Backward-moving Altarelli-Parisi equations for transverse-momentum calculations in jets

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We develop equations for various jet-calculus propagators in which the mass of partons "evolves down;" these are in some sense just the inverse of the normal Altarelli-Parisi equations in which the jet mass "evolves up." The new "backward-moving" equations are preferable for calculations of transverse-momentum distributions because they have a simpler separation of the transverse and longitudinal momentum variables.

I. INTRODUCTION

Ever since Altarelli and Parisi[1] wrote differential equations expressing the QCD evolution of form factors in simple terms, the technique has been widely extended and applied to predictions of properties of quark and gluon jets. For example, the jet calculus of Konishi, Ukawa, and Veneziano uses[2] concatenations of basic "parton propagators," $D_j^i(Q^2,Q_0{}^2;x)$. D_j^i gives the probability that parton i of mass Q_0 will be found in a jet coming from parton j with mass Q. It satisfies the Altarelli-Parisi equation

$$\frac{d}{dY}D_j^i(Q^2,Q_0{}^2;x)=\sum_c \int_x^1 \frac{dz}{z}D_c^i\left[Q^2,Q_0{}^2;\frac{x}{z}\right]P_{cj}(z)\ , \tag{1.1}$$

with

$$Y=\frac{1}{2\pi b}\ln\left[\frac{\alpha_s(\mu^2)}{\alpha_s(Q^2)}\right]$$

and ($12\pi b=11N_c-2N_f$)

$$\alpha_s(Q^2)=\frac{1}{b\ln(Q^2/\Lambda^2)}\ .$$

Equation (1.1) is most simply solved by taking moments in longitudinal momentum

$$\tilde{D}_j^i(Q^2,Q_0{}^2;n)=\int_0^1 x^n D_j^i(Q^2,Q_0{}^2;x)dx\ . \tag{1.2}$$

The equation then has solutions of the form

$$\tilde{D}_j^i(Q^2,Q_0{}^2;n)=(e^{A_n(Y-Y_0)})_{ij} \tag{1.3}$$

with

$$(A_n)_{ij}=\int_0^1 z^n P_{ij}(z)dz\ .$$

Note that because of the very simple form of these equations, the solution also obeys the "backward-moving" equation

$$-\frac{d}{dY_0}D_j^i(Q^2,Q_0{}^2;x)=\sum_c \int_x^1 \frac{dz}{z}D_j^c\left[Q^2,Q_0{}^2;\frac{x}{z}\right]P_{ic}(z)\ . \tag{1.4}$$

We can think of the jet, beginning at Q, successively branching and branching until the parton under observation at the end has mass Q_0.

Because D_j^i is a totally inclusive distribution, the other partons in the jet (those not being enumerated in D_j^i) may evolve as they like. Normally, one speaks of them as though they also have mass Q_0; however, because this function sums over all possible final states, and because the probability that they do *something* is 1, they may in fact be at any stage of their evolution.

In order to study colorless clusters, Bassetto, Ciafaloni, and Marchesini[3] (BCM) made further refinements to jet-calculus ideas. They defined some partially inclusive distributions; we will discuss these in more detail below. They also extended Eq. (1.1) to include the transverse momentum of the observed parton.

If we wish to write the generalization of Eq. (1.1) including transverse momentum, we must note the kinematics of the situation, as described in Fig. 1. Because the jet axis turns as more vertices are added at the large-Q^2 end, momentum *transverse to the jet axis* will contain factors of x, the longitudinal-momentum fraction. Specifically, as given by BCM in Ref. 3(a), Eq. (2.12), the nonsinglet distribution obeys the following equation:

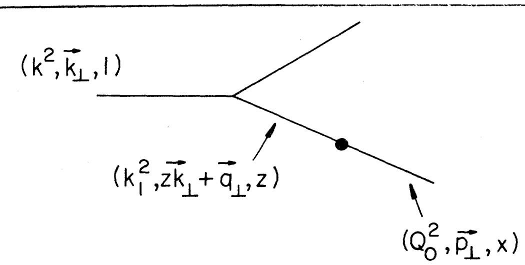

FIG. 1. Mass^2, transverse momentum, and longitudinal fraction of partons in Eq. (1.5), computed relative to *fixed* axes. The transverse momentum of the final parton *relative to the initial parton of mass* k^2 is thus $\vec{p}_\perp-x\vec{k}_\perp$; the momentum of this same final *parton relative to the jet axis after the branch* is

$$\vec{p}_\perp-\frac{x}{z}(z\vec{k}_\perp+\vec{q}_\perp)=\vec{p}_\perp-x\vec{k}_\perp-\frac{x}{z}\vec{q}_\perp\ .$$

$$k^2\frac{d}{dk^2}\mathscr{D}^{\rm NS}(k^2,Q_0{}^2;\vec{p}_\perp - x\vec{k}_\perp,x)=\frac{\alpha(k^2)}{2\pi}\int_x^1\frac{dz}{z}C_F\left[\frac{1+z^2}{1-z}\right]_+\int\frac{d^2\vec{q}_\perp}{\pi}\delta(z(1-z)k^2-Q_0{}^2-q_\perp{}^2)$$
$$\times\mathscr{D}^{\rm NS}\left[k^2,Q_0{}^2;\vec{p}_\perp - x\vec{k}_\perp-\frac{x}{z}\vec{q}_\perp;\frac{x}{z}\right]. \qquad (1.5)$$

(We will use capital script letters for double distributions in transverse momentum. Thus, if D is the distribution in x alone, $\mathscr{D}$ is the distribution in transverse and longitudinal momentum.) Note that in $\mathscr{D}$ we use the momentum transverse to the axis defined by the incoming parton for that $\mathscr{D}$, whereas in Fig. 1 the transverse momentum is measured relative to some axis fixed in space.

Equation (1.5) is annoying because of the mixture of longitudinal and transverse degrees of freedom. BCM dealt with this by using a variant of the Fourier transform for the transverse momentum:

$$\tilde{\mathscr{D}}^{\rm NS}(k^2,Q_0{}^2;\vec{b};x)=\int d^2\vec{p}_\perp e^{-i\vec{b}\cdot\vec{p}_\perp/x}\mathscr{D}^{\rm NS}(k^2,Q_0{}^2;\vec{p}_\perp;x) \qquad (1.6)$$

and then taking moments in x. The resulting equations

$$k^2\frac{d}{dk^2}\tilde{\tilde{\mathscr{D}}}(k^2,Q_0{}^2;\vec{b};n)=\frac{\alpha(k^2)}{2\pi}\int_0^1 z^n dz\, C_F\left[\frac{1+z^2}{1-z}\right]_+\int dq_\perp{}^2\delta(z(1-z)k^2-q_\perp{}^2)J_0\left[\frac{bq_\perp}{z}\right]\tilde{\tilde{\mathscr{D}}}\left[k^2,Q_0{}^2;\frac{\vec{b}}{z};n\right] \qquad (1.7)$$

can be solved numerically. However, this is not particularly simple because the equation for a given b involves all higher values of b.

