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Self-dual monopoles in SU(5)

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Explicit solutions to the self-duality equations of SU(5) gauge theory are obtained for all embeddings of physical interest. This is done by showing that for each case considered of a nonmaximal embedding the equations can be mapped into the self-duality equations for maximal embedding for some SU(N), where $N<5$. The method generalizes to SU(N).

We propose to construct explicit solutions to the Bogomolny self-duality equations in SU(5) for all embeddings[1] of physical interest enumerated by Dokos and Tomaras.[2] For each case of a nonmaximal embedding in this enumeration, that is, the cases $\underline{5}\to\underline{2}+\underline{1}+\underline{1}+\underline{1}$, $\underline{5}\to\underline{3}+\underline{1}+\underline{1}$, and $\underline{5}\to\underline{4}+\underline{1}$, we show that the equations can be mapped into the maximal embedding SU(N) self-duality equations with $N=2$, 3, and 4 for the three cases, respectively. But solution to the problem of maximal embedding is known[3] in SU(N) and known *explicitly* when $N=2,3,4$. Thus our problem is solved. Our method generalizes to SU(N). Let us treat the general case first.

Let the proposed monopole solution by spherically symmetric with respect to $\vec{J}=-i\vec{r}\times\vec{\nabla}+\vec{T}$, where $\vec{T}$ generates some SU(2) subgroup of the gauge group G. In the radial gauge[4] write the gauge field $\vec{W}$ as $gW_{\pm}(r)=\mp i[M_{\pm}(r)-T_{\pm}]/r$, where $W_{\pm}=W^1\pm iW^2$, g is the gauge coupling constant, and 1 and 2 denote the polar and azimuthal (and 3 the radial) directions. Now it is well known[3,4] that when $\vec{T}$ is a maximal embedding of SU(2) in G, then for $G=\mathrm{SU}(N+1)$ the *Ansatz* for the vector and scalar fields may be taken as

$$M_+=\begin{pmatrix} 0 & a_1 & & & \\ & 0 & a_2 & & \\ & & \ddots & \ddots & \\ & & & 0 & a_N \\ & & & & 0 \end{pmatrix}, \tag{1a}$$

$$\Phi=\tfrac{1}{2}\,\mathrm{diag}(\eta_1,\eta_2-\eta_1,\ldots,\eta_N-\eta_{N-1},-\eta_N)\ , \tag{1b}$$

with n_m, a_m real radial functions and $M_-=(M_+)^T$. The resulting Bogomolny equations are[3]

$$r^2\eta_m'=(a_m)^2-m\bar{m}\ , \tag{2a}$$

$$a_m'=(-\tfrac{1}{2}\eta_{m-1}+\eta_m-\tfrac{1}{2}\eta_{m+1})a_m\ , \tag{2b}$$

where $1\le m\le N$, $\bar{m}=N+1-m$, and $\eta_0=\eta_{N+1}=0$, and a prime denotes derivative with respect to the radial variable r. After these preliminaries let us come to the problem at hand. We shall consider a restricted class of embeddings characterized as follows. The lowest-dimensional irreducible representation (irrep) of SU(N) decomposes under $\vec{T}$ as a direct sum of a $\underline{D}$-dimensional irrep plus $(N-D)$ singlets. This means that $\vec{T}$ has the shape $\vec{T}=\mathrm{diag}(\vec{t}_s,0,0,\ldots,0)$, where $\vec{t}_s$ is the standard spin-s representation $(2s+1=D)$ and the number of zero entries is $(N-D)$. Note that we may also permute the position of $\vec{t}_s$ with respect to the zero entries without changing any of the conclusions to follow. The crucial observation to make now is this: for the class of embedding considered the *Ansatz* for the vector field may be taken as

$$M_+=\begin{pmatrix} 0 & b_1 & & & & & \\ & 0 & b_2 & & & & \\ & & \ddots & \ddots & & & \\ & & & 0 & b_{D-1} & & \\ & & & & 0 & & \\ & & & & & \ddots & \\ & & & & & & 0 \end{pmatrix}, \tag{3}$$

