ,$$
$$L_3'=\mathrm{Tr}(\{\lambda_6,\widetilde{Y}_{\mu\nu}\}X^\mu X^\nu)\ ,$$
$$L_3''=\mathrm{Tr}(\lambda_6 X^\mu \widetilde{Y}_{\mu\nu}X^\nu)\ ,$$
$$L_3'''=\mathrm{Tr}(\lambda_6\widetilde{Y}_{\mu\nu})\mathrm{Tr}(X^\mu Y^\nu)\ ,$$
$$L_3''''=\mathrm{Tr}(\lambda_6 X_\mu X_\nu)\mathrm{Tr}\widetilde{Y}^{\mu\nu}\ ,$$
$$L_3'''''=\mathrm{Tr}(\lambda_6 X_\mu Y^{\mu\nu})\mathrm{Tr}X_\nu\ .$$

In this case, L_2'''', L_2''''', L_3'''', and L_3''''' vanish identically since $\mathrm{Tr}\,\widetilde{Y}=\mathrm{Tr}X_\mu=0$. The requirement that the Lagrangian be *CP* even removes most of the rest; L_2', L_2'', L_2''', L_3', L_3'', and L_3''' are all *CP* odd. This leaves only L_2 and L_3 remaining in this class.

In the third class there are many possibilities and we have not identified the minimal number of independent Lagrangians. However, as we shall see in the next section, there are only two kinematic possibilities for $K\to 3\pi$ which can emerge from any Lagrangian of this class. Hence, as far as $K\to 3\pi$ is concerned, we get the maximum possible information from any two independent Lagrangians. Actually, most of what we do is not dependent of any use of specific Lagrangians from this class, but for definiteness we shall sometimes quote amplitudes resulting from the following two examples

$$\begin{aligned} L_4&=\mathrm{Tr}(\lambda_6 X_\mu X^\mu X_\nu X^\nu)\ ,\\ L_5&=\mathrm{Tr}(\lambda_6 X_\mu X_\nu X^\mu X^\nu)\ . \end{aligned} \tag{32}$$

Higher-order momentum dependence can also be generated by extra derivatives appearing in the strong-interaction sector of the theory. The analysis here is simpler than for the weak-interaction sector. In the first class one has only

$$L'_{\mathrm{strong}}=\mathrm{Tr}(\widetilde{Y}\widetilde{Y})\ , \tag{33}$$

and by parity invariance there are no possibilities in the second class (of the form *YXX*). In the third class, there are again many independent Lagrangians; however, we will not need to specify them.

To summarize, we have identified a set of effective Lagrangians (L_0,L_1,L_2,L_3,L_4,L_5) which exhaust the possibilities for the behavior of the weak interactions up to quadratic order (i.e., four factors of derivatives). In the next section we apply these to kaon decays in order to estimate the relative sizes of their coefficients, and hence determine the chiral scale.

III. COMPARISON WITH EXPERIMENT

Chiral symmetry can be used to predict the $K\to 3\pi$ decay parameters (amplitudes and slopes for both $\Delta I=\frac{1}{2}$ and $\Delta I=\frac{3}{2}$) in terms of $K\to 2\pi$ amplitudes.[17] We shall in this paper deal with only the $\Delta I=\frac{1}{2}$ effects as the data is sufficient for our purpose in this sector. The specific numbers are taken from the recent review by Devlin and Dickey (DD).[9] We follow DD in the form of our parametrization of the $\Delta I=\frac{1}{2}$ amplitude (which we take to be the $\Delta I=\frac{1}{2}$ piece of $K_L\to\pi^+\pi^-\pi^0$),

$$A=a+bY+c(Y^2+X^2/3)+d(Y^2-X^2/3) \tag{34}$$

with experiment yielding, in units of 10^{-6},[9]

$$\begin{aligned} a&=0.915\pm0.0024\ ,\\ b&=0.258\pm0.004\ ,\\ c&=-0.0037\pm0.0011\ ,\\ d&=-0.0125\pm0.0012\ . \end{aligned} \tag{35}$$

In the Particle Data Tables, the overall amplitude is divided out, and results are quoted for linear and quadratic dependence of the Dalitz plot (i.e., amplitude squared).[18] We note the clear evidence for quadratic terms.