The work of Odorico and collaborators,[4] and Fox, Wolfram, and Field,[5] on Monte Carlo jet evolution, has demonstrated that the jet cascade can be computed directly as a decay of large-mass partons into partons with smaller mass. When this approach is taken, one can start with a parton traveling at large momentum in a well defined direction; there is every reason to think of this as defining the jet axis. The transverse momentum of the decay products, relative to this axis, is simply additive as the decay chain progresses.

Computation of the successive decays using QCD ideas will give, at each step, a momentum transverse to the decaying parton; simple kinematics then yields momentum transverse to the jet axis. This is conceptually much simpler than the turning of the jet axis which marks the BCM approach.

One might therefore hope that differential equations could be written for the evolution of parton masses downward, analogous to Eq. (1.4) but including transverse momentum. In this paper we show that such equations can indeed be written for all jet-calculus quantities currently used, and that they are easier to cope with than the traditional equations because the transverse and longitudinal degrees of freedom are better separated.

II. BACKWARD-MOVING EQUATIONS FOR $\mathscr{D}^{\rm NS}(Q^2,k^2;\vec{p}_\perp;x)$

The kinematics and concept of the backward-moving equations for $\mathscr{D}$ are given in Fig. 2. We will restrict ourselves to the nonsinglet case here; the extension to the full parton matrix is obvious. From Fig. 2, we obtain[6]

$$-k^2\frac{d}{dk^2}\mathscr{D}^{\rm NS}(Q^2,k^2;\vec{p}_\perp;x)=\int_x^1\frac{\alpha(k^2z(1-z))}{2\pi}\frac{dz}{z^3}P_q^{qg}(z)$$
$$\times\int\frac{d^2\vec{q}_\perp}{\pi}\delta(q_\perp{}^2-k^2g(z))\mathscr{D}^{\rm NS}\left[Q^2,\epsilon(z)k^2;\frac{\vec{p}_\perp-\vec{q}_\perp}{z};\frac{x}{z}\right]. \qquad (2.1)$$

The function $g(z)$ depends on which mass is used in the propagator on the right-hand side. Actually, this mass is not completely determined, although the masses $k_1{}^2$, $k_2{}^2$, and k'^2 at the vertex $k'^2 \to k_1{}^2+k_2{}^2$ are constrained by the kinematic relation

$$k'^2=\frac{k_1{}^2+q_\perp{}^2}{z}+\frac{k_2{}^2+q_\perp{}^2}{1-z}. \qquad (2.2)$$

Hence, if we use $\epsilon(z)k^2$ as the mass of the parent parton in the vertex of Fig. 2, the function

$$g(z)\approx k^1(1-z)[z\epsilon(z)-1].$$

(Note $\epsilon > 1$.)

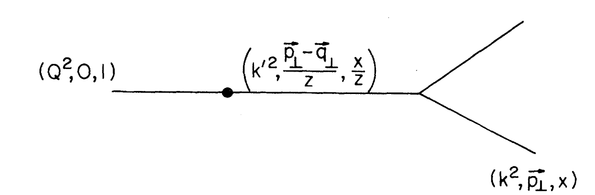

FIG. 2. Kinematics for backward-moving equation. Again, vectors are labeled with mass[2], transverse momentum, and longitudinal fraction relative to a fixed axis. However, here the axis is defined by the incident parton of mass Q, which we take to be traveling very fast.

In the leading-logarithm approximation, one typically neglects the change of the mass in the function being calculated. This may be a poor approximation in practice; however, since we are chiefly concerned with the transverse-momentum aspects of the problem in this paper, we will tend to write the same masses on both sides of the equations and we will use $g(z)$ as a generic function in the δ function specifying $q_\perp{}^2$.

Because this equation contains x only in the longitudinal momentum variable on both sides of the equation, we can take moments easily. A standard Fourier transform in transverse momentum then leads to the equation

$$-k^2\frac{d}{dk^2}\tilde{\tilde{\mathscr{D}}}^{\rm NS}(Q^2,k^2;\vec{b};n)=\int_0^1\frac{\alpha(k^2z(1-z))}{2\pi}dz\,z^nP_q^{qg}(z)\times\int dq_\perp{}^2J_0(bq_\perp)\delta(q_\perp{}^2-k^2g(z))\tilde{\tilde{\mathscr{D}}}^{\rm NS}(Q^2,k^2;z\vec{b};n)\,. \tag{2.3}$$

We see that the equation for $\mathscr{D}$ at one particular b involves only smaller b values. Note that the $b=0$ solution is known at all Q^2: it is the standard Altarelli-Parisi result given in Eq. (1.3). Hence, a grid can be established in b and the solution for higher values of b generated very quickly.

Because Eq. (1.5) and Eq. (2.1) look so different, one might question whether they have the same content. Our next step, therefore, must be to demonstrate that in fact the solutions are the same. To tackle this, we convert the differential equations into integral equations. For instance, Eq. (1.5) may be rewritten as

$$\mathscr{D}^{\rm NS}(k^2,Q_0{}^2;\vec{p}_\perp;x)=\delta(1-x)\delta^2(\vec{p}_\perp)+\int_{Q_0{}^2}^{k^2}\frac{dk'^2}{k'^2}\int_x^1\frac{dz}{z}\frac{\alpha(k'^2z(1-z))}{2\pi}\int\frac{d^2\vec{q}_\perp}{\pi}P_q^{qg}(z)\mathscr{D}^{\rm NS}\left[k'^2,Q_0{}^2;\vec{p}_\perp-\frac{x}{z}\vec{q}_\perp;\frac{x}{z}\right]\times\delta(q_\perp{}^2-k'^2f(z))\,. \tag{2.4}$$

This can now be solved by iteration to give

$$\begin{aligned}\mathscr{D}^{\rm NS}(k^2,Q_0{}^2;\vec{p}_\perp;x)=&\,\delta(1-x)\delta^2(\vec{p}_\perp)+\int_{Q_0{}^2}^{k^2}\frac{dk'^2}{k'^2}\int_x^1\frac{dz}{z}\int\frac{d^2\vec{q}_\perp}{\pi}\Pi(k'^2,z,q_\perp{}^2)\delta\left[1-\frac{x}{z}\right]\delta^2\left[\vec{p}_\perp-\frac{x}{z}\vec{q}_\perp\right]\\&+\int_{Q_0{}^2}^{k^2}\frac{dk'^2}{k'^2}\int_x^1\frac{dz}{z}\int\frac{d^2\vec{q}_\perp}{\pi}\Pi(k'^2,z,q_\perp{}^2)\\&\times\int_{Q_0{}^2}^{k'^2}\frac{dk''^2}{k''^2}\int_{x/z}^1\frac{dz'}{z'}\int\frac{d^2\vec{q}\,'_\perp}{\pi}\Pi(k''^2,z',q_\perp'^2)\\&\times\delta\left[1-\frac{x}{zz'}\right]\delta^2\left[\vec{p}_\perp-\frac{x}{z}\vec{q}_\perp-\frac{x}{zz'}\vec{q}\,'_\perp\right]+\cdots\,,\end{aligned} \tag{2.5}$$

where

$$\Pi(k'^2,z,q_\perp{}^2)=\frac{\alpha(k'^2z(1-z))}{2\pi}P_q^{qg}(z)\delta(q_\perp^2-k'^2f(z))\,.$$

