# Decoupling renormalization and hierarchies

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(Received 27 January 1984)

We introduce the decoupling renormalization scheme for the one-particle irreducible Green's functions of a theory that has a mass hierarchy. The decoupling-subtraction renormalization-group equations respect a tree-level hierarchy. Thus, fine tuning is to be done only at the tree level.

## I. INTRODUCTION

Ever since the observation by Gildener and Weinberg[1] of the so-called hierarchy problem,[2] there has heretofore been no complete field-theoretic solution to this problem. Partly out of sheer frustration with this problem and partly because of the quest for ever "higher" symmetry, people have used this hierarchy problem as good justification for abandoning a grand unified theory (GUT) and going on to a supersymmetric GUT.[3] As a GUT field theorist, I have reason to claim that there is a field-theoretic solution to the hierarchy problem. It lies with the DS (decoupling subtraction) scheme[4] as opposed to the usual $\overline{\text{MS}}$ (minimal subtraction) scheme.[5]

As the name implies, the DS scheme is an outgrowth of the voluminous work on decoupling[6] to which many people have contributed.[2,4,7,8] The difference is, previously, everybody focused their attention exclusively on the light-particle Green's functions and proved the existence of the decoupled theory as $M\to\infty$. In this way, $g_R$, the gauge coupling constant of the unbroken subgroup, was related to $g_r$, the $\overline{\text{MS}}$ renormalized coupling constant of the full theory. As a result, in a minimal SU(5) GUT, the value of $M_X$ can be successfully calculated, to two-loop renormalization-group accuracy.[9]

The DS scheme is a complete renormalization scheme for *all* the Green's functions. When applied to the light-particle-irreducible light Green's function, it reproduces the earlier decoupling result.[8] The renormalization-group equations (RGE's) in the DS scheme respect the *tree-level hierarchy* and hence a perturbative expansion in the DS scheme encounters no hierarchy problem. The fine tuning thus is to be done only at the tree level, since higher-order radiative corrections in the DS scheme no longer act to spoil the hierarchy.

To understand and appreciate this point, it is best to review the hierarchy problem in the $\overline{\text{MS}}$ scheme. For definiteness, let us take the simple Lagrangian

$$-\tfrac{1}{2}(\partial_\mu\phi)^2-\tfrac{1}{2}(\partial_\mu\sigma)^2-\frac{M'^2}{2}\phi^2+\frac{M^2}{2}\sigma^2 -16\pi^2\left[\frac{\lambda_1}{24}\phi^4+\frac{\lambda_2}{24}\sigma^4+\frac{\lambda_3}{4}\phi^2\sigma^2\right] \tag{1.1}$$

and arrange for

$$m^2\equiv M'^2+\frac{3\lambda_3}{\lambda_2}M^2 \ll M^2 \tag{1.2}$$

at the tree level.

In the $\overline{\text{MS}}$ scheme, the one-loop RGE's for $m^2$, $M^2$ read ($t\equiv\ln\mu$)

$$\frac{d}{dt}m^2=\left[\lambda_1-\frac{3\lambda_3{}^2}{\lambda_2}\right]m^2-\left[4\lambda_3-\frac{12\lambda_3{}^2}{\lambda_2}\right]M^2\,, \tag{1.3}$$

$$\frac{d}{dt}M^2=\left[\lambda_2+\frac{3\lambda_3{}^2}{\lambda_2}\right]M^2-\lambda_3 m^2\,, \tag{1.4}$$

and it is the presence of the $M^2$ term on the right-hand side of Eq. (1.3) that spoils the tree-level hierarchy. For if at any given renormalization scale $\mu$, we have fine-tuned

$$m^2(\mu)\ll M^2(\mu) \tag{1.5}$$

by a small change in $\mu$, we find

$$\delta m^2\propto M^2\frac{\delta\mu}{\mu} \tag{1.6}$$

and the tree-level hierarchy becomes completely upset.

