Electromagnetic mass models in general relativity

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The metric coefficients g_{00} and g_{11} of both the Schwarzschild and Reissner-Nordström metrics satisfy the relation $g_{00}\,g_{11}=-1$. A coordinate-independent statement of this relation using the eigenvalues of the Einstein tensor is given. By considering the relation between the metric coefficients to be valid inside a charged perfect-fluid distribution, it is shown that the mass-energy density and the pressure of the distribution are of electromagnetic origin. In the absence of charge, however, there exists no interior solution. A particular solution which confirms the same and matches smoothly with the exterior Reissner-Nordström metric is obtained. This solution represents a charged particle whose mass is entirely of electromagnetic origin.

I. INTRODUCTION

The exterior field of a spherically symmetric charged fluid (dust or perfect-fluid) distribution described by the metric

$$ds^2=g_{ij}dx^i dx^j= e^{\nu}dt^2-e^{\lambda}dr^2-r^2(d\theta^2+\sin^2\theta\, d\phi^2) \quad (i,j=0,1,2,3) \tag{1.1}$$

is the unique Reissner-Nordström solution given by

$$ds^2=\left(1-\frac{2m}{r}+\frac{q^2}{r^2}\right)dt^2-\left(1-\frac{2m}{r}+\frac{q^2}{r^2}\right)^{-1}dr^2 - r^2(d\theta^2+\sin^2\theta\, d\phi^2) \tag{1.2}$$

for which

$$g_{00}\,g_{11}=-1 \ . \tag{1.3}$$

For the metric (1.1), the relation (1.3) is equivalent to

$$\lambda=-\nu \ . \tag{1.4}$$

It is well known that (1.2) in the absence of charge reduces to the Schwarzschild exterior solution

$$ds^2=\left(1-\frac{2m}{r}\right)dt^2-\left(1-\frac{2m}{r}\right)dr^2 - r^2(d\theta^2+\sin^2\theta\, d\phi^2) \ , \tag{1.5}$$

which is also unique and satisfies (1.3). Such uniqueness or simplicity is not possible in the case of spherically symmetric interior fields. Attempts have, therefore, been made by various authors[1-6] to find exact interior solutions in the presence as well as in the absence of charge under different assumptions.

The present work is based on the fact that any interior solution with or without charge must match with either (1.2) or (1.5), as the case may be, on the boundary of the interior field. As such the relation (1.3), which is valid for the exterior fields (1.2) and (1.5) if assumed to be also valid in the interior, will have a natural matching on the boundary. This assumption which is motivated from the unique solutions (1.2) and (1.5) is quite natural and may lead to some solutions which may have interesting physical consequences.

To this end we have examined the case in which the interior is first filled with a perfect fluid. This, however, leads to the vanishing of the pressure and density of the fluid, i.e., the incompatibility of the perfect fluid with assumption (1.3). In the next case, i.e., when the interior is filled with a charged perfect fluid, all the physical quantities, namely, pressure, density, total gravitational mass of the fluid, etc., are dependent on the charge and vanish when the charge vanishes. The interior field thus obtained represents the model of a charged source whose mass is completely of electromagnetic origin. This type of model is of considerable importance in the history of physics.[7] In this paper a particular solution representing such a model is obtained.

In Secs. II and III, the Einstein-Maxwell field equations for a charged perfect fluid and certain general deductions from these field equations are given. Section IV deals with a particular model whose mass is entirely of electromagnetic origin. In Sec. V a coordinate-independent statement of (1.3) is given using the eigenvalues of the Einstein tensor.

II. FIELD EQUATIONS

The spherically symmetric metric in the coordinates t,r,θ,ϕ is given by

$$ds^2=e^{\nu}dt^2-e^{\lambda}dr^2-r^2(d\theta^2+\sin^2\theta\, d\phi^2) \ , \tag{2.1}$$

where λ and ν are functions of the radial coordinate r only. We consider the charged fluid to be confined within a sphere of radius a and is described by the proper mass density $\rho(r)$, the pressure $p(r)$, and the charge density $\sigma(r)$.

The Einstein-Maxwell field equations for the charged fluid distribution are given by

$$G^i{}_j=-8\pi[T^i{}_{j(m)}+T^i{}_{j(\mathrm{em})}] \ , \tag{2.2}$$

$$[(-g)^{1/2}F^{ij}]_{,j}=(-g)^{1/2}J^i \ , \tag{2.3}$$

$$F_{[ij,k]}=0 \ , \tag{2.4}$$

where $G^i{}_j$ is the well-known Einstein tensor and J^i is the current four-vector defined by

$$J^i = \sigma(r) u^i \ , \tag{2.4a}$$

$\sigma(r)$ being the proper charge density of the distribution. In the present case the current four-vector has only one nonvanishing component, viz.,

$$J^0 = \sigma(r)(g_{00})^{-1/2} \ . \tag{2.4b}$$

Here a subscript after a semicolon or a comma denotes covariant or partial differentiation, respectively.