If $c=d=0$, then PCAC requires a unique form of the amplitude[7]

$$A=a\left[1+3\left[\frac{m_\pi}{m_K}\right]^2 Y\right] \tag{36}$$

from which we directly infer

$$\frac{b}{a}=0.252\ . \tag{37}$$

This is quite close to the experimental value,

$$\frac{b}{a}=0.282\pm0.005\ . \tag{38}$$

Likewise the amplitude, as given by Eq. (3) is found to be close but about 20% too low. The uniqueness of the linear form Eq. (36) (with $c=d=0$) is important in our study of higher-order Lagrangians. If, as we shall find to be the case, some of the higher-order Lagrangians do *not* for some reason contain quadratic terms, then they *must* have the general form displayed in Eq. (36) and cannot be distinguished from the lowest-order Lagrangian. Only if "c" and/or "d" are nonzero may we isolate the effect of higher-order Lagrangians.

In calculating the $K\to 3\pi$ decay matrix elements one must include both the direct weak interaction connecting K to three final-state pions plus the pole diagrams where $K\to\pi$ is weak and there follows a strong-interaction process $\pi\to\pi\pi\pi$ or where $K\to K\pi\pi$ is followed by $K\to\pi$. For the lowest-order Lagrangian L_0, Cronin has done the calculation and obtains the required form, Eq. (36).[14] We have repeated his calculation for L_1, L_2, L_3, L_4, and L_5. In what follows we neglect terms of $O((m_\pi/m_K)^2)$. This is justified within our work because terms proportional to $m_\pi{}^2$ are too small to provide any significant comparison with experiment. In addition there is some ambiguity in $K\to 3\pi$ which enters at $O((m_\pi/m_K)^2)$ due to the particular choice of M as an exponential, Eq. (12).[14]

We find that the direct plus pole contributions to $K\to 3\pi$ sum to zero for L_1, and hence L_1 does not contribute to $K\to 2\pi$ or $K\to 3\pi$, except at $O(m_\pi{}^2)$. We find that L_2 and L_3 do not generate any quadratic terms in the amplitude, and hence must have the form of Eq. (36). We have explicitly verified this. These do not lead to any new physics and thus cannot be distinguished from L_0. In addition, we have considered what happens when, in the pole diagrams, the lowest-order strong-interaction Lagrangian is replaced by L'_{strong}. Again one obtains the linear amplitude only.

The above considerations imply that any possible higher-order behavior can originate only from effective Lagrangians (either strong or weak) which contain four factors of X_μ. Since the power series for X_μ starts off with the terms

$$X_\mu = \frac{i}{F_\pi}\partial_\mu \phi^A \lambda^A + \cdots \tag{39}$$

and $K \to 3\pi$ involves four meson fields, the resulting amplitude must contain a factor of the momentum for *each* of the four fields. There are thus only two possible combinations for $K_L \to \pi^+\pi^-\pi^0$ consistent with the symmetry of the $\pi^+\pi^-$ required by CP invariance,

$$\begin{aligned} A^{\text{quad}} = {} & 4a_1 k\cdot p_0 p_+\cdot p_- \\ & + 4a_2(k\cdot p_+ p_0\cdot p_- + k\cdot p_- p_0\cdot p_+) . \end{aligned} \tag{40}$$

As it must, A^{quad} vanishes when any of the pions is taken soft. Working out the kinematics, we obtain the result given previously in Eq. (4). As an example of how this may be generated, let us consider

$$L^{\text{quad}} = \frac{g}{\Lambda_4^2} L_4 + \frac{g}{\Lambda_5^2} L_5 \tag{41}$$

which leads to

$$\begin{aligned} a_1 &= \frac{-g}{\Lambda_4^2} - \frac{g}{\Lambda_5^2} , \\ a_2 &= \frac{-g}{\Lambda_5^2} . \end{aligned} \tag{42}$$

The above result is quite important in the consideration of quadratic terms in $K \to 3\pi$, as it expresses the PCAC constraints on these elements. If our proof were more elegant, rather than exhaustive, we would be tempted to call Eq. (40) a theorem.