In the DS scheme, the one-loop RGE's for the $m$, $M$ read

$$\frac{d}{dt}m_R{}^2=\left[\lambda_{1R}-\frac{9\lambda_{3R}{}^2}{\lambda_{2R}}\right]m_R{}^2\,, \tag{1.7}$$

$$\frac{d}{dt}M_R{}^2=-\lambda_{3R}m_R{}^2+\frac{3\lambda_{3R}{}^2}{\lambda_{2R}}M_R{}^2\,, \tag{1.8}$$

$$\frac{d}{dt}\lambda_{1R}=3\left[\lambda_{1R}-\frac{6\lambda_{3R}{}^2}{\lambda_{2R}}\right]^2, \tag{1.9}$$

$$\frac{d}{dt}\lambda_{2R}=3\lambda_{3R}{}^2\,, \tag{1.10}$$

$$\frac{d}{dt}\lambda_{3R}=\left[\lambda_{1R}-\frac{6\lambda_{3R}{}^2}{\lambda_{2R}}\right]\lambda_{3R}\,, \tag{1.11}$$

and, as advertised, the RGE's for $m_R{}^2$ now respects the tree-level hierarchy ($m_R{}^2\ll M_R{}^2$). If you form the linear combination

$$\lambda^*\equiv\lambda_{1R}-\frac{9\lambda_{3R}{}^2}{\lambda_{2R}} \tag{1.12}$$

you will find naturally the RGE's for the decoupled theory

$$\frac{d}{dt}\lambda^*=3\lambda^{*2}\,, \tag{1.13}$$

$$\frac{d}{dt}m_R{}^2=\lambda^* m_R{}^2\ , \tag{1.14}$$

with $\lambda^*$ as the effective coupling constant of the decoupled theory. In fact, $\lambda^*$ is precisely the low-energy limit of the light-particle-irreducible light four-point function, already investigated earlier in decoupling studies.[8] Thus, the DS scheme includes the decoupling result for the light sector.

The DS scheme gives the RGE's for the $\lambda_{iR}$ *separately* and thus can be used to follow the evolution of the Higgs potential from $\mu=M_X$, say, down to low energy scales. It is only through such a study that we would be able to investigate the stability of any hierarchy chain.

## II. HIERARCHY WITHOUT SPONTANEOUS BREAKING

In this section, we shall deal with the hierarchy problem in its simplest manifestation, viz., without spontaneous symmetry breaking. Throughout this paper, we shall use the dimensional regularization of 't Hooft and Veltman,[5] with this definition

$$n=4-\epsilon\ . \tag{2.1}$$

Consider now the tree Lagrangian, interpolated to $n$ dimensions, with 't Hooft scale $\mu$,

$$\mathscr{L}_t=-\tfrac{1}{2}(\partial_\nu\phi)^2-\tfrac{1}{2}(\partial_\nu\sigma)^2-\frac{m^2}{2}\phi^2-\frac{M^2}{2}\sigma^2 -(16\pi^2)\left[\frac{\mu^2e^\gamma}{4\pi}\right]^{\epsilon/2}\left[\frac{\lambda_1}{24}\phi^4+\frac{\lambda_2}{24}\sigma^4+\frac{\lambda_3}{4}\phi^2\sigma^2\right]. \tag{2.2}$$

In (2.1), $\gamma$ is Euler's constant $=0.577\,21\ldots$. It and the $4\pi$ factor have been chosen (much as in $\overline{\text{MS}}$) for later convenience. In principle all the coupling and mass parameters in (2.2) depend implicitly on $\mu$. Equation (2.2) yields, upon perturbative expansion, Green's functions which have simple poles in the complex $n$ plane. To define the renormalized Green's function, it is necessary to add $\mathscr{L}_c$, the counterterm Lagrangian, to the tree Lagrangian. We choose to write the counterterms as