If we change the symbols for the masses of the partons under observation, and also reorder the integrations, this result can be cast into the form

$$\begin{aligned}\mathscr{D}^{\rm NS}(Q^2,k^2;\vec{p}_\perp;x)=&\,\delta(1-x)\delta^2(\vec{p}_\perp)+\int_{k^2}^{Q^2}\frac{dk'^2}{k'^2}\int_x^1\frac{dz}{z}\int\frac{d^2\vec{q}_\perp}{\pi}\delta\left[1-\frac{x}{z}\right]\delta^2\left[\vec{p}_\perp-\frac{x}{z}\vec{q}_\perp\right]\Pi(k'^2,z,q_\perp{}^2)\\&+\int_{k^2}^{Q^2}\frac{dk'^2}{k'^2}\int_x^1\frac{dz}{z}\int\frac{d^2\vec{q}_\perp}{\pi}\Pi(k''^2,z,q_\perp{}^2)\\&\times\int_{k''^2}^{Q^2}\frac{dk'^2}{k'^2}\int_{x/z}^1\frac{dz'}{z'}\int\frac{d^2\vec{q}\,'_\perp}{\pi}\Pi(k'^2,z',q_\perp'^2)\\&\times\delta\left[1-\frac{x}{zz'}\right]\delta^2\left[\vec{p}_\perp-\frac{x}{z'}\vec{q}\,'-\frac{x}{zz'}\vec{q}_\perp\right]+\cdots\,.\end{aligned} \tag{2.6}$$

Similarly, Eq. (2.1) becomes the integral equation

$$\mathscr{D}^{\rm NS}(Q^2,k^2;\vec{p}_\perp;x)=\delta(1-x)\delta^2(\vec{p}_\perp)+\int_{k^2}^{Q^2}\frac{dk'^2}{k'^2}\int_x^1\frac{dz}{z^3}\frac{\alpha(k'^2z(1-z))}{2\pi}P_q^{qg}(z)\int\frac{d^2\vec{q}_\perp}{\pi}\delta(q_\perp{}^2-k'^2g(z))\mathscr{D}^{\rm NS}\left[Q^2,k'^2;\frac{\vec{p}_\perp-\vec{q}_\perp}{z};\frac{x}{z}\right] \tag{2.7}$$

with solution

$$\begin{aligned}\mathscr{D}^{\rm NS}(Q^2,k^2;\vec{p}_\perp;x)=&\;\delta(1-x)\delta^2(\vec{p}_\perp)\\&+\int_{k^2}^{Q^2}\frac{dk'^2}{k'^2}\int_x^1\frac{dz}{z^3}\int\frac{d^2\vec{q}_\perp}{\pi}\Pi'(k'^2,z,q_\perp{}^2)\delta\left[1-\frac{x}{z}\right]\delta^2\left[\frac{\vec{p}_\perp-\vec{q}_\perp}{z}\right]\\&+\int_{k^2}^{Q^2}\frac{dk'^2}{k'^2}\int_x^1\frac{dz}{z^3}\int\frac{d^2\vec{q}_\perp}{\pi}\Pi'(k'^2,z,q_\perp{}^2)\\&\times\int_{k'^2}^{Q^2}\frac{dk''^2}{k''^2}\int_{x/z}^1\frac{dz'}{z'^3}\int\frac{d^2\vec{q}\,'_\perp}{\pi}\Pi'(k''^2,z',q'^2_\perp)\delta\left[1-\frac{x}{zz'}\right]\\&\times\delta^2\left[\frac{(\vec{p}_\perp-\vec{q}_\perp)/z-\vec{q}\,'_\perp}{z'}\right]+\cdots\,,\end{aligned} \tag{2.8}$$

where

$$\Pi'(k'^2,z,q_\perp{}^2)=\frac{\alpha(k'^2z(1-z))}{2\pi}P_q^{qg}(z)\delta(q_\perp{}^2-k'^2g(z))\,.$$

We see that (2.6) and (2.8) are the same term by term provided that $f(z)$ and $g(z)$ are the same. Hence, the backward-moving equations are indeed the inverse of the forward-moving equations and we may use them if we wish.

III. BACKWARD-MOVING EQUATIONS FOR THE Γ^i

A. Longitudinal momentum only

Bassetto, Ciafaloni, and Marchesini[3] found it convenient to define some new distributions called Γ^i. These have the property that i is the "first" quark coming out of the incident parton; only gluons have been emitted prior to this. These are convenient if one wishes to construct colorless clusters consisting of a quark, an antiquark, and multiple gluons. The forward-moving equations for these propagators have been given in Ref. 3, and solutions were displayed in Ref. 7.

Because these equations are not so simple as those for the D's, the solutions are not given analytically. It is thus not possible to differentiate directly with respect to the final mass Q_0. Actually, this is just as well; as we will now explain, it is necessary to distinguish between the mass of the final parton under observation, namely, i, and the masses of the gluons which are emitted prior to it in the cascade.

The masses of these gluons are important, because we wish to avoid decays of these into $q\bar{q}$ pairs. The restriction that certain gluons go only into gluons is imposed within these equations by the function σ: the probability that gluons go only into gluons.

In order to obtain backward-moving equations which are easy to interpret, we thus consider the situation in which (a) the incoming parton has mass Q, (b) the outgoing parton under consideration has mass k, and (c) all the gluons in the final state "between" the incident parton and the parton under observation are at Q_0. We thus define new functions $G_a{}^b(Q^2,k^2;Q_0{}^2;x)$, with a being the incoming parton and b the outgoing parton. Both a and b may be either gluons or quarks.

The equations obeyed by these G's are depicted in Fig. 3. They are

$$-k^2\frac{d}{dk^2}G_i^g(Q^2,k^2;Q_0{}^2;x)=G_i^g(x)V_g(k^2)+\sum_j\int_x^1\frac{\alpha(\;)}{2\pi}\frac{dz}{z}\hat{P}_q^{gq}(z)G_i^j\left[\frac{x}{z}\right]+\tfrac{1}{2}\int_x^1\frac{\alpha(\;)}{2\pi}\frac{dz}{z}\hat{P}_g^{gg}(z)G_i^g\left[\frac{x}{z}\right]\sigma(\lambda(1-z)k^2,Q_0{}^2)+\tfrac{1}{2}\int_x^1\frac{\alpha(\;)}{2\pi}\frac{dz}{z}\hat{P}_g^{gg}(z)G_i^g\left[\frac{x}{z}\right]\,, \tag{3.1a}$$

