Axially symmetric, static self-dual Yang-Mills and stationary Einstein-gauge field equations

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It is shown that a restricted class of static, axially symmetric self-dual $SU(n+1)$ Yang-Mills field equations are equivalent to the stationary, axially symmetric Einstein–$(n-1)$-Maxwell field equations.

Equivalence between the axially symmetric (AS), static self-dual Yang-Mills and the As stationary Einstein field equations was first demonstrated by Witten.[1] When the space-time is AS and stationary, the essential part of the vacuum field equations can be reduced to a complex nonlinear elliptic partial differential equation, known as the Ernst[2] equation. Witten has shown that members of a special class of the AS static self-dual SU(2) Yang-Mills equations, in Yang's R gauge, also reduce to the same equation. It is also known that the Ernst equation can be formulated as a σ model on the symmetric space SU(1,1)/U(1). Recently,[3] it was shown that this analogy exists also between the AS static self-dual SU(3) Yang-Mills and the AS stationary Einstein-Maxwell field equations. Here the key equations are the electrovacuum Ernst[4] equations, which may be formulated as a σ model on $SU(2,1)/SU(2) \otimes U(1)$.

In this work we shall show that the above analogy between the field equations of two theories can be further extended. The restricted class of the AS static self-dual $SU(n+1)$ Yang-Mills and the AS stationary Einstein–Abelian-gauge field equations will be both formulated as a σ model on the Kähler manifold $SU(n+1)/SU(n) \otimes U(1)$.

The Einstein–Abelian-gauge field equations are given by

$$G_{\mu\nu}=\gamma_{ab}(F^a{}_{\mu\alpha}F_\nu{}^{b\alpha}-\tfrac{1}{4}g_{\mu\nu}F^a{}_{\alpha\beta}F^{b\alpha\beta}) \ , \tag{1}$$

$$F^{a\mu\nu}{}_{;\nu}=0 \ , \tag{2}$$

$$F^a{}_{\mu\nu}=\partial_\mu A^a{}_\nu-\partial_\nu A^a{}_\mu \ , \tag{3}$$

where γ_{ab} is a diagonal matrix (with positive-definite entries), $a,b=1,2,\ldots,n-1$ $(n>0)$. A semicolon denotes covariant differentiation with respect to the Christoffel-Riemann connection. The metric γ_{ab} of the Abelian-gauge group can be taken as the Kronecker symbol δ_{ab} by rescaling the gauge potentials $A^a{}_\mu$. When the space-time is stationary and axially symmetric the line element can be expressed as

$$ds^2=-f(dt-\omega d\phi)^2+f^{-1}[e^{2\gamma}(d\rho^2+dz^2)+\rho^2d\phi^2] \ , \tag{4}$$

where t, ρ, z, and ϕ are the local coordinates and the functions f, ω, and γ depend only on ρ and z. We shall assume that the gauge potential one-form A^a has two components,

$$A^a\equiv A^a{}_\mu dx^\mu=A^a{}_t\,dt+A^a{}_\phi\,d\phi \ , \tag{5}$$

where $A^a{}_t$ and $A^a{}_\phi$ are the components of $A^a{}_\mu$ in the direction of time (t) and the azimuthal angle (ϕ) coordinates, respectively. They also depend on ρ and z. Using (4) and (5), the gauge (2) and the Einstein (1) field equations reduce to

$$\vec{\nabla}\cdot[\rho^{-2}(\vec{\nabla}A^a{}_\phi-\omega\vec{\nabla}A^a{}_t)]=0 \ , \tag{6a}$$

$$\vec{\nabla}\cdot[f^{-1}\vec{\nabla}A^a{}_t+\rho^{-2}f\omega(\vec{\nabla}A^a{}_\phi-\omega\vec{\nabla}A^a{}_t)]=0 \ , \tag{6b}$$