$$\mathscr{L}_c=-\tfrac{1}{2}(Z_\phi-1)(\partial_\nu\phi)^2-\tfrac{1}{2}(Z_\sigma-1)(\partial_\nu\sigma)^2-\tfrac{1}{2}(Z_mZ_\phi-1)m^2\phi^2-\tfrac{1}{2}(Z_MZ_\sigma-1)M^2\sigma^2 -(16\pi^2)\left[\frac{\mu^2e^\gamma}{4\pi}\right]^{\epsilon/2}\left[(Z_1-1)\frac{\lambda_1}{24}\phi^4+(Z_2-1)\frac{\lambda_2}{24}\sigma^4+(Z_3-1)\frac{\lambda_3}{4}\phi^2\sigma^2\right] \tag{2.3}$$

so that

$$\sqrt{Z_\phi}\phi=\phi_B\ ,\quad \sqrt{Z_\sigma}\sigma=\sigma_B\ , \tag{2.4}$$

$$\left[\frac{\mu^2e^\gamma}{4\pi}\right]^{\epsilon/2} Z_1Z_\phi{}^{-2}\lambda_1=\lambda_{1B}\ , \tag{2.5}$$

$$\left[\frac{\mu^2e^\gamma}{4\pi}\right]^{\epsilon/2} Z_2Z_\sigma{}^{-2}\lambda_2=\lambda_{2B}\ , \tag{2.6}$$

$$\left[\frac{\mu^2e^\gamma}{4\pi}\right]^{\epsilon/2} Z_3Z_\phi{}^{-1}Z_\sigma{}^{-1}\lambda_3=\lambda_{3B}\ , \tag{2.7}$$

$$Z_mm^2=m_B{}^2\ ,\quad Z_MM^2=M_B{}^2\ . \tag{2.8}$$

So far, we have not yet specified the renormalization scheme. In the $\overline{\text{MS}}$ scheme, we simply choose the counterterms to be *purely* pole terms in order to cancel the pole terms that arise from the radiative corrections due to (2.2) (along the way we have also gotten rid of $e^\gamma/4\pi$ from our renormalized Green's functions).

In $\overline{\text{MS}}$, to two-loop accuracy, the renormalization constants ($\hat{Z}$) are

$$\hat{Z}_\phi=1-\frac{\lambda_1{}^2}{12\epsilon}-\frac{\lambda_3{}^2}{4\epsilon}\ , \tag{2.9}$$

$$\hat{Z}_\sigma=1-\frac{\lambda_2{}^2}{12\epsilon}-\frac{\lambda_3{}^2}{4\epsilon}\ , \tag{2.10}$$

$$\hat{Z}_m=1+\frac{\lambda_1}{\epsilon}+\frac{2(\lambda_1{}^2+\lambda_3{}^2)}{\epsilon^2}-\frac{\lambda_1{}^2+\lambda_3{}^2}{2\epsilon}+\frac{M^2}{m^2}\left[\frac{\lambda_3}{\epsilon}+\frac{2\lambda_3{}^2+(\lambda_1+\lambda_2)\lambda_3}{\epsilon^2}-\frac{\lambda_3{}^2}{\epsilon}\right], \tag{2.11}$$

$$\hat{Z}_M=1+\frac{\lambda_2}{\epsilon}+\frac{2(\lambda_2{}^2+\lambda_3{}^2)}{\epsilon^2}-\frac{\lambda_2{}^2+\lambda_3{}^2}{\epsilon}+\frac{m^2}{M^2}\left[\frac{\lambda_3}{\epsilon}+\frac{2\lambda_3{}^2+(\lambda_1+\lambda_2)\lambda_3}{\epsilon^2}-\frac{\lambda_3{}^2}{\epsilon}\right], \tag{2.12}$$

$$\hat{Z}_1\lambda_1=\lambda_1+\frac{3\lambda_1{}^2+3\lambda_3{}^2}{\epsilon}+\frac{9\lambda_1{}^3+12\lambda_1\lambda_3{}^2+12\lambda_3{}^3+3\lambda_2\lambda_3{}^2}{\epsilon^2}-\frac{3\lambda_1{}^3+3\lambda_1\lambda_3{}^2+6\lambda_3{}^3}{\epsilon}\ , \tag{2.13}$$