$$-k^2\frac{d}{dk^2}G_i^j(Q^2,k^2;Q_0{}^2;x)=G_i^j(x)V_q(k^2)+\int_x^1\frac{\alpha(\;)}{2\pi}\frac{dz}{z}\hat{P}_g^{j\bar{j}}(z)G_i^g\left[\frac{x}{z}\right]$$
$$+\int_x^1\frac{\alpha(\;)}{2\pi}\frac{dz}{z}\hat{P}_q^{qg}(z)\sigma(\lambda(1-z)k^2,Q_0{}^2)G_i^j\left[\frac{x}{z}\right], \tag{3.1b}$$

$$-k^2\frac{d}{dk^2}G_g^g(Q^2,k^2;Q_0{}^2;x)=G_g^g(x)V_g(k^2)+\sum_i\int_x^1\frac{\alpha(\;)}{2\pi}\frac{dz}{z}G_g^i\left[\frac{x}{z}\right]\hat{P}_q^{gq}(z)$$
$$+\tfrac{1}{2}\int_x^1\frac{\alpha(\;)}{2\pi}\frac{dz}{z}\hat{P}_g^{gg}(z)G_g^g\left[\frac{x}{z}\right]\sigma(\lambda(1-z)k^2,Q_0{}^2)+\tfrac{1}{2}\int\frac{\alpha(\;)}{2\pi}\frac{dz}{z}\hat{P}_g^{gg}(z)G_g^g\left[\frac{x}{z}\right], \tag{3.1c}$$

$$-k^2\frac{d}{dk^2}G_g^i(Q^2,k^2;Q_0{}^2;x)=G_g^i(x)V_q(k^2)+\int_x^1\frac{\alpha(\;)}{2\pi}\frac{dz}{z}G_g^g\left[\frac{x}{z}\right]\hat{P}_g^{i\bar{i}}(z)$$
$$+\int_x^1\frac{\alpha(\;)}{2\pi}\frac{dz}{z}G_g^i\left[\frac{x}{z}\right]\hat{P}_q^{qg}(z)\sigma(\lambda(1-z)k^2,Q_0{}^2)\,. \tag{3.1d}$$

We abbreviate the arguments in G functions on the right-hand side of the equations, showing only those which are different from the left-hand side. Also, the argument of $\alpha(\;)$ is understood to be $k^2z(1-z)$. The splitting functions $\hat{P}$ used here are the usual Altarelli-Parisi P functions without the δ-function singularities at the end points. These have been accounted for in the "virtual potentials" $V(k^2)$, defined in the caption to Fig. 3.

We then expect that $G_a^b(Q^2,Q_0{}^2;Q_0{}^2,x)$ will be the same as the BCM $\Gamma_a^b(Q^2,Q_0{}^2;x)$ function for the two cases with outgoing quarks; the cases with outgoing gluons are not used by BCM. Notice that by defining the new functions in which only one final-state parton moves in mass, we are able to write equations which look just like the BCM equations and use the same function σ.

We must now prove our assertion that these equations have the same content as the forward-moving equations of BCM. For this it is simplest to go into moment space; Eqs. (3.1) then take the form

$$-k^2\frac{d}{dk^2}\tilde{G}_i^g(Q^2,k^2;Q_0{}^2;n)=M(k^2,Q_0{}^2;n)\tilde{G}_i^g$$
$$+N(k^2,Q_0{}^2;n)\sum_j\tilde{G}_i^j\,, \tag{3.2a}$$

$$-k^2\frac{d}{dk^2}\tilde{G}_i^j(Q^2,k^2;Q_0{}^2;n)=P(k^2,Q_0{}^2;n)\tilde{G}_i^j$$
$$+S(k^2,Q_0{}^2;n)\tilde{G}_i^g\,, \tag{3.2b}$$

$$-k^2\frac{d}{dk^2}\tilde{G}_g^g(Q^2,k^2;Q_0{}^2;n)=M(k^2,Q_0{}^2;n)\tilde{G}_g^g$$
$$+N(k^2,Q_0{}^2;n)\sum_i\tilde{G}_g^i\,, \tag{3.2c}$$

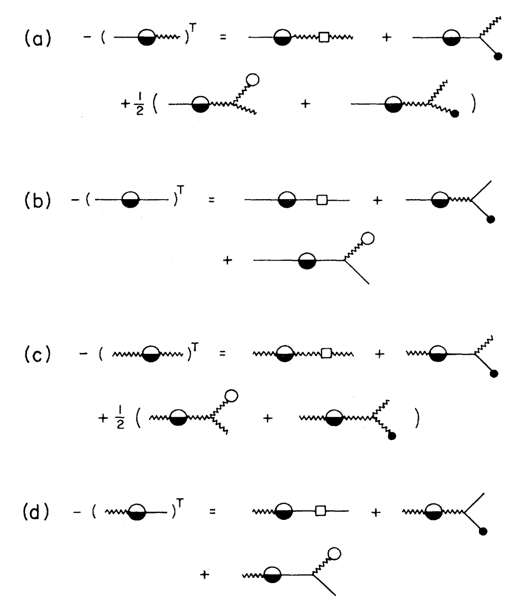

FIG. 3. Backward-moving equations for the G functions. T denotes $k^2(d/dk^2)$; differentiation with respect to the final observed mass. ∼○ denotes σ, the probability that gluons go only into gluons. ∼□∼ and —□— denote the virtual quark and gluon potentials, respectively:

$$V_g(k^2)=-\int_\epsilon^{1-\epsilon}dz\frac{\alpha[z(1-z)k^2]}{2\pi}+[\tfrac{1}{2}\hat{P}_g^{gg}(z)+\hat{P}_g^{q\bar{q}}(z)]\,,$$

$$V_q(k^2)=-\int_\epsilon^{1-\epsilon}dz\frac{\alpha[z(1-z)k^2]}{2\pi}[\tfrac{1}{2}\hat{P}_q^{qg}(z)+\tfrac{1}{2}\hat{P}_q^{gq}(z)]\,.$$

Note that the same graphs describe backward motion of the W function; one must just keep track of both sets of momentum in writing the equations.

$$-k^2\frac{d}{dk^2}\tilde{G}^{\,i}_{g}(Q^2,k^2;Q_0{}^2;n)=P(k^2,Q_0{}^2;n)\tilde{G}^{\,i}_{g}+S(k^2,Q_0{}^2;n)\tilde{G}^{\,g}_{g}\ , \quad (3.2\text{d})$$

where the functions $M(k^2)$, $N(k^2)$, $P(k^2)$, and $S(k^2)$ are defined by

$$M(k^2,Q_0{}^2;n)=-\frac{C_A}{2\pi}\int_{Q_0{}^2}^{k^2}\frac{\alpha(t')[1-\sigma(t')]}{t'}dt' +\frac{\alpha(k^2)b}{2}[1-\sigma(k^2,Q_0{}^2)] +\frac{\alpha(k^2)}{4\pi}A_g^{gg}(n)[1+\sigma(k^2,Q_0{}^2)]\ ,$$

$$N(k^2,Q_0{}^2;n)=\frac{\alpha(k^2)}{2\pi}A_q^{gq}(n)\ , \quad (3.3)$$

$$P(k^2,Q_0{}^2;n)=-\frac{C_F}{\pi}\int_{Q_0{}^2}^{k^2}\frac{\alpha(t')}{t'}[1-\sigma(t')]dt' +\frac{3}{4\pi}C_F\alpha(k^2)[1-\sigma(k^2,Q_0{}^2)] +\frac{\alpha(k^2)}{2\pi}\sigma(k^2,Q_0{}^2)A_q^{qg}(n)\ ,$$

$$S(k^2,Q_0{}^2;n)=\frac{\alpha(k^2)}{2\pi}\frac{1}{N_f}A_g^{q\bar{q}}(n) =\frac{\alpha(k^2)}{2\pi}A_g^{i\bar{i}}(n)\ .$$