where b_i are real radial functions and $M_-=(M_+)^T$. This conclusion is reached by following the same line of arguments as the one that led to Eq. (1a). Indeed, M_+ can be written[4] in terms of anticommutators involving T_+ and powers of T_3. We can always express it in the form $M_{\pm}=\tilde{M}_{\pm}T_{\pm}$, i.e., shift $T_{\pm}$ to the right. Next we diagonalize T_3 so that $\tilde{M}_{\pm}$ becomes a diagonal matrix. Then writing $T_{\pm}$ using the standard form of the matrix $\vec{t}_s$ we arrive at Eq. (3). Q.E.D. As for the scalar field, it has the structure of a traceless diagonal matrix with entries that are again real radial functions. Inspection of Eq. (3) reveals the following. The self-duality equations resulting from it will fall into two parts. The nontrivial part will have a structure identical with that of Eq. (2); that is, with the equations for maximal embedding of $\vec{T}$ in SU(D), while the remaining set of equations will say that certain $(N-D)$ scalar field components have zero spatial derivative. Thus our problem gets mapped into another problem whose solution is already known.

We shall write the various monopole *Ansätze* for SU(5) using the notation of Ref. 2. We shall, moreover, impose the requirement of invariance under simultaneous inversion of $\hat{r}$ and $\vec{T}$ to simplify the expressions for the vector field. The asymptotic Higgs field will be taken to be $\Phi_\infty = \text{diag}(1,1,1,-\frac{3}{2},-\frac{3}{2})$. Let us now consider various cases.

(a) $\underline{5} \to \underline{2}+\underline{1}+\underline{1}+\underline{1}$. This embedding has $\vec{T}=\text{diag}(0,0,\vec{\tau}/2,0)$. The fields are[2]

$$g\vec{W}(r)=\frac{K(r)-1}{r}(\vec{T}\times\hat{r}) \ , \tag{4}$$

$$\Phi(r)=\text{diag}(\phi_1,\phi_1,\phi_2+\phi_3\hat{r}\cdot\vec{\tau},-2(\phi_1+\phi_2)) \ . \tag{5}$$

The self-duality equations that result from the above are

$$\phi_1'=\phi_2'=0 \ , \tag{6a}$$

$$\frac{K'}{K}=-2\phi_3, \quad r^2(2\phi_3)'=1-K^2 \ . \tag{6b}$$

Equation (6b) is solved by a rescaled version of the celebrated Prasad-Sommerfield[5] solution

$$K=\frac{ar}{\sinh ar}, \quad 2\phi_3=a\coth ar-\frac{1}{r} \tag{7}$$

with $a=\frac{5}{2}$ and $\phi_1=1$, $\phi_2=-\frac{1}{4}$ to satisfy the desired $r\to\infty$ behavior of Φ.

(b) $\underline{5}\to\underline{3}+\underline{1}+\underline{1}$. This case has $\vec{T}=\text{diag}(0,0,\vec{T}_1)$, with $\vec{T}_1$ the standard spin-1 representation. The fields are[2]

$$g\vec{W}(r)=-\frac{\vec{T}\times\hat{r}}{r}+\frac{1}{r}\{K_0(r)+K_1(r)\hat{r}\cdot\vec{T} + K_2(r)(\hat{r}\cdot\vec{T})^2,\vec{T}\times\hat{r}\} \ , \tag{8}$$

$$\Phi(\vec{r})=\text{diag}(-\tfrac{3}{2}\phi_1-\phi_3,-\tfrac{3}{2}\phi_1-\phi_3,\phi_1,\phi_1,\phi_1) + \phi_2(r)\hat{r}\cdot\vec{T}+\phi_3(\hat{r}\cdot\vec{T})^2 \ , \tag{9}$$