It is easy to see that the addition of the quadratic terms yield a clear improvement for the predicted $K \to 3\pi$ amplitudes. For example, if we fit a_1 and a_2 to the quadratic coefficients c and d, we obtain

$$a_1 = -(3.1 \pm 0.3) \times 10^{-6} m_K^{-4} ,$$

$$a_2 = (0.95 \pm 0.16) \times 10^{-6} m_K^{-4} ,$$

and a total amplitude,

$$\begin{aligned} A^{\text{tot}} = {} & [(0.95 \pm 0.07) + (0.31 \pm 0.01) Y \\ & -(0.0037 \pm 0.0011)(Y^2 + X^2/3) \\ & -(0.0125 \pm 0.0012)(Y^2 - X^2/3)] \end{aligned} \tag{43}$$

in good agreement with experiment. Here the quoted uncertainty arises solely from the statistical errors given for the empirical coefficients c and d.

A second procedure which, in general, is independent of the first is to perform a least-squares fit to the *full* amplitude. The experimental errors on the constant plus linear terms are much smaller than the theoretical errors on the PCAC extrapolation. We have thus chosen to expand the error to 10% in order to approximate the theoretical uncertainties. In the case of c and d, this 10% was added directly on the experimental errors. The result of this fit is

$$a_1 = 2.6 \times 10^{-6} / m_K^2 ,$$

$$a_2 = -0.76 \times 10^{-6} / m_K^2$$

with the total amplitude

$$\begin{aligned} A^{\text{tot}} = {} & 0.94 + 0.29 Y - 0.034 (Y^2 + X^2/3) \\ & -0.0105 (Y^2 - X^2/3) . \end{aligned} \tag{44}$$

We see that the two procedures agree very well. From the quality of the fit ($\chi^2 = 1.78/2$ DOF) we can say that PCAC with quadratic terms agrees well with experiment at the 10% level. Thus we have shown that inclusion of higher-order chiral terms has the potential of clearing up the longstanding discrepancy between the experimentally determined $\Delta I = \frac{1}{2}$ $K \to 3\pi$ decay amplitude Eq. (5), and that predicted via PCAC/current-algebra techniques Eq. (3) and, in addition, of yielding quadratic terms in the amplitude of the size observed experimentally.

Although one could terminate the discussion here with this rather successful phenomenology, it is tempting to attempt to use this result in order to draw conclusions concerning the chiral scale Λ_χ. What we desire is an answer to the somewhat vague question: How large are the effects of higher-order (in the particle momentum) terms in the chiral expansion? Using the coefficients of phenomenological Lagrangians is ambiguous, as we show below. Thus, we feel that the best measure of the size of higher-order effects is obtained by calculating how much of the full amplitude comes from lowest order and how much arises from higher order. When combined with the characteristic scale of the process (m_K), this may be converted into a scale Λ_χ. We therefore *define*

$$(m_K/\Lambda_\chi)^2 = A^{\text{quad}} / A^{\text{linear}}$$

and we find

$$\Lambda_\chi = 970 \pm 140 \text{ MeV} \quad \text{(fit to quadratic dependence)}$$

$$\Lambda_\chi = 993 \pm 50 \text{ MeV} \quad \text{(fit to the full amplitude)} .$$

Another way to quantify the results is to say that the higher-order effects enter at the 25% level in kaon decays.