Explicitly, for three flavors, we may define the matrix

$$Z(k^2)=\begin{bmatrix} P & 0 & 0 & S\\ 0 & P & 0 & S\\ 0 & 0 & P & S\\ N & N & N & M\end{bmatrix}\ . \quad (3.4)$$

Then Eqs. (3.2a) and (3.2b) may be summarized as

$$-k^2\frac{d}{dk^2}\begin{bmatrix}\tilde{G}^{\,1}_{1}\\ \tilde{G}^{\,2}_{1}\\ \tilde{G}^{\,3}_{1}\\ \tilde{G}^{\,g}_{1}\end{bmatrix}=Z\begin{bmatrix}\tilde{G}^{\,1}_{1}\\ \tilde{G}^{\,2}_{1}\\ \tilde{G}^{\,3}_{1}\\ \tilde{G}^{\,g}_{1}\end{bmatrix}\ . \quad (3.5)$$

We again cast this into the form of an integral equation and solve by iteration. The result may be rewritten as

$$(\tilde{G}^{\,1}_{1},\tilde{G}^{\,2}_{1},\tilde{G}^{\,3}_{1},\tilde{G}^{\,g}_{1})=(1,0,0,0)X(Q^2,k^2)\ , \quad (3.6)$$

where the matrix X is

$$X(Q^2,k^2)=1+\int_{k^2}^{Q^2}\frac{dk'^2}{k'^2}Z^{+}(k'^2)+\int_{k^2}^{Q^2}\frac{dk''^2}{k''^2}Z^{+}(k''^2)\int_{k^2}^{k''^2}\frac{dk'^2}{k'^2}Z^{+}(k'^2) +\int_{k^2}^{Q^2}\frac{dk'''^2}{k'''^2}Z^{+}(k'''^2)\int_{k^2}^{k'''^2}\frac{dk''^2}{k''^2}Z^{+}(k''^2)\int_{k^2}^{k''^2}\frac{dk'^2}{k'^2}Z^{+}(k'^2)+\cdots\ . \quad (3.7)$$

As was pointed out in Ref. 7, the equations for the Γ^i take the form

$$Q^2\frac{d}{dQ^2}\begin{bmatrix}\tilde{\Gamma}^{\,1}_{1}(Q^2,Q_0{}^2)\\ \tilde{\Gamma}^{\,1}_{2}(Q^2,Q_0{}^2)\\ \tilde{\Gamma}^{\,1}_{3}(Q^2,Q_0{}^2)\\ \tilde{\Gamma}^{\,1}_{g}(Q^2,Q_0{}^2)\end{bmatrix}=Z^{+}(Q^2)\begin{bmatrix}\tilde{\Gamma}^{\,1}_{1}\\ \tilde{\Gamma}^{\,1}_{2}\\ \tilde{\Gamma}^{\,1}_{3}\\ \tilde{\Gamma}^{\,1}_{g}\end{bmatrix} \quad (3.8)$$

with the same Z as above. Their solution thus becomes

$$\begin{bmatrix}\tilde{\Gamma}^{\,1}_{1}(Q^2,Q_0{}^2)\\ \tilde{\Gamma}^{\,1}_{2}(Q^2,Q_0{}^2)\\ \tilde{\Gamma}^{\,1}_{3}(Q^2,Q_0{}^2)\\ \tilde{\Gamma}^{\,1}_{g}(Q^2,Q_0{}^2)\end{bmatrix}=X(Q^2,Q_0{}^2)\begin{bmatrix}1\\0\\0\\0\end{bmatrix}\ . \quad (3.9)$$

We note that the results of the equations may be written in the form

$$\tilde{G}^{\,i}_{j}(Q^2,k^2;Q_0{}^2;n)=X(Q^2,k^2;n)_{ji}\ ,\qquad \tilde{G}^{\,g}_{i}(Q^2,k^2;Q_0{}^2;n)=X(Q^2,k^2;n)_{i4}\ , \quad (3.10\text{a})$$

$$\tilde{G}^{\,i}_{g}(Q^2,k^2;Q_0{}^2;n)=X(Q^2,k^2;n)_{4i}\ ,\qquad \tilde{G}^{\,g}_{g}(Q^2,k^2;Q_0{}^2;n)=X(Q^2,k^2;n)_{44}\ , \quad (3.10\text{b})$$

$$\tilde{\Gamma}^{\,i}_{g}(Q^2,Q_0{}^2;n)=X(Q^2,Q_0{}^2;n)_{4i}\ ,\qquad \tilde{\Gamma}^{\,i}_{j}(Q^2,Q_0{}^2;n)=X(Q^2,Q_0{}^2;n)_{ji}\ , \quad (3.10\text{c})$$

and hence the $G^i_a(Q^2,Q_0{}^2;Q_0{}^2;n)$ agree with the Γ^i_a for the appropriate cases with quarks going out, as claimed.

B. Inclusion of transverse momentum

The inclusion of transverse momentum is very straightforward since the kinematics of this case is exactly the same as in the case of the D functions. We thus can write immediately

$$-k^2\frac{d}{dk^2}\mathcal{G}_i^j(Q^2,k^2;Q_0^2;\vec{p}_\perp;x)=V_q(k^2)\mathcal{G}_i^j(\vec{p}_\perp,x)+\int_x^1\frac{dz}{z^3}\int\frac{d^2\vec{q}_\perp}{\pi}\frac{\alpha(\ )}{2\pi}\hat{P}_g^{jj}(z)\delta(q_\perp{}^2-k^2g(z))\mathcal{G}_i^g\left[\frac{\vec{p}_\perp-\vec{q}_\perp}{z};\frac{x}{z}\right]$$
$$+\int_x^1\frac{dz}{z^3}\int\frac{d^2\vec{q}_\perp}{\pi}\frac{\alpha(\ )}{2\pi}\hat{P}_q^{qg}(z)\delta(q_\perp{}^2-k^2g(z))\sigma(\lambda(1-z)k^2,Q_0{}^2)\mathcal{G}_i^j\left[\frac{\vec{p}_\perp-\vec{q}_\perp}{z};\frac{x}{z}\right]$$
(3.11)

and the method of solution will be the same as for the case of $\mathcal{D}$'s, with the slight complication of the added matrix algebra and the auxiliary function σ.