$$\vec{\nabla}\cdot[\rho^{-2}f^2\vec{\nabla}\omega-4\rho^{-2}fA^a{}_t(\vec{\nabla}A^a{}_\phi-\omega\vec{\nabla}A^a{}_t)]=0 \ , \tag{6c}$$

$$f\nabla^2f=(\vec{\nabla}f)^2-\rho^{-2}f^4(\vec{\nabla}\omega)^2+2f\vec{\nabla}A^a{}_t\cdot\vec{\nabla}A^a{}_t+2\rho^{-2}f^3(\vec{\nabla}A^a{}_\phi-\omega\vec{\nabla}A^a{}_t)\cdot(\vec{\nabla}A^a{}_\phi-\omega\vec{\nabla}A^a{}_t) \ , \tag{6d}$$

$$\begin{aligned}\gamma_{,\rho}=&\tfrac{1}{4}f^{-2}[(f_{,\rho})^2-(f_{,z})^2]-\tfrac{1}{4}\rho^{-1}f^2[(\omega_{,\rho})^2-(\omega_{,z})^2]-\rho f^{-1}(A^a{}_{t,\rho}A^a{}_{t,\rho}-A^a{}_{t,z}A^a{}_{t,z})\\&+\rho^{-1}f[(A^a{}_{\phi,\rho}-\omega A^a{}_{t,\rho})(A^a{}_{\phi,\rho}-\omega A^a{}_{t,\rho})-(A^a{}_{\phi,z}-\omega A^a{}_{t,z})(A^a{}_{\phi,z}-\omega A^a{}_{t,z})] \ ,\end{aligned} \tag{6e}$$

$$\gamma_{,z}=\tfrac{1}{2}\rho f^{-2}f_{,\rho}f_{,z}-\tfrac{1}{2}\rho^{-1}f^2\omega_{,\rho}\omega_{,z}-2\rho f^{-1}A^a{}_{t,\rho}A^a{}_{t,z}+2\rho^{-1}f(A^a{}_{\phi,\rho}-\omega A^a{}_{t,\rho})(A^a{}_{\phi,z}-\omega A^a{}_{t,z}) \ , \tag{6f}$$

$$\gamma_{\rho\rho}+\gamma_{zz}+\tfrac{1}{4}f^{-2}(\vec{\nabla}f)^2+\tfrac{1}{4}\rho^{-2}f^2(\vec{\nabla}\omega)^2-f^{-1}\vec{\nabla}A^a{}_t\cdot\vec{\nabla}A^a{}_t-\rho^{-2}(\vec{\nabla}A^a{}_\phi-\omega\vec{\nabla}A^a{}_t)\cdot(\vec{\nabla}A^a{}_\phi-\omega\vec{\nabla}A^a{}_t)=0 \ , \tag{6g}$$

where $\vec{\nabla}$, $\vec{\nabla}\cdot$, and ∇^2 are the grad, divergence, and the Laplacian operators in the flat three-dimensional cylindrical coordinates, and those terms with repeated indices are summed up. Out of these seven equations only the first six are independent. Equations (6e) and (6f) imply the last equation (6g). On the other hand, Eqs. (6e) and (6f) determine the function γ if f, ω, $A^a{}_t$, and $A^a{}_\phi$ are known. Hence the essential part of the Einstein- and Abelian-gauge field equations consist of the first four equations (6a)–(6d). Introducing n complex scalars ϵ and Φ^a these equations are written as

$$f\nabla^2\epsilon=(\vec{\nabla}\epsilon+2\bar{\Phi}^a\vec{\nabla}\Phi^a)\cdot\vec{\nabla}\epsilon \ , \tag{7}$$

$$f\nabla^2\Phi^a=(\vec{\nabla}\epsilon+2\bar{\Phi}^b\vec{\nabla}\Phi^b)\cdot\vec{\nabla}\Phi^a \ , \tag{8}$$