where the curly brackets in (8) denote an anticommutator. The resulting self-duality equations are

$$\begin{aligned} &K_+'=-K_+\phi_+, \quad r^2\phi_+'=1+K_-^2-2K_+^2 \ , \\ &K_-'=-K_-\phi_-, \quad r^2\phi_-'=1+K_+^2-2K_-^2, \\ &(\tfrac{3}{2}\phi_1+\phi_3)'=0 \ , \end{aligned} \tag{10}$$

where

$$\phi_\pm=\phi_2\pm\phi_3, \quad K_\pm=K\pm K_1, \quad K=2K_0+K_2 \ . \tag{11}$$

We notice that although three radial functions K_0,K_1,K_2 appear in Eq. (8), the vector field involves only two radial functions since K_0 and K_2 enter always via a fixed linear combination given in Eq. (11). The expression for M_+ involves only two b entries which are K_+ and K_-. This is a general situation found for other embeddings as well (see below). Now it is clear that the nontrivial part of Eq. (10) is of identical structure as Eq. (2) for the SU(3) gauge theory. Thus we obtain the solution

$$\begin{aligned} &K_+^2=\tfrac{9}{2}a^2r^2\frac{(3ar-1)e^{ar}+e^{-2ar}}{[e^{2ar}-(3ar+1)e^{-ar}]^2} \ , \\ &K_-^2(r)=K_+^2(-r) \\ &\phi_+(r)=-\frac{1}{r}+a\frac{2e^{2ar}+(3ar-2)e^{-ar}}{e^{2ar}-(3ar+1)e^{-ar}} \\ &\qquad -\frac{a}{2}\frac{(3ar+2)e^{ar}-2e^{-2ar}}{(3ar-1)e^{ar}+e^{-2ar}} \ , \\ &\phi_-(r)=-\phi_+(-r), \quad \phi_1=\tfrac{2}{3}c-\tfrac{1}{3}(\phi_+-\phi_-) \ , \end{aligned} \tag{12}$$

where $a=\frac{5}{3}$ and the constant $c=-1$ again to match the $r\to\infty$ behavior of $\Phi(r)$. The above is essentially a rescaled version of the solution for SU(3) gauge theory.[3,6]

(c) $\underline{5}\to\underline{4}+\underline{1}$. Here we have $\vec{T}=\text{diag}(\vec{t}_{3/2},0)$ and $\vec{t}_{3/2}$ are the standard spin-$\frac{3}{2}$ representation of SU(2) generators. The *Ansätze* for the fields are[2]

$$g\vec{W}(r)=-\frac{\vec{T}\times\hat{r}}{r}+\frac{1}{r}\{K(r)+K_1(r)\hat{r}\cdot\vec{T}+K_2(r)(\hat{r}\cdot\vec{T})^2 + K_3(r)(\hat{r}\cdot\vec{T})^3,\vec{T}\times\hat{r}\} \ , \tag{13}$$

$$\Phi(r)=\text{diag}(\phi_0,\phi_0,\phi_0,\phi_0,-4\phi_0-5\phi_2) + \phi_1\hat{r}\cdot\vec{T}+\phi_2(\hat{r}\cdot\vec{T})^2+\phi_3(\hat{r}\cdot\vec{T})^3 \ . \tag{14}$$

The self-duality equations are found to be

$$\begin{aligned} &(\phi_0+\tfrac{5}{4}\phi_2)'=0, \quad J_1'=-J_1\psi_1 \ , \\ &J_2'=-J_2\psi_2, \quad J_3'=-J_3\psi_3 \ , \end{aligned} \tag{15}$$

$$\begin{aligned} &r^2\psi_1'=1+2J_2^2-3J_1^2, \quad r^2\psi_3'=1+2J_2^2-3J_3^2 \ , \\ &r^2\psi_2'=1+\tfrac{3}{2}(J_1^2+J_3^2)-4J_2^2 \ , \end{aligned} \tag{16}$$

