The Gribov horizon

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Properties of the Coulomb-gauge Gribov horizon are discussed. In a non-Abelian gauge model, transverse field amplitudes must lie in the region bounded by the horizon. Natural enforcement of this restriction is provided by the Coulomb energy associated with color-charge fluctuations. To isolate the most important color-charge terms at the horizon and investigate their influence on quantum eigenstates, a simple nonlinear transformation is introduced which removes the horizon to infinity. The results of some numerical calculations on the shape of the horizon are also reported.

I. INTRODUCTION

The Gribov horizon is the locus of the first zero of the Faddeev-Popov determinant; gauge-field amplitudes must lie within the open region G bounded by the horizon.[1-3] Gribov pointed out that this constraint provides a mechanism for color-charge confinement.[1] In essence, the idea is that modes with short wavelengths are not affected by the constraint, while the amplitudes of modes with long wavelengths are greatly reduced from their free-field values. As a consequence, the excitation energies associated with long wavelengths are increased, giving rise to a spectrum with a mass gap. Gribov's analysis made use of several approximations, and numerical calculations have recently been undertaken to study Gribov's confinement scenario nonperturbatively, using both direct[4] and lattice Monte Carlo[5] methods.

A new numerical method for studying eigenstates of a quantum field model was introduced in a recent paper.[4] To obtain a manageable number of degrees of freedom, the model is formulated on a restricted spatial domain, taken to be the surface of a four-dimensional hypersphere, and a cutoff is introduced at a wave number Λ. [The wave number is the O(4) principal quantum number, and corresponds to the magnitude of the momentum.] In this scheme, numerical calculations have manifest rotational invariance in all degrees of approximation, and group-theoretical methods involving O(4) can be used to classify states and evaluate matrix elements.[6] This method was applied to a study of the SU(2) gauge-field model, in the Coulomb gauge.[4] The determination of the locus of the horizon was not approximated, but the results were in close qualitative agreement with those of Gribov[1] and supported his suggestions about confinement. Approximations similar to Gribov's were used in solving for the energies of the vacuum and excited states. Results reported from the Monte Carlo studies also show clearly the existence of the horizon.[5]

The most serious approximation in the previous work[1,4] was neglect of the Coulomb energy associated with color-charge fluctuations. The Coulomb energy is especially important for field configurations very near the horizon, and its effect on the horizon structure of eigenstates is explored in this paper. It is shown to be possible to introduce a nonlinear mapping which removes the horizon to infinity; this mapping provides a simple way to isolate and examine leading terms in the vicinity of the horizon. It is found that the leading terms in the energy automatically keep the field configuration away from the horizon. Since all gauge orbits have intersections in G^2, any field configuration which lies beyond the horizon is a "Gribov copy" of some configuration within G, and should not be included independently. The fact that the Coulomb energy confines the field variables to the interior of G indicates that the horizon constraint is not to be considered as a separate requirement, but arises automatically from the structure of the Hamiltonian.

The organization of this paper is as follows. After a brief section on notation, some general properties of the horizon surface will be reviewed and discussed. Then the horizon-expansion transformation will be defined and the behavior of eigenstates in the vicinity of the horizon will be examined. This part of the paper is independent of the hyperspherical formalism. Finally, some new numerical results on the shape of the horizon in the SU(2) hyperspherical model will be presented.

A. Notation

The fields are expanded in normal modes defined over a unit hypersphere. A mode will be labeled by a Greek index, for example, $\alpha=(\hat{\alpha},\tilde{\alpha})=(A,a)$, where $\hat{\alpha}$ denotes the color index, and $\tilde{\alpha}$ denotes the spatial mode. The corresponding Roman letters, upper and lower case, denote, respectively, the wave number and the O(4) representation component (including helicity and color). Thus, for example, the transverse potential is $\vec{A}^{\hat{\alpha}}(x)=\sum_{\tilde{\alpha}} a_\alpha \vec{V}_{\tilde{\alpha}}(x)$. The electric field amplitude for mode α is $e_\alpha=i\partial/\partial a_\alpha$. The magnetic field amplitude b_α includes the usual self-coupling term. Implied summation over paired indices will be used, along with matrix notation. The Hamiltonian is[1,7]

$$H=\tfrac{1}{2}(F^{-1}e_\alpha F e_\alpha + b_\alpha b_\alpha + F^{-1}\sigma_\alpha F C_{\alpha\beta}\sigma_\beta)\ , \tag{1}$$

where

$$\sigma^{\hat{\alpha}}(x)=gk^{-1}\vec{A}^{\hat{\beta}}(x)\cdot\vec{E}^{\hat{\gamma}}(x)f^{\hat{\alpha}\hat{\beta}\hat{\gamma}}\ ,$$
$$D^{\hat{\alpha}\hat{\beta}}=gk^{-1}f^{\hat{\alpha}\hat{\beta}\hat{\gamma}}\vec{A}^{\hat{\gamma}}\cdot\nabla k^{-1}\ , \tag{2}$$

and where $k^2=(-\nabla^2)$; in terms of normal modes,

$$\sigma^{\alpha}=ga_{\beta}e_{\gamma}Y^{\alpha\beta\gamma}\ ,$$
$$D^{\alpha\beta}=gX^{\alpha\beta\gamma}a_{\gamma}\ , \tag{3}$$

where X and Y are certain constants—products of the structure constants, 3-j symbols, and reduced matrix elements.[6] The Coulomb energy is given by the matrix $C=(1-D)^{-2}$, and can be expressed in terms of the scalar potential $\Phi=k^{-1}(1-D)^{-1}\sigma$ as $\frac{1}{2}\Phi^{\dagger}k^2\Phi$. The Faddeev-Popov determinant is $F=\det(1-D)$. Zeros of F correspond to unit eigenvalues of D: $d(a)=1$, where $D\phi=d\phi$, and ϕ is the eigenvector. In these definitions, factors of k^{-1} have been inserted in a way which is convenient for numerical calculations, but not conventional.