where

$$\epsilon = f - \Phi^a \bar{\Phi}^a + i\psi \ , \tag{9}$$

$$\Phi^a = A^a{}_t + iB^a \ , \tag{10}$$

with

$$\vec{\nabla} B^a = -\rho^{-1} f \hat{n} \times (\vec{\nabla} A^a{}_\phi - \omega \vec{\nabla} A^a{}_t) \ , \tag{11}$$

$$\vec{\nabla} \psi = -[\rho^{-1} f^2 \hat{n} \times \vec{\nabla} \omega + 2\,\mathrm{Im}(\bar{\Phi}^2 \vec{\nabla} \Phi^a)] \ , \tag{12}$$

where $\hat{n}$ is the unit vector along the azimuthal direction and Im denotes the imaginary part. Although there is no interaction (by construction) between the potentials $A^a{}_\mu$ it appears that the complex scalars Φ^a intract with one another. Equations (7) and (8) may be obtained from a variational principle where the Lagrangian density reads

$$\mathscr{L} = 4f^{-2}[(\vec{\nabla}\epsilon + 2\bar{\Phi}^a \vec{\nabla}\Phi^a) \cdot (\vec{\nabla}\bar{\epsilon} + 2\Phi^b \vec{\nabla}\bar{\Phi}^b) - 2f \vec{\nabla}\Phi^a \cdot \vec{\nabla}\bar{\Phi}^a] \ , \tag{13}$$

with

$$f = \frac{\epsilon + \bar{\epsilon}}{2} + \Phi^a \bar{\Phi}^a \ . \tag{14}$$

This suggests that we should utilize the theory of harmonic mappings. The mapping $F{:}N \to M$ between the Riemann manifolds M and N with metrics

$$M\colon ds^2 = 4f^{-2}[(d\epsilon + 2\Phi^\dagger d\Phi)(d\bar{\epsilon} + 2d\Phi^\dagger \Phi) - 2fd\Phi^\dagger d\Phi] \ , \tag{15}$$

$$N\colon ds^2 = d\rho^2 + dz^2 + \rho^2 d\phi^2 \tag{16}$$

is harmonic if the field equations (7) and (8) are satisfied. Here Φ denotes an $(n-1)$-dimensional complex column vector with components Φ^a and a dagger denotes Hermitian conjugation. Defining a new column vector ω by

$$\omega = \begin{pmatrix} 1 \\ \Phi \\ i\epsilon \end{pmatrix}, \quad \Phi = \begin{pmatrix} \Phi_1 \\ \vdots \\ \Phi_{n-1} \end{pmatrix} , \tag{17}$$

and a Hermitian unimodular matrix γ by

$$\gamma = \begin{pmatrix} 0 & 0 & \cdots & -i \\ 0 & & & 0 \\ \vdots & & I_{n-1} & \vdots \\ i & 0 & \cdots & 0 \end{pmatrix} , \tag{18}$$

where I_{n-1} is the $(n-1)\times(n-1)$ unit matrix, then the metric of M becomes

$$M\colon ds^2 = 4f^{-2}(|d\omega^\dagger \gamma \omega|^2 - 2fd\omega^\dagger \gamma d\omega) \ , \tag{19}$$

with

$$2f = \omega^\dagger \gamma \omega \ . \tag{20}$$

This metric describes the complex hyperbolic space[5] $SU(n,1)/SU(n) \otimes U(1)$ and it can be further simplified. We define a Hermitian matrix

$$P_e = \gamma - 2\frac{\omega\omega^\dagger}{\omega^\dagger \gamma \omega} \ , \tag{21}$$

which has unit determinant. In addition, it also satisfies

$$P_e \gamma P_e = \gamma \ . \tag{22}$$

Then the metric in (19), when expressed in terms of P takes the form

$$M\colon ds^2 = \mathrm{tr}(P_e{}^{-1} dP_e \otimes P_e{}^{-1} dP_e) \ . \tag{23}$$