where

$$\begin{aligned} &J_1=2K_0+\tfrac{5}{2}K_2+2K_1+\tfrac{7}{2}K_3, \quad J_2=2K_0+\tfrac{1}{2}K_2 \ , \\ &J_3=2K_0+\tfrac{5}{2}K_2-2K_1-\tfrac{7}{2}K_3 \ , \end{aligned} \tag{17}$$

and

$$\begin{aligned} &\psi_1=\phi_1+2\phi_2+\tfrac{13}{4}\phi_3, \quad \psi_2=\phi_1+\tfrac{1}{4}\phi_3 \ , \\ &\psi_3=\phi_1-2\phi_2+\tfrac{13}{4}\phi_3 \ . \end{aligned} \tag{18}$$

As before, we obtain the solution of the above equations:

$$\begin{aligned} &J_1^2=\tfrac{64}{3}a^2r^2\frac{Q_2}{Q_1^2}, \quad J_2^2=\frac{a^2}{4}r^2\frac{Q_1Q_3}{Q_2^2}, \quad J_3^2=\tfrac{64}{3}a^2r^2\frac{Q_2}{Q_3^2} \ , \\ &\psi_1=-\frac{1}{r}+a\frac{P_1}{Q_1}-\frac{a}{2}\frac{P_2}{Q_2}, \quad \psi_3=-\frac{1}{r}+a\frac{P_3}{Q_3}-\frac{a}{2}\frac{P_2}{Q_2} \ , \\ &\psi_2=-\frac{1}{r}+a\frac{P_2}{Q_2}-\frac{a}{2}\left[\frac{P_1}{Q_1}+\frac{P_3}{Q_3}\right], \quad \phi_0=\tfrac{3}{8}-\tfrac{5}{16}(\psi_1-\psi_3) \ , \end{aligned} \tag{19}$$

where

$$\begin{aligned} &Q_1=(8a^2r^2-4ar+1)e^{ar}-e^{-3ar}, \quad Q_3(r)=-Q_1(-r) \ , \\ &Q_2=ar(2ar-1)e^{2ar}+ar(2ar+1)e^{-2ar} \ , \end{aligned} \tag{20}$$

and

$$\begin{aligned} &P_1=(8a^2r^2+12ar-3)e^{ar}+3e^{-3ar}, \quad P_3(r)=P_1(-r) \ , \\ &P_2=(4a^2r^2+2ar-1)e^{2ar}-(4a^2r^2-2ar-1)e^{-2ar} \ . \end{aligned} \tag{21}$$

The constant $a=\frac{5}{4}$ to match the $r\to\infty$ behavior of Φ. Thus our solution is a rescaled version of the SU(4) solution of Bais and Weldon.[5]

(d) $\underline{5}\to\underline{5}$. This case corresponds to the maximal embedding of $\vec{T}$ in SU(5), with $\vec{T}$ the standard spin-2 representation. The *Ansätze* for the fields are[2]

$$g\vec{W}(r)=-\frac{\vec{T}\times\hat{r}}{r}+\frac{1}{r}\{K_0+K_1\hat{r}\cdot\vec{T}+K_2(\hat{r}\cdot\vec{T})^2 + K_3(\hat{r}\cdot\vec{T})^3+K_4(\hat{r}\cdot\vec{T})^4,\vec{T}\times\hat{r}\} \ , \tag{22}$$

$$\Phi(r)=-2\phi_2-\tfrac{34}{5}\phi_4+\phi_1\hat{r}\cdot\vec{T}+\phi_2(\hat{r}\cdot\vec{T})^2 + \phi_3(\hat{r}\cdot\vec{T})^3+\phi_4(\hat{r}\cdot\vec{T})^4 \ . \tag{23}$$