The factors k^{-1} in Eq. (2) are infinite for the scalar mode $A=0$, which is constant over the hypersphere. This implies that finite-energy eigenstates can have no net color charge; only color-singlet states are permitted. If the Laplacian were altered by explicit inclusion of an infrared parameter, the main effect would be to allow nonsinglet states to have finite energies. If only color-singlet states are considered, the mode $A=0$ can be dropped. There is no coupling between the mode $A=0$ and other scalar modes, because D contains a gradient operator.

B. General properties of the horizon

Zwanziger[2] has shown that in any direction in the space of field variables a, the horizon surface H lies at a finite distance from the origin; he has also shown that H is convex. For the Coulomb-gauge hyperspherical model, these properties are easy to verify. First, note that for any a, Tr $D^2>0$; more specifically, Tr $D^2=\sum_{\alpha}T_A{a_{Aa}}^2$, where, using sum rules for 3-j symbols, T_A is given by a sum over squares of reduced matrix elements. The general expression is rather complicated, but for $\Lambda\gg A\gg 1$ there is a simple limiting formula: $T_A=\alpha\Lambda/3\pi$, where $\alpha=g^2/4\pi$. Furthermore, Tr $D=0$, since D is antisymmetric in color and spatial variables separately. Thus, for any a, $D(a)$ has at least one nonzero eigenvalue of each sign, showing that in every direction $d(a)=1$ occurs for finite values of the a_{α}.

To examine convexity, let $a_{\rm I}$ and $a_{\rm II}$ be two points on H. Then, for normalized ϕ, $\mathrm{Max}(\phi^{\dagger}D_{\rm I}\phi)=\mathrm{Max}(\phi^{\dagger}D_{\rm II}\phi)=1$, the maxima occurring for $\phi_{\rm I}$ and $\phi_{\rm II}$, respectively. Since $D(a)$ is linear in a, for $a=ta_{\rm I}+(1-t)a_{\rm II}$ we have $D(a)=tD_{\rm I}+(1-t)D_{\rm II}$, and hence for $0\le t\le 1$, $\phi^{\dagger}D(a)\phi\le 1$. Thus, a does not lie beyond H, i.e., H is convex.

Numerical calculations reported earlier[4] (and in more detail later in this paper) show that H is rather smooth, and that the following "ellipsoidal approximation" is surprisingly accurate:

$$z(a)=d(a)^2\sim\tilde{z}=\alpha\sum_{\alpha}U_A{a_{\alpha}}^2/N_A\ , \tag{4}$$

where N_A is the degeneracy for wave number A; for SU(2), $N_A=6A(A+2)$. The U_A are constants, which for SU(2) are all close to 0.5. For a more accurate description, the U_A can be considered to depend weakly on the a_{α}.

Local gauge transformations, applied to some initial field configuration, may lead to another configuration in the Coulomb gauge, a Gribov copy of the first. While it is known that configurations beyond H always have copies in G^2, it is an open question whether any configurations in G have copies that are also in G. Local transformations would not respect the condition $A\le\Lambda$, so the Hamiltonian (1) does not allow such copies to be discussed directly. It is possible to pose some related but different questions, which illustrate some special features of the Coulomb gauge.[4] Two transverse potentials, $a_{\rm I}$ and $a_{\rm II}$, can be said to be "charge equivalent" if they lead to the same (classical) self-energy for any external charge distribution σ_0. In an Abelian gauge model, all transverse potentials are charge equivalent, but this is of no consequence, because it is known on other grounds that each set of potentials is physically distinct; the question of copies does not arise. In the non-Abelian case, $a_{\rm I}$ and $a_{\rm II}$ are charge equivalent if $C_{\rm I}=C_{\rm II}$. In the relation $D=1-C^{-1/2}$, the square root of the eigenvalues of C, $c=(1-d)^{-2}$, could have either sign. In G, however, all eigenvalues satisfy $d<1$, so only the positive root is allowed. This implies that $D_{\rm I}=D_{\rm II}$, and hence $a_{\rm I}=a_{\rm II}$, if both are inside G. If $a_{\rm I}$ lies beyond the horizon H, so there are some $d_{\rm I}>1$, there is an equivalent matrix $D_{\rm II}$ with all $d_{\rm II}<1$. It does not follow generally that there would be a charge-equivalent $a_{\rm II}$ inside G, because the a's are not complete in the space of matrices D. However, if $a_{\rm I}$ is infinitesimally beyond H, there is a charge-equivalent $a_{\rm II}$ infinitesimally close to $a_{\rm I}$ and inside H; it is also known that a Gribov copy of $a_{\rm I}$ is similarly located.[1,3]