IV. KEEPING TRACT OF GLUON MOMENTUM

A. The H functions

As discussed at length in Refs. 3 and 8, we would like to form colorless clusters containing a quark, antiquark, and multiple gluons, and know the momentum of the entire cluster. This requires that, in addition to the quark momentum used by the BCM Γ^i functions, we need the momentum of the gluons which come off before the quark in question. In Ref. 8, a set of functions called H^i were defined, and the forward-moving equations for these written down. For convenience, we repeat these [here in $\mathfrak{H}(\alpha,\beta,\gamma,\delta)$, β is the mass^2 of the (gluons + quarks) whose longitudinal and transverse momenta are γ and δ]:

$$\left[\frac{d}{d\ln k^2}-V_g(k^2)\right]\mathfrak{H}_g^q(k^2,p^2,x,\vec{p}_\perp-x\vec{k}_\perp)$$
$$=\int\frac{dz}{z}\frac{\alpha(\ )}{2\pi}\hat{P}_g^{q\bar{q}}(z)\int\frac{d^2\vec{q}_\perp}{\pi}\delta(z(1-z)k^2-q_\perp{}^2)\mathfrak{H}_q^q\left[\lambda(z)k^2,p^2,\frac{x}{z},\vec{p}_\perp-x\vec{k}_\perp-\frac{x}{z}\vec{q}_\perp\right]$$
$$+\tfrac{1}{2}\int\frac{dz}{z}\frac{\alpha(\ )}{2\pi}\hat{P}_g^{gg}(z)\int\frac{d^2\vec{q}_\perp}{\pi}\delta[z(1-z)k^2-q_\perp{}^2]\mathfrak{H}_g^q\left[\lambda(z)k^2,p^2,\frac{x}{z},\vec{p}_\perp-x\vec{k}_\perp-\frac{x}{z}\vec{q}_\perp\right]$$
$$+\tfrac{1}{2}\int\frac{dz}{1-z}\frac{\alpha(\ )}{2\pi}\hat{P}_g^{gg}(z)\int dk_1{}^2dk_2{}^2\frac{d^2\vec{q}_\perp}{\pi}\left|\frac{x-z}{x}\right|\delta(z(1-z)k^2-zk_2{}^2-(1-z)k_1{}^2-\vec{q}_\perp{}^2)$$
$$\times\frac{d}{dk_1{}^2}\sigma_g(k_1{}^2,Q_0{}^2)\frac{d}{dk_2{}^2}\mathfrak{H}_g^q\left[k_2{}^2,(p-k_1)^2,\frac{x-z}{1-z},\vec{p}_\perp-x\vec{k}_\perp-\frac{1-x}{1-z}\vec{q}_\perp\right],\quad(4.1\text{a})$$

$$\left[\frac{d}{d\ln k^2}-V_q(k^2)\right]\mathfrak{H}_q^q(k^2,p^2,x,\vec{p}_\perp-x\vec{k}_\perp)$$
$$=\int\frac{dz}{z}\frac{\alpha(\ )}{2\pi}\hat{P}_q^{gq}(z)\int\frac{d^2\vec{q}_\perp}{\pi}\delta(z(1-z)k^2-q_\perp{}^2)\mathfrak{H}_g^q\left[\lambda(z)k^2,p^2,\frac{x}{z},\vec{p}_\perp-x\vec{k}_\perp-\frac{x}{z}\vec{q}_\perp\right]$$
$$+\int\frac{dz}{1-z}\frac{\alpha(\ )}{2\pi}\hat{P}_q^{gq}\int dk_1{}^2dk_2{}^2\frac{d^2\vec{q}_\perp}{\pi}\left|\frac{x-z}{x}\right|\delta(z(1-z)k^2-zk_2{}^2-(1-z)k_1{}^2-q_\perp{}^2)$$
$$\times\frac{d}{dk_1{}^2}\sigma_g(k_1{}^2,Q_0{}^2)\frac{d}{dk_2{}^2}\mathfrak{H}_q^q\left[k_2{}^2,(p-k_1)^2,\frac{x-z}{1-z},\vec{p}_\perp-x\vec{k}_\perp-\frac{1-x}{1-z}\vec{q}_\perp\right].\quad(4.1\text{b})$$

In Ref. 7 the longitudinal-momentum distributions for the

$$H^i=\int d\beta\int d^2\vec{p}_\perp\mathfrak{H}^i(k^2,\beta,x,\vec{p}_\perp)$$

were computed as well as various related mass distributions. One feature of these longitudinal-momentum equations which makes them more complicated than the ones for Γ and D is that after moments of the distribution are computed, equations for the nth moment involve all lower moments. However, solution by normal techniques is still possible.

When the full equations including transverse momentum (4.1) are considered, however, it becomes clear that the trick used by BCM to separate the Fourier-transform variable from the moment-transform variable will not work here. If we define

$$\bar{\mathcal{H}}(k^2,x,\vec{b})=\int dp^2\int d^2\vec{p}_\perp e^{-i\vec{b}\cdot\vec{p}_\perp/x}\mathcal{H}(k^2,p^2;x;\vec{p}_\perp)$$

we find that the transform of (4.1a) becomes

$$\left[\frac{d}{d\ln k^2}-V_g(k^2)\right]\overline{\mathscr{H}}{}^q_g(k^2;x,\vec{b})$$

$$=\int\frac{dz}{z}\frac{\alpha(\;)}{2\pi}\hat{P}^{q\bar{q}}_g(z)\int\frac{d^2\vec{q}_\perp}{\pi}\delta[z(1-z)k^2-q_\perp{}^2]\overline{\mathscr{H}}{}^q_q\left[\lambda(z)k^2;\frac{x}{z};\frac{\vec{b}}{z}\right]e^{-i\vec{b}\cdot\vec{q}_\perp/z}$$

$$+\tfrac{1}{2}\int\frac{dz}{z}\frac{\alpha(\;)}{2\pi}\hat{P}^{gg}_g(z)\int\frac{d^2\vec{q}_\perp}{\pi}\delta[z(1-z)k^2-q_\perp{}^2]\overline{\mathscr{H}}{}^q_g\left[\lambda(z)k^2;\frac{x}{z};\frac{\vec{b}}{z}\right]e^{-i\vec{b}\cdot\vec{q}_\perp/z}$$

$$+\tfrac{1}{2}\int\frac{dz}{1-z}\frac{\alpha(\;)}{2\pi}\hat{P}^{gg}_g(z)\int\frac{d^2\vec{q}_\perp}{\pi}\delta[z(1-z)k^2-q_\perp{}^2]\overline{\mathscr{H}}{}^q_g\left[\lambda(1-z)k^2;\frac{x-z}{1-z};\frac{\vec{b}}{x}\left[\frac{x-z}{1-z}\right]\right]$$

$$\times\exp\left[-i\vec{b}\cdot\vec{q}_\perp\left[\frac{1-x}{1-z}\right]\Big/x\right]. \qquad (4.2)$$

This so mixes the transverse and longitudinal momentum that straightforward solution is not possible.

We need an approach through the backward-moving equations to obtain any clue about the transverse-momentum behavior.