We get the self-duality equations in the form

$$J_1' = -J_1\psi_1,\quad J_2' = -J_2\psi_2,\quad J_3' = -J_3\psi_3,\quad J_4' = -J_4\psi_4, \tag{24}$$

$$\begin{aligned} r^2\psi_1' &= 1+3J_2^2-4J_1^3,\quad r^2\psi_4' = 1+3J_3^2-4J_4^2\ ,\\ r^2\psi_2' &= 1+2J_1^2+3J_3^2-6J_2^2,\\ r^2\psi_3' &= 1+2J_4^2+3J_2^2-6J_3^2\ , \end{aligned} \tag{25}$$

where

$$\begin{aligned} &J_i(i=1,4)=2K_0+5K_2+17K_4+\eta(3K_1+9K_3)\ ,\\ &J_i(i=2,3)=2K_0+K_2+K_4+\eta(K_1+K_3)\ ,\\ &\psi_i(i=1,4)=\phi_1+7\phi_3+\eta(3\phi_2+15\phi_4)\ ,\\ &\psi_i(i=2,3)=\phi_1+\phi_3+\eta(\phi_2+\phi_4)\ , \end{aligned} \tag{26}$$

and $\eta=+1$ for $i=1,2$ and $\eta=-1$ for $i=3,4$. Solution of the self-duality equations for this case has been found elsewhere[7] as part of a general approach to extract explicit solutions from the formalism developed recently by Ganoulis, Goddard, and Olive[8] based on the modified Toda molecule equations. From the results of Ref. 7, it is straightforward, if somewhat tedious, to obtain the following expressions:

$$J_1^2=\frac{25a^2r^2}{D_1^2}D_2,\quad J_2^2=\tfrac{25}{24}a^2r^2\frac{D_1D_3}{D_2^2}\ , \tag{27}$$

$$J_3^2(r)=J_2^2(-r),\quad J_4^2(r)=J_1^2(-r)\ , \tag{28}$$

where

$$D_1=(50a^2r^2-80ar+48)e^{ar}-(40ar+48)e^{-3ar/2}\ , \tag{29}$$

$$\begin{aligned} D_2=&\ (50a^2r^2-80ar+16)e^{2ar}\\ &+(125a^3r^3+50a^2r^2+80ar-32)e^{-ar/2}+16e^{-3ar} \end{aligned} \tag{30}$$

and

$$D_3(r)=D_2(-r)\ . \tag{31}$$

As for the Higgs field, it is convenient to quote the result in terms of the functions χ_i $(i=1,2,3,4)$ defined as

$$\chi_i=2\sum_{j=1}^{4}A_{ij}\left(\psi_j+\frac{1}{r}\right)\ , \tag{32}$$

where A_{ij} is the inverse of the Cartan matrix[9] of SU(5). We then have

$$\begin{aligned} \chi_i=\frac{a}{D_1}\{&(50a^2r^2+20ar-32)e^{ar}\\ &+(60ar+32)e^{-3ar/2}\}\ , \end{aligned} \tag{33}$$

$$\begin{aligned} \chi_2=\frac{a}{D_2}\{&(100a^2r^2-60ar-48)e^{2ar}\\ &+(-\tfrac{125}{2}a^3r^3+350a^2r^2\\ &+60ar+96)e^{-ar/2}-48e^{-3ar}\}\ , \end{aligned} \tag{34}$$

$$\chi_3(r)=-\chi_2(-r),\quad \chi_4(r)=-\chi_1(-r)\ , \tag{35}$$

with the constant $a=2$ to match the chosen $r\to\infty$ behavior of Φ.