In axial gauges, the Faddeev-Popov determinant is a constant, and the independent field variables are not constrained. Although the structure of a gauge model might therefore seem to be exhibited somewhat more simply in space-axial gauges, the lack of manifest rotational invariance is a drawback in explicit calculations. The independent variables also have a more direct physical interpretation in the Coulomb gauge. In transforming between the axial and Coulomb gauges, the unboundedness of axial-gauge potentials is associated with the fact that the scalar potential $\Phi=k^{-1}(1-D)^{-1}\sigma$ is not bounded, and, depending on σ, may diverge as H is approached. This suggests that the horizon constraint on the transverse potentials is closely related to the scalar potential and the Coulomb energy. Conversely, inside G, Φ is finite if σ is finite and smooth, so there is no intrinsic term in the energy which can separate one part of G from another.

C. Decoupled-oscillator approximations

The horizon surface is elongated in directions corresponding to large wave numbers. On the other hand, for the vacuum state of a free field, $\langle {a_{\beta}}^2\rangle=1/(2\omega_B)$ (on a unit hypersphere, $\omega_B=B+1$) so that modes with small

wave numbers have large vacuum amplitudes. As pointed out by Gribov, use of the free-field value for $A_B{}^2=\langle a_B{}^2\rangle$ gives an estimate for z which will surely exceed 1 if Λ and α are large enough; the approximation (4) gives $z=\alpha\sum_B U_B/(B+1)$. The resulting situation is as depicted in Fig. 1; the mean amplitudes for a free field would lie at a point (F) outside the horizon. Existence of the horizon constrains the mean gauge-field amplitudes to some point (G) inside H. Roughly speaking, we expect the amplitudes to be pushed in the direction perpendicular to H. Thus, if the wave number B is small (so the normal to H has a large component in the B direction), the mode is constrained to have a small amplitude. On the other hand, if B is large, so the surface is nearly independent of the value of A_B, the existence of the horizon has little effect.[1] These qualitative observations can be made more precise by simple approximate calculations which just take into account the existence of the horizon and ignore other nonlinear dynamical effects. These calculations also verify that the excitation energies associated with constrained modes are increased.

To estimate the influence of the horizon, Gribov used an approximation in which the mean-field configuration was required to lie exactly on the horizon. The Coulomb energy and the magnetic self-coupling were neglected. This gives harmonic-oscillator equations with modified frequencies

$$W_A=(\omega_A{}^2+\lambda\zeta_A)^{1/2}\,, \tag{5}$$

where, using the approximation (4),

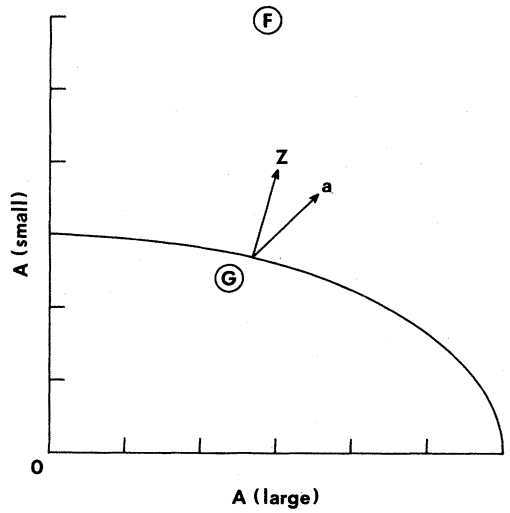

FIG. 1. Schematic illustration of the shape of the Gribov horizon, and of the influence of the horizon on the amplitudes of modes with a large wave number (abscissa) and a small wave number (ordinate). The point labeled F indicates the mean values of the amplitudes for a free field. The point labeled G indicates the mean values for a gauge field constrained by the horizon. The vectors a and z denote, respectively, the direction of G from the origin, and the corresponding normal to the horizon.

$$\zeta_A=\alpha U_A/N_A\,, \tag{6}$$

and where λ is the Lagrange multiplier, obtained from the constraint

$$1=\alpha\sum_A U_A/2W_A\,. \tag{7}$$

The energy of the vacuum, in this approximation, is given by the expectation value of the free-particle Hamiltonian,

$$E=\tfrac{1}{2}\sum_A(W_A-\lambda\zeta_A/2W_A)N_A\,. \tag{8}$$

Results which are qualitatively similar can be obtained from a scheme in which F is approximated by an exponential. The logarithmic derivatives are equated to those of the zero factor $(1-z)$ at a self-consistently determined mean value $\langle a_\alpha{}^2\rangle$:

$$F=F_0\exp\left[-\sum_\alpha\mu\zeta_A a_\alpha{}^2\right], \tag{9}$$

where F_0 is a constant, and where the proportionality factor μ is given by

$$\begin{aligned}\mu^{-1}&=1-\sum_\alpha\zeta_A\langle a_\alpha{}^2\rangle\\&=1-\alpha\sum_A U_A/2W_A\,.\end{aligned} \tag{10}$$

This also leads to harmonic-oscillator equations, but the frequencies and the vacuum energy are given by

$$\begin{aligned}W_A&=[\omega_A{}^2+(\mu\zeta_A)^2]^{1/2}\,,\\E&=\tfrac{1}{2}\sum_A(W_A-\mu\zeta_A)N_A\,,\end{aligned} \tag{11}$$

instead of Eqs. (5) and (8). Numerical calculations previously reported used a more complicated version of this approximation, in which F_{approx} was required to vanish for $\sum_\alpha a_\alpha{}^2>N_A/(\alpha U_A)$, and in which additional factors in F were included in matching the logarithmic derivatives.[4] These additional factors arise, in part, from the additional zeros of F. The simpler exponential approximation described here gives quite similar results.