We feel that a definition of Λ_χ such as given in Eq. (7) is perhaps superior to employing the phenomenological Lagrangians L_4, L_5 directly. If we were to pursue the latter course, it is straightforward to deduce that

$$L_{\Delta I = 1/2} = 3.6 \times 10^{-8} m_\pi^2 \left[\text{Tr}\lambda_6 X_\mu X^\mu + \frac{\text{Tr}\lambda_6 X_\mu X^\mu X_\nu X^\nu}{(0.51 \text{ GeV})^2} - \frac{\text{Tr}\lambda_6 X_\mu X_\nu X^\mu X^\nu}{(0.76 \text{ GeV})^2} \right] . \tag{45}$$

Clearly, the "scale factors" appearing in this approach, although still larger than the kaon mass, are rather below 1 GeV. The problem with investing such an estimate with undue significance is that the definition and normalization of operators appearing in phenomenological Lagrangians is somewhat arbitrary. For example, instead of the operators L_4, L_5 of Eq. (32), one could choose a different set and obtain which appear quite different, such as

$$L_{\Delta I=1/2}=3.6\times 10^{-8} m_\pi^2 \left[\mathrm{Tr}\lambda_6 X_\mu X^\mu + \frac{\mathrm{Tr}(\lambda_6 [X_\mu, X_\nu][X^\mu, X^\nu])}{(0.84\ \mathrm{GeV})^2} + \frac{\mathrm{Tr}(\lambda_6 \{X_\mu, X_\nu\}\{X^\mu, X^\nu\})}{(1.2\ \mathrm{GeV})^2} \right].$$

An additional ambiguity occurs because of cancellation between the two amplitudes a_1, a_2. Since these coefficients have opposite sign, there is considerable cancellation which takes place in the overall decay amplitude, less so in the case of the quadratic terms. For both these reasons then, it seems reasonable to utilize the experimental decay amplitude directly in estimating chiral scale Λ_χ.

IV. CONCLUSIONS

We can summarize our results as follows. For kaon decay amplitudes, the constraints of current algebra and PCAC involve rather substantial extrapolations away from the physical region. It is insufficient to posit that momentum dependence is absent from the decay amplitude; otherwise contradictions occur between alternative soft-pion limits. In fact, given the chiral $(8_L, 1_R)$ content of the underlying effective Lagrangian, the constant term, in a power-series expansion of the amplitude in the particle momenta, is absent. The next order, quadratic in momenta, turns out to provide a reasonable though imperfect description of the decay data. In this paper we have shown that the contributions which are *quartic* in momenta can be inferred from the X^2, Y^2 terms in the measured $K\to 3\pi$ amplitudes. Moreover, we argue that it is natural to interpret the relative size of the leading and next-to-leading contributions in terms of a scale parameter Λ_χ as in Eq. (7),

$$m_K^2/\Lambda_\chi^2 \equiv A^{\mathrm{quad}}/A^{\mathrm{linear}} .$$

Our two determinations yield $\Lambda_\chi = 0.97$ GeV (fit to quadratic terms only) or $\Lambda_\chi = 0.99$ GeV (fit to full amplitude). In either case, the rather substantial value of Λ_χ suggests why the leading order in amplitudes describing low-energy phenomena such as kaon decay accords a reasonable model of the associated data. (We note that the empirical scale Λ_χ is qualitatively in accord with the theoretical value $4\pi F_\pi \simeq 1$ GeV of Ref. 4.)