B. Longitudinal momentum only

Again we wish to write backward-moving equations which involve only simple QCD vertices and the BCM function σ. Apparently, the best way to do this is to define yet another set of distributions, $W(Q^2,k^2;Q_0{}^2;x_1,x_2)$, in which the momentum x_1 of all the gluons (between the two partons at Q^2 and k^2) and the momentum x_2 of the parton in question are separately handled. Both the function G and the function H can then be computed from the W's, using the relations

$$G^b_a(Q^2,k^2;Q_0{}^2;x_2)=\int_0^{1-x_2}dx_1\,W^b_a(Q^2,k^2;Q_0{}^2;x_1,x_2)\,, \qquad (4.3a)$$

$$H^b_a(Q^2,k^2;Q_0{}^2;x)=\int_0^1 dx_2\int_0^{1-x_2}dx_1\,\delta(x-x_1-x_2)W^b_a(Q^2,k^2;Q_0{}^2;x_1,x_2)\,. \qquad (4.3b)$$

The W's then obey the equations (see Fig. 3) (again we suppress the first three arguments of W on the right-hand side of the equation)

$$\left[-\frac{d}{d\ln k^2}-V_g(k^2)\right]W^g_i(Q^2,k^2;Q_0{}^2;x_1,x_2)$$

$$=\sum_j\int\frac{\alpha(\;)}{2\pi}W^j_i\left[x_1,\frac{x_2}{z}\right]\hat{P}^{gq}_q(z)\frac{dz}{z}$$

$$+\tfrac{1}{2}\int\frac{\alpha(\;)}{2\pi}W^g_i\left[x_1-x_2\frac{(1-z)}{z},\frac{x_2}{z}\right]\sigma(\lambda(1-z)k^2,Q_0{}^2)\hat{P}^{gg}_g(z)\frac{dz}{z}+\tfrac{1}{2}\int\frac{\alpha(\;)}{2\pi}W^g_i\left[x_1,\frac{x_2}{z}\right]\hat{P}^{gg}_g(z)\frac{dz}{z}\,, \qquad (4.4a)$$

$$\left[-\frac{d}{d\ln k^2}-V_q(k^2)\right]W^j_i(Q^2,k^2;Q_0{}^2;x_1,x_2)$$

$$=\int\frac{\alpha(\;)}{2\pi}\frac{dz}{z}\hat{P}^{j\bar{j}}_g(z)W^g_i\left[x_1;\frac{x_2}{z}\right]+\int\frac{\alpha(\;)}{2\pi}\frac{dz}{z}\hat{P}^{qg}_q(z)W^j_i\left[x_1-x_2\frac{(1-z)}{z};\frac{x_2}{z}\right]\sigma(\lambda(1-z)k^2;Q_0{}^2)\,, \qquad (4.4b)$$

$$\left[-\frac{d}{d\ln k^2}-V_g(k^2)\right]W^g_g(Q^2,k^2;Q_0{}^2;x_1,x_2)$$

$$=\sum_i\int\frac{\alpha(\;)}{2\pi}W^i_g\left[x_1,\frac{x_2}{z}\right]\hat{P}^{gq}_q(z)\frac{dz}{z}$$

$$+\tfrac{1}{2}\int\frac{\alpha(\;)}{2\pi}W^g_g\left[x_1-x_2\frac{(1-z)}{z},\frac{x_2}{z}\right]\hat{P}^{gg}_g(z)\frac{dz}{z}\sigma(\lambda(1-z)k^2,Q_0{}^2)+\tfrac{1}{2}\int\frac{\alpha(\;)}{2\pi}W^g_g\left[x_1,\frac{x_2}{z}\right]\hat{P}^{gg}_g(z)\frac{dz}{z}\,, \qquad (4.4c)$$

$$\left[-\frac{d}{d\ln k^2}-V_q(k^2)\right]W^i_g(Q^2,k^2;Q_0{}^2;x_1,x_2)$$

$$=\int\frac{\alpha(\;)}{2\pi}\hat{P}^{q\bar{q}}_g(z)W^g_g\left[x_1,\frac{x_2}{z}\right]\frac{dz}{z}+\int\frac{\alpha(\;)}{2\pi}\hat{P}^{qg}_q(z)W^i_g\left[x_1-x_2\frac{(1-z)}{z};\frac{x_2}{z}\right]\sigma(\lambda(1-z)k^2,Q_0{}^2)\frac{dz}{z}\,, \qquad (4.4d)$$

which can be solved in the usual way by taking moments.

If we take double moments of Eq. (4.4b), we obtain $(m,n>0)$

$$\left[-\frac{d}{d\ln k^2}-V_q(k^2)\right]\tilde{\tilde{W}}_i^j(Q^2,k^2;Q_0^2;m,n)$$

$$=\frac{\alpha(k^2)}{2\pi}A_g^{i\bar{i}}(n)\tilde{\tilde{W}}_i^g(m,n)+\frac{\alpha(k^2)}{2\pi}P_q^{qg}(n,m)\sigma(k^2,Q_0^2)\tilde{G}_i^j(n+m)$$
$$+\frac{\alpha(k^2)}{2\pi}\sum_{j=1}^{m-1}\begin{bmatrix}m\\j\end{bmatrix}P_q^{qg}(n,j)\sigma(k^2,Q_0^2)\tilde{\tilde{W}}_i^j(m-j,n+j)$$
$$+\left[\frac{\alpha(k^2)\sigma(k^2,Q_0^2)}{2\pi}A_q^{qg}(n)-\frac{3}{2}\frac{\alpha\sigma}{2\pi}C_F+C_F\Xi(k^2)\right]\tilde{\tilde{W}}_i^j(m,n) \quad (4.5)$$

with

$$\Xi(k^2)=\frac{1}{\pi}\int_{Q_0^2}^{k^2}\frac{dk'^2}{k'^2}\alpha(k'^2)\sigma(k'^2)\ ,\quad P(n_1,n_2)=\int_0^1 dz\, z^{n_1}(1-z)^{n_2}P(z)\ .$$

This is similar to the equation for H in that it mixes moments. Solution for $W(l_1,l_2)$ requires solving for all G's up to $G(l_1+l_2)$; for all $W(l,j)$ up to $j=l_1+l_2-l$; and for all $W(i,j)$ for $i<l_1$; $j<l_1+l_2-i$. The moments of H can then be obtained from those of W by

$$\int_0^1 x^n H_a{}^b(Q^2,k^2;Q_0^2;x)dx=\sum_{i=0}^{n}\begin{bmatrix}n\\i\end{bmatrix}\tilde{\tilde{W}}_a^b(Q^2,k^2;Q_0^2;n-i;i)\ . \quad (4.6)$$