Since these two approximations involve uncoupled harmonic oscillators, the classification of excited states is relatively simple. The allowed excited states require at least a double excitation (2 "gluons") coupled together to give a color singlet. Let the excited modes have wave numbers B and C; $B=C$ is allowed and gives the lowest energy. The excitation energy, Ω_{BC}, is approximately equal to W_B+W_C. More precisely, although the frequencies in excited states are still given by Eqs. (5) or (11), these frequencies also obtain slightly altered values, because the constants λ or μ must be reevaluated. In Gribov's approximation, (7) is replaced by

$$1=\sum_A\zeta_A(\tfrac{1}{2}N_A+\delta_{AB}+\delta_{AC})/W_A\,, \tag{12}$$

and the energy of the excited state is

$$E_{BC}=\sum_A(W_A-\lambda\zeta_A/2W_A)(\tfrac{1}{2}N_A+\delta_{AB}+\delta_{AC})\,. \tag{13}$$

In the exponential approximation, Eq. (10) is replaced by

$$\mu^{-1}=1-\sum_A \zeta_A(\tfrac{1}{2}N_A+\delta_{AB}+\delta_{AC})/W_A\ , \tag{14}$$

and the energy of the excited state is

$$E_{BC}=W_B+W_C+\tfrac{1}{2}\sum_A (W_A-\mu\zeta_A)N_A\ . \tag{15}$$

In either case, the excitation energy is given by $\Omega_{BC}=E_{BC}-E$.

For large wave numbers A, Eqs. (5) and (11) show that W_A is dominated by the free-field energy; this is in accordance with the principle of asymptotic freedom. If α is not too small, W_A takes its minimum value $W_{\min}$ at an intermediate value $A_{\min}$, rather than at $A=1$, and $W_{\min}$ is substantially larger than the minimum free-field energy $\omega_1=2$. The reason is that ζ_A has a factor N_A in the denominator, so for small wave numbers, W_A is large. Thus, either approximation gives a spectrum with a mass gap. Furthermore, if either Λ or the "bare" coupling strength α is increased, $W_{\min}$ is increased, and also $A_{\min}$ is nondecreasing. Thus, for a given effective coupling strength (or, more precisely, for a fixed value of $W_{\min}$), if Λ is increased, α should be decreased. These effects are all somewhat accentuated in the exponential approximation, in part because the mean-field configuration lies closer to the origin, but also because Eq. (5) contains ζ_A in the argument of the square root, while (11) contains ${\zeta_A}^2$.

II. EXPANDED-HORIZON TRANSFORMATION

In the simplified calculations described in Sec. I, the horizon constraint was approximated. For a more accurate treatment of the constraint, it is convenient to use a transformation which removes the horizon to infinity. A simple transformation is introduced here which leads to a well-behaved Hamiltonian involving new, unbounded variables. This transformation is also useful for examining the properties of the Hamiltonian in the neighborhood of the horizon. A quite general form for the transformation is first discussed, followed by consideration of a specific functional form.

The transformation is defined as

$$a_\alpha=q_\alpha R(z(q))\ , \tag{16}$$

where $z(q)=d(q)^2$, and where d is the largest positive eigenvalue of D, as in Eq. (4). Since $D(q)$ is linear in q, z is homogeneous of degree 2, and therefore $z(a)=z(Rq)=R(z(q))^2z(q)$. Thus, if $R^2\sim z^{-1}$ for large z, then $q\to\infty$ will correspond to $z(a)\to 1$. In other words, in terms of the new variables q_α, the horizon is at ∞.

The partial derivative matrix is

$$S_{\alpha\beta}=\partial a_\beta/\partial q_\alpha=R\delta_{\alpha\beta}+R'z_\alpha q_\beta\ , \tag{17}$$

where

$$z_\alpha=\partial z/\partial q_\alpha\ . \tag{18}$$

This transformation has several features which simplify its use in numerical as well as analytical calculations. First, for any function $R(z)$, given a vector a, the transformation is specified once the distance to the horizon has been determined. Second, the derivatives z_α can be determined by a first-order perturbation calculation utilizing the eigenvector ϕ associated with the horizon point. Finally, Euler's theorem, $z_\alpha q_\alpha=2z$, can be used to simplify many formal expressions. In particular, z_β is an eigenvector of $S_{\alpha\beta}$; using this fact, the Jacobian is found to be

$$\begin{aligned} J&=\det(S)=R^N j\ ,\\ j&=1+2zR'/R\ , \end{aligned} \tag{19}$$

where $N=\sum N_A$ is the total number of degrees of freedom. Similarly, the inverse matrix is

$${S^{-1}}_{\alpha\beta}=\partial q_\beta/\partial a_\alpha=R^{-1}(\delta_{\alpha\beta}+\rho z_\alpha q_\beta)\ , \tag{20}$$

where

$$\rho=-R'/(jR)\ .$$

In terms of the new variables q_α, the electric field operator is

$$e_\alpha=i{S^{-1}}_{\alpha\beta}\partial/\partial q_\beta\ . \tag{21}$$

Note that the second term in ${S^{-1}}_{\alpha\beta}$ enters only through the radial derivatives $\partial/\partial q$.