What can be said about nonleptonic transition operators with chiral transformation properties distinct from the $(8_L, 1_R)$ studied here? The presence of weak $\Delta I = \frac{3}{2}$ effects and also electromagnetism implies the existence of operators transforming according to $(27_L, 1_R)$ and $(8_L, 8_R)$ chiral representations.[19] In leading order, the $(27_L, 1_R)$ effective Lagrangian has two derivatives

$$L^{(27,1)} \sim C \begin{bmatrix} 8 & 8 & 27 \\ i & j & 6 \end{bmatrix} \mathrm{Tr}\lambda_i X_\mu \lambda_j X^{\mu\dagger} , \qquad (46)$$

where C is a Clebsch-Gordan coefficient and indices i,j are summed. As with $(8_L, 1_R)$ amplitudes, the leading terms are quadratic in particle momenta. The (8,8) operator requires no derivatives in leading order

$$L^{(8,8)} \sim \mathrm{Tr}\lambda_6 M \lambda_3 M^\dagger . \qquad (47)$$

Finally, let us consider how our calculation clarifies the issue of chiral uncertainty in a phenomenological determination of the B parameter.[20] The B parameter is a dimensionless quantity appearing in the K^0-$\bar{K}^0$ complex[21] and is defined by

$$B = \frac{\langle K^0 | O | \bar{K}^0 \rangle}{\frac{16}{3} F_K^2 M_K^2} , \qquad (48)$$

where O is the chiral four-quark operator $\bar{d}\Gamma_L^\mu s\, \bar{d}\Gamma_{L\mu} s$ $[\Gamma_L^\mu \equiv \gamma^\mu(1+\gamma^5)]$. There has recently been substantial interest in the value of B due to its importance to estimates which bound the t-quark mass.[22] Unfortunately, B is sensitive to details of quark wave functions so its theoretical determination is subject to the usual difficulties of strong-interaction dynamics. Its extraction from experimental data therefore takes on added significance. The basis for a phenomenological determination is the observation that the $\Delta S = 2$ operator O belongs to the same 27-plet as the $\Delta I = \frac{3}{2}$ operator which induces the weak decay $K^- \to \pi^0 \pi^-$.[20] An intermediate step in the analysis involves the reduction: $K^-\pi^0\pi^-$ amplitude $\to \bar{K}^0\pi^0$ amplitude. This is accomplished with current algebra and PCAC. In doing so, a quadratic momentum dependence was assumed for the $\Delta I = \frac{3}{2}$ $K\pi\pi$ amplitude[20]

$$\langle \pi^-(q_-)\pi^0(q_0) | H_w^{\Delta I = 3/2} | K^-(k) \rangle = iA(3k^2 + q_0^2 - 4q_-^2) , \qquad (49)$$

where $A = 1.2\times 10^{-8} m_K^{-1}$ is inferred from the $K^-\pi^0\pi^-$ decay rate with $\pi^0\eta$ mixing subtracted out. The question of a "chiral uncertainty" in this procedure pertains to whether the employment of quadratic momentum dependence as in Eq. (49) is reasonable. Our conclusion, based on the study of the $\Delta I = \frac{1}{2}$ $K\pi\pi\pi$ amplitude described in this paper, is that a good first approximation valid to about 25%, is indeed given by Eq. (49).

Of course, the quantities germane to the B parameter are $\Delta I = \frac{3}{2}$ and $K\pi\pi$ instead of $\Delta I = \frac{1}{2}$ and $K\pi\pi\pi$. In spite of this, we feel relatively secure in claiming that chiral uncertainty in the phenomenological determination of B is not qualitatively substantial. Surely, one would expect that momentum dependence in the exotic $\Delta I = \frac{3}{2}$ channel is, if anything, modest compared to that in the $\Delta I = \frac{1}{2}$ channel. On a dynamical basis, there is no reason to suspect rapid momentum variation (as, for example, due to low-lying resonant states) for $\Delta I = \frac{3}{2}$. The use of chiral symmetry is one of the most reliable tools in low-energy dynamics and the data studied here does not give any reason to doubt its applicability in the determination of the B parameter.

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$$L \sim \mathrm{Tr}\lambda_6 \partial_\mu M \partial_\mu M^\dagger ,$$

where M is a matrix which transforms as $U_L M U_R{}^{-1}$ under chiral $SU(3)\otimes SU(3)_R$.

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