$T^i{}_{j(m)}$ and $T^i{}_{j(\mathrm{em})}$ are the energy-momentum tensors of the perfect fluid and the electromagnetic field, respectively, and are given by

$$T^i{}_{j(m)} = (\rho + p) u^i u_j - p\delta^i{}_j \tag{2.5}$$

and

$$T^i{}_{j(\mathrm{em})} = \frac{1}{4\pi}\left(-F_{jl}F^{il} + \tfrac{1}{4}\delta^i{}_j F_{lm}F^{lm}\right) \ , \tag{2.6}$$

where u^i is the four-velocity of the fluid satisfying

$$u^i u_i = 1, \quad u^0 = (g_{00})^{-1/2} \ , \tag{2.7}$$

and F_{ij} is the Maxwell tensor defined in terms of the four-potential A_i as

$$F_{ij} = A_{i,j} - A_{j,i} \ . \tag{2.8}$$

In our case the four-potential is given by

$$A_i = [\Phi(r), 0, 0, 0], \quad i = 0, 1, 2, 3 \ .$$

Due to spherical symmetry the only nonvanishing component of the Maxwell tensor is

$$F_{01} = -F_{10} = \Phi' \ . \tag{2.9}$$

Here and in what follows a prime will denote differentiation with respect to r only.

The field equations (2.2)–(2.4) finally reduce to

$$e^{-\lambda}\left[\frac{\lambda'}{r} - \frac{1}{r^2}\right] + \frac{1}{r^2} = 8\pi T^0{}_0 = 8\pi\rho + E^2 \ , \tag{2.10}$$

$$e^{-\lambda}\left[\frac{\nu'}{r} + \frac{1}{r^2}\right] - \frac{1}{r^2} = 8\pi p - E^2 \ , \tag{2.11}$$

$$e^{-\lambda}\left[\frac{\nu''}{2} - \frac{\lambda'\nu'}{4} + \frac{\nu'^2}{4} + \frac{\nu' - \lambda'}{2r}\right] = 8\pi p + E^2 \ , \tag{2.12}$$

$$[r^2 E(r)]' = 4\pi r^2 \sigma(r) e^{\lambda/2} \ , \tag{2.13}$$

where the "electric field strength" $E(r)$ is defined as

$$E(r) = -e^{-(\nu+\lambda)/2}\Phi' \ . \tag{2.14}$$

III. CERTAIN GENERAL RESULTS

From Eq. (2.10), one gets

$$e^{-\lambda} = 1 - \frac{2M(r)}{r} \ , \tag{3.1}$$

where

$$M(r) = \int_0^r 4\pi r^2 T^0{}_0 \, dr = 4\pi \int_0^r r^2\left[\rho + \frac{E^2}{8\pi}\right] dr \ . \tag{3.2}$$

We take the integration constant to be zero in order to ensure the regularity of the metric throughout the fluid distribution. Since the radius of the charged fluid extends up to a, for $r > a$,

$$e^{-\lambda} = 1 - \frac{2m}{r} + \frac{q^2}{r^2} \ , \tag{3.3}$$

where m and q are the total gravitational mass and the total charge, respectively.

For $r > a$, from (3.1), (3.2), and (3.3) we get

$$m = \frac{q^2}{2r} + 4\pi \int_0^r r^2\left[\rho + \frac{E^2}{8\pi}\right] dr \ . \tag{3.4}$$

Equation (2.14) gives

$$E(r) = \frac{1}{r^2}\int_0^r 4\pi r^2 \sigma(r) e^{\lambda/2} dr = \frac{q(r)}{r^2} \ . \tag{3.5}$$

Differentiating (2.11) with respect to r and adding this to (2.11) and then subtracting (2.12), we get

$$\frac{dp}{dr} = -(\rho + p)\frac{\nu'}{2} + \frac{1}{8\pi r^4}\frac{d}{dr}[q^2(r)] \ , \tag{3.6}$$

where

$$\frac{\nu'}{2} = \left[M(r) + 4\pi r^3 p - \frac{q^2(r)}{2r}\right] \Big/ r[r - 2M(r)] \ .$$

Equation (3.6) is the generalization of the well-known Tolman-Oppenheimer-Volkoff (TOV) equation of hydrostatic equilibrium to the case when charge is present. This can also be derived from the conservation equation $T^j{}_{1;j} = 0$, where

$$T^j{}_1 = T^j{}_{1(\mathrm{em})} + T^j{}_{1(m)} \ .$$

Adding Eqs. (2.10) and (2.11) we get

$$e^{-\lambda}(\lambda' + \nu') = 8\pi r(\rho + p) \ . \tag{3.7}$$

The relation (3.7) contains no charge term and is one of the characteristics of the present model. It holds irrespective of whether the body is charged or not. The presence of charge is detected by the generalized TOV equation (3.6).