Now transform to polar coordinates in q space, and let $x=q^2$. The weighting function for integration with $dx\,d\Omega_N$ is then

$$G=\tfrac{1}{2}x^{N/2-1}JF\ . \tag{22}$$

The function $z(q)$ can be written as $z(q)=x\zeta(\theta)$, where θ denotes the direction. The electric field operator (21) takes the form

$$e_\alpha=i\epsilon_\alpha\partial_x+\text{"angular terms ,"} \tag{23}$$

where

$$\epsilon_\alpha=2{S^{-1}}_{\alpha\beta}q_\beta=2R^{-1}(q_\alpha+\rho q^2 z_\alpha)\ . \tag{24}$$

It will be seen that ϵ_α becomes large as the horizon is approached, so the radial part of e dominates the angular part.

The operator ordering used here is defined by first rewriting the Hamiltonian (1) in a manifestly self-adjoint form, using a renormalized wave function $\Phi=F^{1/2}\Psi$. The transformation is then applied to matrix elements, and at the end the new weight function $G^{1/2}$ is eliminated. The radial part of the transverse electric energy becomes

$$U_{Ex}=-\tfrac{1}{2}G^{-1}\partial_x G\partial_x\ , \tag{25}$$

and the radial part of the Coulomb energy is

$$U_{Cx}=\tfrac{1}{2}G^{-1}\sigma^\dagger_{\alpha x}GC^{\alpha\beta}\sigma_{\beta x}\ , \tag{26}$$

where

$$\sigma_{\alpha x}=gRq_\beta\epsilon_\gamma Y^{\alpha\beta\gamma}\partial_x\ . \tag{27}$$

Along with the radial terms (25) and (26), the transverse and Coulomb energies contain purely angular terms and

also mixed terms containing ∂_x and one angular derivative.

A. Exponential transformation

A convenient specific transformation is given by

$$R(z)^2=(1-e^{-z})/z\ , \tag{28}$$

which expands exponentially the neighborhood of the horizon:

$$1-z(a)=e^{-z(q)}\ . \tag{29}$$

The Jacobian is given by

$$j=R^{-2}e^{-z},\quad J=R^{N-2}e^{-z}\ , \tag{30}$$

and is nonsingular for finite x. The leading terms for $x\to\infty$ are

$$R\sim z^{-1/2},\quad J\sim z^{-(N/2-1)}e^{-z}\ . \tag{31}$$

The "radial" electric field involves the vector

$$\begin{aligned}&\epsilon_\alpha\sim z_\alpha q^2 z^{-3/2}e^z\ ,\\ &\epsilon^2=\sum\epsilon_\alpha{}^2\sim Zq^4z^{-3}e^{2z}\ ,\end{aligned} \tag{32}$$

where $Z=\sum z_\alpha{}^2$. The leading parts of G and ϵ^2 have the form

$$G\sim f(\theta)e^{-2z},\quad \epsilon^2\sim h(\theta)e^{2z}\ , \tag{33}$$

where $f(\theta)$ and $h(\theta)$ are functions only of direction.

The leading term in the transverse electric energy is

$$U_{Ex}\sim-\tfrac{1}{2}h(\theta)e^{2z}\partial_x{}^2\ . \tag{34}$$

Near the horizon, the Coulomb energy matrix C is dominated by the leading eigenvalue $(1-d)^{-2}$ and the corresponding ghost eigenvector ϕ, giving

$$C^{\alpha\beta}\sim 4e^{2z}\phi^\alpha\phi^\beta\ . \tag{35}$$

With the definition

$$y(\theta)=2g\phi_\alpha q_\beta z_\gamma Y^{\alpha\beta\gamma}q^2z^{-2}\ , \tag{36}$$

it is found that

$$2\phi^\alpha\sigma^\alpha\sim y(\theta)e^z\partial_x\ , \tag{37}$$

giving

$$U_{Cx}\sim-\tfrac{1}{2}y(\theta)^2e^{2z}\partial_x e^{2z}\partial_x\ . \tag{38}$$

In U_E and U_C, the derivative ∂_x always appears multiplied by an extra factor $e^z=e^{x\zeta}$, which is absent from the angular derivative terms. The angular derivatives are therefore less important near the horizon, except possibly at certain exceptional points.

Although $y(\theta)$ [Eq. (36)] contains an explicit factor g, the hidden factors in z make $y\sim g^{-1}$ for small g, and also make $h(\theta)\sim g^{-2}$. In terms of the original field variables a, the horizon is at $a\sim g^{-1}$, which provides a source of nonanalyticity in g. After the transformation, these nonanalytic effects appear through the leading radial derivative terms in the Hamiltonian.

Numerical calculations show that y varies rapidly as θ is changed, and there are directions (of dimensionality $N-2$) in which y vanishes. The sign of y can be defined by requiring that the eigenfunction ϕ be continuous, but there are exceptional directions with degenerate eigenvalues, which are branch points for y. For pairs of randomly chosen directions, the values of y become weakly correlated if the angular separations are more than about 30°.