C. Inclusion of transverse momentum

The equation for W including transverse momentum analogous to (4.4b) is

$$\left[-\frac{d}{d\ln k^2}-V_q(k^2)\right]\mathscr{W}_i^j(Q^2,k^2;Q_0^2;x_1,\vec{p}_\perp^{\,1};x_2,\vec{p}_\perp^{\,2})$$

$$=\int\frac{\alpha(\ )}{2\pi}\frac{dz}{z^3}\hat{P}_q^{qg}(z)\int\frac{d^2\vec{q}_\perp}{\pi}\sigma(\lambda(1-z)k^2,Q_0^2)\delta(q_\perp^2-k^2g(z))$$
$$\times\mathscr{W}_i^j\left[x_1-x_2\frac{(1-z)}{z},\vec{p}_\perp^{\,1}-\left[\frac{1-z}{z}\right](\vec{p}_\perp^{\,2}-\vec{q}_\perp)+\vec{q}_\perp;\frac{x_2}{z},\frac{\vec{p}_\perp^{\,2}-\vec{q}_\perp}{z}\right]$$
$$+\int\frac{\alpha(\ )}{2\pi}\frac{dz}{z^3}\hat{P}_g^{jj}(z)\int\frac{d^2\vec{q}_\perp}{\pi}\delta(q_\perp^2-k^2g(z))\mathscr{W}_i^g\left[x_1,\vec{p}_\perp^{\,1};\frac{x_2}{z},\frac{\vec{p}_\perp^{\,2}-\vec{q}_\perp}{z}\right]. \quad (4.7)$$

Although this looks somewhat messy, we again have the feature that no x's are included in the transverse-momentum distributions. Notice that the full $\mathscr{G}$ and $\mathscr{H}$ functions may be computed from

$$\mathscr{H}_a^b(Q^2,k^2;Q_0^2;\vec{p}_\perp,x)=\int d^2\vec{p}_\perp^{\,1}\int d^2\vec{p}_\perp^{\,2}\delta^2(\vec{p}_\perp-\vec{p}_\perp^{\,1}-\vec{p}_\perp^{\,2})$$
$$\times\int_0^1 dx_1\int_0^{1-x_2}dx_2\delta(x-x_1-x_2)\mathscr{W}_a^b(Q^2,k^2;Q_0^2;x_1,\vec{p}_\perp^{\,1};x_2,\vec{p}_\perp^{\,2})\ ,$$
$$\mathscr{G}_a^b(Q^2,k^2;Q_0^2;\vec{p}_\perp,x)=\int d^2\vec{p}_\perp^{\,1}\int_0^{1-x}dx_1\mathscr{W}_a^b(Q^2,k^2;Q_0^2;x_1,\vec{p}_\perp^{\,1};x,\vec{p}_\perp)\ .$$

After computing moments, and Fourier transforming the two transverse-momentum variables separately, we have

$$\left[-\frac{d}{d\ln k^2}-V_q(k^2)\right]\mathscr{W}_i^j(Q^2,k^2;Q_0^2;m,\vec{b}_1;n,\vec{b}_2)$$
$$=\sum_{j=0}^{m}\begin{bmatrix}m\\j\end{bmatrix}\int\frac{\alpha(\ )}{2\pi}dz\,\sigma(\lambda(1-z)k^2,Q_0^2)\hat{P}_q^{qg}(z)J_0(k\sqrt{g(z)}\,|\,\vec{b}_1-\vec{b}_2\,|\,)z^n(1-z)^j\mathscr{W}_i^j(m-j,\vec{b}_1;n+j,\vec{b}_2z+\vec{b}_1(1-z))$$
$$+\int\frac{\alpha(\ )}{2\pi}\hat{P}_g^{jj}(z)dz\,J_0(b_2k\sqrt{g(z)})z^n\mathscr{W}_i^g(m,\vec{b}_1;n,z\vec{b}_2)\ . \quad (4.8)$$

Since the moments needed for $\mathscr{H}$ are ones with both b values equal, we do not need to solve these equations in general to deal with the H's. Instead we note that with $b_1=b$, $b_2=b\xi$ ($\xi<1$), we obtain

$$\left[-\frac{d}{d\ln k^2}-V_q(k^2)\right]\mathscr{W}_i^j(Q^2,k^2;Q_0^2;m,\vec{b};n,\xi\vec{b})$$

$$=\sum_{j=0}^{m}\begin{bmatrix} m \\ j \end{bmatrix}\int\frac{\alpha(\ )}{2\pi}dz\,\sigma(\lambda(1-z)k^2,Q_0^2)\hat{P}_q^{qg}(z)J_0(k\sqrt{g(z)}b(1-\xi))z^n(1-z)^j\mathscr{W}_i^j(m-j,\vec{b};n+j,\vec{b}[1-(1-\xi)z])$$

$$+\int\frac{\alpha(\ )}{2\pi}\hat{P}_g^{jj}(z)dz\,J_0(\xi bk\sqrt{g(z)})z^n\mathscr{W}_i^g(m,\vec{b};n,z\xi\vec{b})\ . \tag{4.9}$$

One can then set up a grid in b_2, and solve as for the $\mathscr{D}$'s.

V. SUMMARY AND CONCLUSIONS

In order to be able to separate the longitudinal and transverse degrees of freedom in the jet behavior, the backward-moving equations must be used. These are more easily solved numerically than the forward moving equations. However, they have the same physical content and the answers can be shown to be identical. Solution is in progress, and results will be presented in a later paper.

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[1]G. Altarelli and G. Parisi, Nucl. Phys. **B126**, 298 (1977).

[2]K. Konishi, A. Ukawa, and G. Veneziano, Nucl. Phys. **B157**, 45 (1979).

[3](a) A. Bassetto, M. Ciafaloni, and G. Marchesini, Nucl. Phys. **B163**, 477 (1980); (b) Phys. Lett. **83B**, 207 (1979); (c) *ibid.* **86B**, 366 (1979); (d) Phys. Rep. **100**, 201 (1983).

[4]R. Odorico, Nucl. Phys. **B172**, 157 (1980); Phys. Lett. **103B**, 465 (1981); P. Mazzanti and R. Odorico, Z. Phys. C **7**, 61 (1980); P. Mazzanti, R. Odorico, and V. Roberto, Nucl. Phys. **B193**, 541 (1981).

[5]G. C. Fox and S. Wolfram, Nucl. Phys. **B168**, 285 (1980); R. D. Field, in *Multiparticle Dynamics 1982,* proceedings of the XIII International Symposium on Multiparticle Dynamics, Volendam, Netherlands, edited by W. Kittel, W. Metzger, and A. Stergiou (World Scientific, Singapore, 1983); R. D. Field and G. C. Fox, invited talks presented at the Europhysics Study Conference on Jet Dynamics in Quark and Lepton Interactions, Erice, Sicily, 1982 (unpublished).

[6]Henceforth we implement the improvement introduced by D. Amati, A. Bassetto, M. Ciafaloni, G. Marchesini, and G. Veneziano [Nucl. Phys. **B173**, 429 (1980)], who found that replacing $a(k^2)$ by $a(k^2z(1-z))$ gives a better treatment of the infrared singularities of the problem.

[7]L. Jones and R. Migneron, Phys. Rev. D **27**, 2715 (1983).

[8]B. Crespi and L. M. Jones, Phys. Rev. D **28**, 2840 (1983).