We shall consider the asymptotic behavior as $r\to\infty$ of the B field, that is, of Φ'. It is easy to show that for the maximal $\vec{\mathrm{T}}$-embedding, case D, the asymptotic B field is parallel to the asymptotic Higgs field; and thus the color magnetic field vanishes. For the three cases A, B, and C of the nonmaximal $\vec{\mathrm{T}}$ embeddings considered, the color magnetic field does not vanish. From the asymptotic B field we find that the magnetic charge of the solutions B, C, and D is twice, thrice, and six times, respectively, the magnetic charge of the solution A. We have also checked this conclusion by making a direct calculation of the monopole masses. The monopole mass for the cases B, C, and D are found to be, respectively, twice, thrice, and six times the mass of the lightest monopole, case A. The monopole mass is a common constant times a certain one-dimensional integral which can be evaluated by expressing the integrand, by use of the self-duality equations, as a perfect derivative. Let us illustrate this procedure for our case D, which is "most difficult" of all the cases. First we obtain

$$\int \mathrm{Tr}(\vec{\mathrm{D}}\Phi)^2d^3x=4\pi I,\quad I=\tfrac{1}{5}\int_0^\infty F(r)dr\ , \tag{36}$$

where

$$\begin{aligned} F(r)=r^2\{&4[(\psi_1')^2+(\psi_4')^2]+6[(\psi_2')^2+(\psi_3')^2]\\ &+2\psi_1'\psi_4'+8\psi_2'\psi_3'+4(\psi_1'\psi_3'+\psi_2'\psi_4'+6(\psi_1'\psi_2'+\psi_3'\psi_4')\}\\ &+20(J_1^2\psi_1^2+J_4^2\psi_4^2)+30(J_2^2\psi_2^2+J_3^2\psi_3^2)\ . \end{aligned} \tag{37}$$

Next, we derive, using Eqs. (24) and (25), the relation

$$\begin{aligned} F(r)=\frac{d}{dr}\{&10\psi_1(1-J_1^2)+15\psi_2(1-J_2^2)\\ &+15\psi_3(1-J_3^2)+10\psi_4(1-J_4^2)\}\ . \end{aligned} \tag{38}$$

Recalling the asymptotic behavior [that result from Eqs. (27)–(35)]

$$\psi_1(\infty)=\psi_2(\infty)=\psi_4(\infty)=0,\quad \psi_3(\infty)=\tfrac{5}{2}\ ,$$

$$J_1^2(\infty)=\tfrac{1}{2},\quad J_2^2(\infty)=\tfrac{1}{3},\quad J_3^2(\infty)=0,\quad J_4^2(\infty)=\tfrac{1}{4}\ ,$$

the integral I is now easily evaluated to be $\frac{15}{2}$ (the lower limit gives no contribution). The remaining cases are treated similarly; the corresponding integral is found to be $\frac{5}{4}$ (case A), $\frac{5}{2}$ (case B), and $\frac{15}{4}$ (case C).

We end this note with a remark on the role of the fundamental Higgs field H for two cases characterized by the $\vec{\mathrm{T}}$ embeddings $\underline{5}\to\underline{2}+\underline{1}+\underline{1}+\underline{1}$ (case A) and $\underline{5}\to\underline{4}+\underline{1}$ (case C). For these two cases the *Ansatz*[2] for H is $H=\mathrm{col}(0,0,0,0,h(r))$. We may now obtain the desired field equations by the usual procedure of minimizing the energy integral. It is now clear, from the form of the covariant derivative of H, that equations governing H will be decoupled from the equations governing the adjoint Higgs field Φ and the gauge field $\vec{\mathrm{W}}$. Specifically, we find that in the Prasad-Sommerfield limit of no Higgs potential, the resulting equations have the following structure. They consist of a set of second-order differential equations, which are precisely the ones that would result from the corresponding self-duality equations upon differentiation, plus the additional equation for $h(r)$:

$$h''+\frac{2}{h}h'=0\ . \tag{39}$$

The only solution of the above equation with the correct behavior at the origin (finite energy) is $h=\mathrm{const}=h(\infty)$. Thus the fundamental Higgs field plays no role in the Prasad-Sommerfield limit for the two cases considered.

Note added. Since writing this paper, I have learned that Carl Gardner (private communication) has also studied self-dual monopole solutions in SU(5). He has independently obtained our solutions A and D and two other, different, solutions.

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