B. Asymptotic wave functions

In the neighborhood of the horizon, radial derivative terms asymptotically have larger coefficients than angular derivatives. In general, therefore, the eigenstates of the Hamiltonian may be considered to adjust adiabatically to angular changes in ζ, y, and h, as the field configuration is varied in the vicinity of H. There are however, special directions in which this approximation may not be accurate. We begin by examining the asymptotic radial differential equation in a fixed direction,

$$-\tfrac{1}{2}G^{-1}\partial_x GM^{-1}\partial_x\Psi+U\Psi=0\ , \tag{39}$$

where $G\sim\exp(-2x\zeta)$, $M\sim\exp(-4x\zeta)$, and $U\sim\exp(2x\zeta)$. The mixed x-θ derivative terms have been absorbed by considering the nonleading parts of G and M to contain angular derivatives. The leading part of U also contains angular derivatives. For a preliminary discussion, this θ differentiation will be ignored. A formal solution of (39) is then

$$\Psi=\exp\left[-\int\kappa(x')dx'\right]\ , \tag{40}$$

where

$$\begin{aligned}&\kappa=\beta/2+(\beta^2/4+2MU)^{1/2}\ ,\\ &\beta=\partial_x\ln(G\kappa/2M)\ .\end{aligned} \tag{41}$$

The leading terms give $\kappa=\beta=2\zeta$, and $\Psi=G_0M/GM_0\sim\exp(-2x\zeta)$. Since U is positive-definite, the corrections increase the rate at which Ψ decreases for large x.

For $y\sim0$, ignoring the angular derivative terms—which also contain a factor of y—we can write $M^{-1}\sim\exp(2x\zeta)[a+by^2\exp(2x\zeta)]$, where a and b are positive constants. For $by^2\ll ae^{-2x\zeta}$, we then have $\beta\sim0$ and $\kappa\sim(2MU)^{1/2}$; this is essentially independent of x, and corresponds to the usual WKB solution. However, we cannot expect the adiabatic approximation to be accurate near these directions.

In directions where y does not vanish, the leading term of integrands which give the matrix elements of the energy comes from the first term of (39), and is roughly of the form $M/G\sim\exp(-2x\zeta)$. If $y=0$, both terms are of the same order: $\exp[-2x(2MU)^{1/2}]$. In either case, the Coulomb energy leads to wave functions which vanish at the horizon, and in fact, vanish strongly enough to give convergent integrals.

To include the effects of the angular derivatives, we could still use the formal expressions of Eqs. (39) and (41), provided the order of the operators were chosen appropriately. In (39), we should use an "x ordering" so that κ for

larger x' would be to the left of κ for smaller x'. In general, we may expect that inclusion of the angular derivative effects would smooth out the θ dependence of the wave function. It may be anticipated, therefore, that the estimate $\Psi \sim \exp(-2x\zeta)$ would be valid even for $y=0$.

As discussed in Sec. III, there are certain directions in which the surface H has corners, arising from the crossing of eigenvalues. In these directions, the derivatives z_α have discontinuities, and hence the angular derivative terms might become more important. Since these directions correspond to degenerate eigenvalues, the estimates based on a single first zero also underestimate the numerical coefficient of the radial derivative terms. In the direction of a corner, therefore, Ψ may be overestimated.

Expressed in terms of the proper field variables a_α, the distance to the horizon is given by $\eta \sim \exp(-x\zeta)$. In any given direction θ, we have $F \sim \eta$ and $\Psi \sim \eta^2$. Integrands which give the various contributions to matrix elements of the Hamiltonian have the following behavior near the horizon: magnetic energy, $\sim\eta^5$; transverse electric energy, $\sim\eta^3$; Coulomb energy, $\sim\eta$.

III. HORIZON SHAPE

A simple approximate relation between the wave numbers and the horizon distances was depicted in Fig. 1 and incorporated into Eq. (4). As discussed in Sec. I C, simplified calculations based on this relation lead directly to a spectrum with a mass gap. In this section, the results of numerical calculations of horizon distances will be presented and discussed. These calculations were undertaken to explore the degree of validity of the approximation (4), and to estimate the coefficients appearing in that formula. Furthermore, as an aid in finding good approximations for solving the eigenvalue problem, it is useful to understand the degree of smoothness of the horizon surface and the nature of exceptional points on it. Additional information about the structure of the Faddeev-Popov determinant F in the immediate vicinity of the horizon is given by the distance between the first and subsequent zeros.

The horizon surface depends on the cutoff Λ as well as on the gauge group. An understanding of this dependence might suggest how the spectra will depend on these variables. However, only the simplest gauge group, SU(2), is considered here, because the small number of color variables allows larger Λ to be considered.

A sampling technique was used to explore the field configuration space, which has a large number of dimensions. For each $\Lambda \leq 6$, 50 random field configurations were used. To define these field configurations, for each $B \leq \Lambda$, vectors a_{Bb} were chosen with random directions and fixed lengths given by $\sum_b a_{Bb}{}^2 = N_B A_B{}^2$. The A_B, listed in Table I, are the self-consistent rms values determined through calculations reported earlier,[4] and correspond to the values of α also listed in Table I. These values of α are very approximate, and are listed only as a rough guide. The calculations presented here do not involve α, and only the ratios of the A_B are relevant, not their absolute magnitudes. The random sampling of directions was supplemented by other calculations, in particular, by a search

TABLE I. RMS values, A_B, of transverse amplitudes.