We now assume the relation (1.3) or (1.4) to be valid within the charged fluid distribution. This when substituted in (3.7) gives[8]

$$\rho = -p \ . \tag{3.8}$$

Using Eq. (3.8) in Eq. (3.6), we get

$$\frac{dp}{dr} = \frac{1}{8\pi r^4}\frac{d}{dr}[q^2(r)] \ . \tag{3.9}$$

From (3.8) it is evident that the pressure p must be negative inside the charged sphere; that is, the body must be under tension. Also since the pressure vanishes at $r = a$, it must be an increasing function of r: i.e., dp/dr in (3.9) is positive in $0 < r < a$. Obviously the mass density decreases as r increases.

In the absence of charge Eq. (3.9), using the condition that the pressure vanishes on the boundary $r = a$, leads to the vanishing of ρ and p identically. Hence an uncharged perfect fluid is incompatible with the condition (1.3).

However, the situation is altogether different in the presence of charge. In this case Eqs. (3.4), (3.5), (3.8), and

(3.9) which determine the total gravitational mass, the pressure, and the mass density show that all these three physical quantities are dependent on charge density σ and vanish when the charge vanishes. Thus, the total gravitational mass of this model comes out to be of electromagnetic origin only. This is also obvious from the arguments given in the preceding paragraph. The case (1.3) thus gives a class of models where mass is derived completely from the charge of the electromagnetic fields. Such models whose mass is built from electromagnetism alone are of considerable importance in physics.[7]

IV. A PARTICULAR SOLUTION

We assume here the charge distribution to be known and the condition (1.4) to be valid throughout the interior; i.e., we take

$$\sigma(r)=\sigma_0 e^{-\lambda/2} \ , \tag{4.1}$$

where σ_0 is the constant charge density at $r=0$, the center of the charged fluid. Equation (4.1) when substituted in (3.4), (3.5), (3.8), and (3.9) gives the solution

$$E(r)=\frac{4\pi}{3}\sigma_0 r \ ,$$

$$q(r)=\frac{4\pi}{3}\sigma_0 r^3 \ ,$$

$$p(r)=\frac{2\pi}{3}\sigma_0^2(r^2-a^2) \ ,$$

$$\rho(r)=\frac{2\pi}{3}\sigma_0^2(a^2-r^2) \ ,$$

$$e^{\nu}=e^{-\lambda}=1-2M(r)/r \ ,$$

where

$$M(r)=\frac{8\pi^2}{45}\sigma_0^2 r^3(5a^2-2r^2) \ .$$

The total gravitational mass m is given by

$$m=\tfrac{64}{45}\pi^2\sigma_0^2 a^5 \ .$$

The above solution illustrates the conclusions of the preceding section.

V. A COORDINATE-INDEPENDENT STATEMENT FOR EQ. (1.3)

A coordinate-independent statement of the relation (1.3) [equivalently of (1.4)] is obtained by using the eigenvalues of the Einstein tensor $G^i{}_j$. The eigenvalues are determined by the equation

$$\det[G^2{}_j-\mu\delta^i{}_j]=0 \ . \tag{5.1}$$

In the case of a static spherically symmetric space-time (2.1) only the diagonal components of the Einstein tensor survive and it can be seen from (5.1) that each of these components is itself an eigenvalue. Thus we have

$$\mu_0=G^0{}_0,\quad \mu_1=G^1{}_1,\quad \mu_2=G^2{}_2,\quad \mu_3=G^3{}_3 \ . \tag{5.2}$$

On substituting the values of $G^0{}_0$ and $G^1{}_1$ in Eq. (5.2) one can easily verify that

$$\mu_1-\mu_0=G^1{}_1-G^0{}_0=\frac{e^{-\lambda}}{r}(\lambda'+\nu') \ . \tag{5.3}$$

Using Eq. (1.4) in Eq. (5.3), we get

$$\mu_1-\mu_0=0 \ . \tag{5.4}$$

On the other hand the condition (5.4) together with (5.3) and the boundary condition ($\lambda=-\nu$) on the surface of the fluid distribution gives the relation (1.3). Hence Eq. (5.4) is a necessary and sufficient condition for a static spherically symmetric space-time to satisfy the relation (1.3). As μ_0 and μ_1 are coordinate-independent scalars the condition (5.4) is a coordinate-independent statement of (1.3).

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[1]W. B. Bonnor, Z. Phys. 160, 727 (1960).
[2]C. F. Kyle and A. W. Martin, Nuovo Cimento 50, 583 (1967).
[3]S. J. Wilson, Can. J. Phys. 47, 2401 (1969).
[4]K. D. Krori and J. Barua, J. Phys. A 8, 508 (1975).
[5]A. Nduka, Gen. Relativ. Gravit. 7, 501 (1976).
[6]T. Singh and R. B. S. Yadav, Acta Phys. Pol. B 9, 475 (1978).
[7]R. P. Feynman, R. B. Leighton, and M. Sands, *The Feynman Lectures on Physics* (Addison-Wesley, Palo Alto, 1964), Vol. II, Chap. 28.
[8]It is interesting to note that when $\rho=-p$, by using boundary conditions we automatically get $\lambda=-\nu$; so that $\lambda=-\nu \Leftrightarrow \rho=-p$.