The field equations (7) and (8) may also be expressed in terms of P_e. They become

$$d(aP_e{}^{-1} {}^*dP_e) = 0 \ . \tag{24}$$

Here we changed the manifold N to M_0 with the metric

$$M_0\colon ds^2 = d\rho^2 + dz^2 \ . \tag{25}$$

The exterior derivative d and the Hodge dual operation * in (24) are defined on M_0. The function a is a harmonic function in M_0, i.e.,

$$d^*da = 0 \ . \tag{26}$$

Replacing the manifold N with M_0 is in a sense replacing ρ^2 appearing as a coefficient of $d\phi^2$ in the space-time metric (4) with a^2. Harmonicity of the function a follows from the field equations. Hence, in summary, the AS stationary Einstein-Abelian-gauge field equations (6a)–(6d) may be reduced to a single differential equation for the matrix P_e subject to some constraints. These equations read

$$d(aP_e{}^{-1*} dP_e) = 0 \ , \tag{27}$$

$$(\gamma P_e)^2 = I, \quad P = P^\dagger \ , \tag{28}$$

where P_e is an $(n+1)\times(n+1)$ matrix.

In the R gauge the self-dual $SU(N)$ Yang-Mills field equations are given by[6]

$$d(P^{-1\sim} dP) = 0 \ , \tag{29}$$

where

$$P = P^\dagger, \quad \det P = 1 \ , \tag{30}$$

and

$$d = dy\frac{\partial}{\partial y} + dz\frac{\partial}{\partial z}, \quad {}^\sim d = dy\frac{\partial}{\partial \bar{z}} - dz\frac{\partial}{\partial \bar{y}} \ . \tag{31}$$

Here, $y = (1/\sqrt{2})(x^1 + ix^2)$, $z = (1/\sqrt{2})(x^3 - ix^4)$, and the bar operation denotes complex conjugation. Axially symmetric static fields do not depend on the azimuthal angle (ϕ) and time (t), where $y = \rho e^{i\phi}$ and $z = \bar{z} = t$. In this case (29) becomes

$$d(aP^{-1*} dP) = 0 \ , \tag{32}$$

where d and * are defined on M_0 with metric (25) as before. We notice that the AS static self-dual $SU(N)$ Yang-Mills equations (32) are more general than the AS stationary–Abelian-gauge $[(n-1)\times\text{Maxwell}]$ field equations. They become identical when $N = n+1$ and $P = P_e$, which implies $(\gamma P)^2 = I$. This constraint is, of course, compatible with the field equations (32).

The group $SU(n,1)$ is the isometry group of the metric (23) of M. Hence the transformation

$$P_e \to P_e' = SP_eS^\dagger \ , \tag{33}$$

with

$$S\gamma S^{\dagger}=\gamma \ ; \qquad (34)$$

i.e., $S \in SU(n,1)$, leaves the metric (23) and the field equation (24) invariant. This may be utilized to generate new solutions of the field equations. Such global transformations (S = constant) produce gague-equivalent Yang-Mills fields.[6] On the other hand, the $SU(n,1)$ transformation (33) leads to distinct solutions in the Einstein case. For instance, starting from a vacuum solution ($n=1$), say the Kerr metric, it is possible to obtain a solution of the Einstein-$(n-1)$-Maxwell field equations, say an $(n-1)$ Abelian-gauge charged-Kerr metric.

It is known that the vacuum ($n=1$) and the electrovacuum ($n=2$) field equations can be integrated via the Zakharov-Shabat[7] and the Belinski-Zakharov[8] techniques. These methods can be extended to integrate the field equations [(27), (28)]. Such an extension has been given by several authors.[9]

Quite recently,[10] it has been shown that the Kerr-Newman metric ($n=2$) is the unique stationary black-hole solution of the Einstein-Maxwell field equations. If a black hole carries $(n-1)$ Abelian-gauge charges, the above theorem can be paraphrased. The $(n-1)$ Abelian-gauge charged-Kerr metric is the unique stationary black-hole solution of the Einstein-Abelian-gauge field equations.

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