B \ Λ / α	1	2	3 4.0	4 3.5	5 3.2	6 3.0
1	0.4	0.4	0.391	0.374	0.353	0.333
2		0.4	0.397	0.385	0.372	0.360
3			0.352	0.349	0.345	0.341
4				0.316	0.314	0.313
5					0.288	0.288
6						0.267

for directions in which the horizon is either especially close or especially distant. The same rms values A_B were used in searching for exceptional points.

The calculations were done as follows. For each random vector a_α, the distance to the horizon was determined by finding the largest eigenvalue of D and the corresponding eigenvector. The slope of the horizon surface at this point was then determined by a first-order perturbation calculation. The second- and third-largest eigenvalues were also found. A normalized measure of the (squared reciprocal) zero distance is given by the quantity $U = z(a)/(\alpha A^2)$, where $A^2 = \sum_B A_B{}^2$. Mean values of this quantity U are shown in Fig. 2, for the first three zeros. Note that for the first zero, U is essentially independent of Λ. The zeros also seem to get closer together, as Λ is increased, relative to their distances from the origin. The rms fluctuations in the zero distances, for randomly chosen directions, are about half of the mean separations between the zeros.

Table II compares, for various Λ, the averaged normalized horizon distances $\langle \eta \rangle = U^{-1/2}$ with the largest ($\eta_{\max}$) and smallest ($\eta_{\min}$) values found. The smallest values are all associated with nondegenerate eigenvalues of D, and are places where the horizon surface is nearly flat. These points are also exceptional in that the distance to the second zero, η', is about twice the distance ($\eta_{\min}$) to the

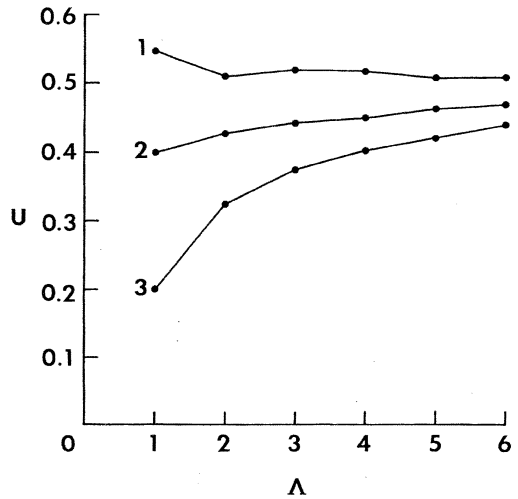

FIG. 2. Mean values of U (normalized squared reciprocal distance) for the first three Faddeev-Popov zeros.

TABLE II. Distances to horizon.

Λ	$\eta_{\min}$	η'	$\langle\eta\rangle$	$\eta_{\max}$	deg.
1	1.085	2.171	1.35	2.506	4
2	0.885	1.769	1.40	2.50	7
3	0.681	1.338	1.39	2.3	10
4	0.544	1.054	1.39		
5	0.443	0.849	1.41		
6	0.371	0.704	1.40		

first zero, as shown in Table II. In contrast, the largest values are associated with degenerate eigenvalues, as tabulated, and correspond to corners arising from the intersection of several different zero surfaces. (In these numerical calculations, the degeneracy is interpreted as the number of zeros in a 1% interval of distance.) Because of the high degeneracy and the fact that the extrema are corners, searching for the most distant horizon points is very time consuming and has been carried out only for small Λ. The values listed in Table II have been calculated with the additional constraint that the positive and negative helicity modes have the same rms values: $\sum_{b(+)} a_{Bb}{}^2 = \sum_{b(-)} a_{Bb}{}^2$, or $A_{B+} = A_{B-}$. If this constraint is relaxed, the most distant corners recede somewhat, but only by about 1% for $\Lambda=2$ or 3. For $\Lambda=1$, there is a bigger difference; the degeneracy is also increased, to 8. Because of the difficulty of calculation, these results for distant corners are somewhat provisional.

It is possible that the field configurations which correspond to horizon extrema have a special significance in the structure of eigenstates, but no simple way to characterize these configurations has been found. One possible characterization would be through the ratios $R_B = (A_{B+} - A_{B-})^2 / A_B{}^2$. The approximation (4) assumes that the horizon distance is essentially independent of these ratios. Except for the special case $\Lambda=1$, this is not inconsistent with the results on horizon extrema. It has also been found, for the randomly chosen directions, that there is no significant correlation between the values of the R_B and values of U. The values of certain cubic combinations of the a_α have also been examined, namely, the combinations which enter into the cubic term in the magnetic energy. These values, also, are not significantly correlated with the horizon distance.

While corners, or multiple zeros, certainly exist on the horizon surface, the dimensionality of these loci has not been determined. In the course of exploring properties of the charge function $y(\theta)$, calculations were made at closely spaced angles on a number of randomly chosen planes through the origin; in these calculations, the orbit of the horizon never crossed the locus of a multiple zero. However, the orbit often passed quite close to a double zero. This indicates that the dimensionality of a corner is less than $N-2$.

Figure 3 shows the quantities U_B introduced in Eq. (4). These have been calculated from the mean value of the normal to the horizon; they represent the parameters of the approximating ellipse which is tangent to the mean horizon surface in the neighborhood of the self-consistent mean-field configuration. If the elliptical approximation

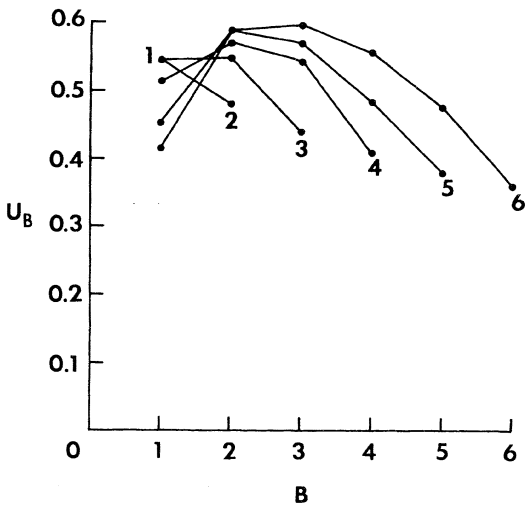

FIG. 3. Mean values of the U_B (parameters in the elliptical approximation) for the indicated values of Λ.

were exact, the points in Fig. 3 which correspond to different Λ would all coincide. The differences are modest, but are not negligible. An indication of the average degree of smoothness of H is provided by Fig. 4, which compares various contributions to the quantity $Z = \sum z_\beta{}^2$. The solid lines in Fig. 4 give, for each B, the mean contributions Z_{B1} associated with changes in the values of the A_B, with fixed ratios for the a_{Bb}. The dashed lines give the mean contributions Z_{B2} associated with angular variations, that is, with the A_B held fixed. (Factors of α have

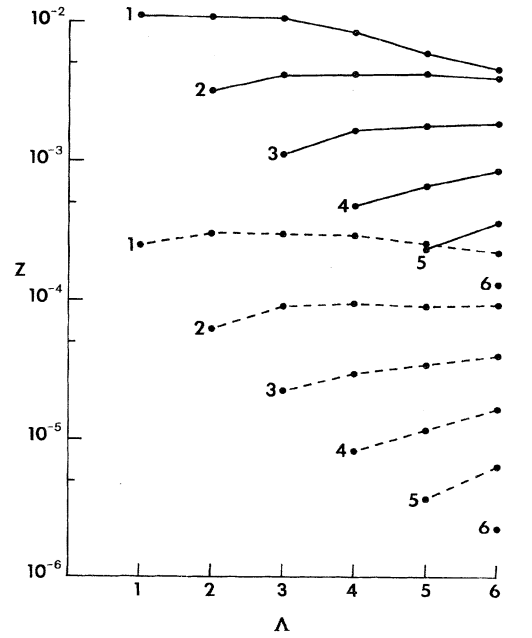

FIG. 4. Comparison of the "radial" (solid lines) and "angular" (dashed lines) mean contributions to Z, the square of the gradient $z(a)$. The lines connect points corresponding to the same wave number.

been omitted.) These results show that the main contributions to the horizon distances are, at least near a self-consistent value, adequately summarized by the elliptical approximation, and that fluctuations around this approximate formula are, on the average, not large. However, this observation leaves open the question, whether the structure of eigenstates might not be influenced by the nature of the extreme deviations.

IV. SUMMARY

We have considered here various properties of the Gribov horizon in non-Abelian gauge models, giving particular attention to the chromoelectrostatic potential. It is well understood, on very general grounds, that transverse amplitudes must be confined to the convex region G bounded by the first zero of the Faddeev-Popov determinant. It has been pointed out here that this constraint on the amplitude does not need to be enforced as an additional separate requirement, but emerges naturally from the behavior of the Coulomb energy at the horizon. Additional restrictions on the amplitudes do not arise naturally, and we have presented an argument that all points in G can be considered to be physically distinct, so no additional restriction to a subregion of G should be imposed.

It has been shown that a simple mapping can be introduced which removes the horizon to infinity in all directions. This map is used as a convenient device for isolating and studying leading behavior at the horizon. The fact that it is possible to introduce such a mapping, and obtain suitable boundary conditions at infinity, by itself shows that models are consistently formulated entirely within the domain G.

Gribov pointed out that the horizon constraint provides a confinement mechanism. The calculations of the horizon shape reported here verify that the main qualitative features of the shape, which have been incorporated into Eq. (4), agree with the assumptions made by Gribov. Simple approximate calculations which use these qualitative features, as reviewed in the Introduction, lead directly to the existence of a mass gap in the spectrum. These simplified calculations are also consistent with various aspects of the principle of asymptotic freedom.

The numerical calculations show that the horizon surface is basically smooth and regular; qualitative features of the spectrum should be adequately described by use of properties of the "averaged" horizon surface. There are also small wriggles and corners which must be taken into account in an accurate calculation. Even using the simple approximation (4), it is technically complicated to impose the horizon constraint exactly, for example, in evaluating the integrals which would arise in a variational calculation. The existing calculations have therefore further approximated, as described above, the treatment of the horizon. The horizon-expansion transformation introduced in Sec. II could provide a useful way to overcome these complications in future numerical calculations, because the range of the mapped variable q is not constrained in any way, and because the general form of the mapping is not affected by the existence of wriggles in the horizon.

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