$$\hat{Z}_2\lambda_2=\lambda_2+\frac{3\lambda_2^2+3\lambda_3^2}{\epsilon}+\frac{9\lambda_2^3+12\lambda_2\lambda_3^2+12\lambda_3^3+3\lambda_1\lambda_3^2}{\epsilon^2}-\frac{3\lambda_2^3+3\lambda_2\lambda_3^2+6\lambda_3^3}{\epsilon}, \tag{2.14}$$

$$\hat{Z}_3\lambda_3=\lambda_3+\frac{4\lambda_3^2+(\lambda_1+\lambda_2)\lambda_3}{\epsilon}+\frac{19\lambda_3^3+6(\lambda_1+\lambda_2)\lambda_3^2+2(\lambda_1^2+\lambda_2^2)\lambda_3+\lambda_1\lambda_2\lambda_3}{\epsilon^2}$$
$$-\frac{5\lambda_3^3+3(\lambda_1+\lambda_2)\lambda_3^2+\frac{1}{2}(\lambda_1^2+\lambda_2^2)\lambda_3}{\epsilon}. \tag{2.15}$$

In Eqs. (2.9)—(2.15) all the $\lambda_i$, $m^2$, $M^2$ are $\overline{\mathrm{MS}}$ parameters and depend on the renormalization scale $\mu$. In contrast, the bare parameters in Eqs. (2.5)—(2.8) do not depend on $\mu$. From them it is possible to deduce the renormalization-group equations[10]

$$\mu\frac{\partial}{\partial\mu}\lambda_1=-\epsilon\lambda_1+3(\lambda_1^2+\lambda_3^2)-\tfrac{17}{3}\lambda_1^3-5\lambda_1\lambda_3^2-12\lambda_3^3, \tag{2.16}$$

$$\mu\frac{\partial}{\partial\mu}\lambda_2=-\epsilon\lambda_2+3(\lambda_2^2+\lambda_3^2)-\tfrac{17}{3}\lambda_2^3+5\lambda_2\lambda_3^2-12\lambda_3^3, \tag{2.17}$$

$$\mu\frac{\partial}{\partial\mu}\lambda_3=-\epsilon\lambda_3+4\lambda_3^2+(\lambda_1+\lambda_2)\lambda_3-9\lambda_3^3-6(\lambda_1+\lambda_2)\lambda_3^2-\tfrac{5}{6}(\lambda_1^2+\lambda_2^2)\lambda_3, \tag{2.18}$$

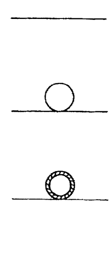
$$\mu\frac{\partial}{\partial\mu}m^2=(\lambda_1-\tfrac{5}{6}\lambda_1^2-\tfrac{1}{2}\lambda_3^2)m^2+(\lambda_3-2\lambda_3^2)M^2, \tag{2.19}$$

$$\mu\frac{\partial}{\partial\mu}M^2=(\lambda_3-2\lambda_3^2)m^2+(\lambda_2-\tfrac{5}{6}\lambda_2^2-\tfrac{1}{2}\lambda_3^2)M^2. \tag{2.20}$$

So far everything is standard, and because the light and heavy particles are not decoupled we have the hierarchy problem. However as was already noted in our earlier paper on decoupling, there is another subtraction scheme which directly implements the decoupling.

In order to better understand and appreciate this DS scheme, we perform first the DS renormalization to one loop accuracy. Consider the two-point Green's function for the $\phi$ field. It is given by

$$\Gamma_R^{(2\phi)}=-i\Big[p^2+m_R^2$$
$$+\tfrac{1}{2}\lambda_{1R}m_R^2\left[-\frac{2}{\epsilon}+\ln\frac{m_R^2}{\mu^2}-1\right]$$
$$+\tfrac{1}{2}\lambda_{3R}M_R^2\left[-\frac{2}{\epsilon}+\ln\frac{M_R^2}{\mu^2}-1\right]$$
$$+(Z_\phi-1)p^2+(Z_mZ_\phi-1)m_R^2\Big]. \tag{2.21}$$



The DS scheme simply subtracts away the contribution due to the heavy graph. We thus define (accurate to first order in $\lambda_R$)

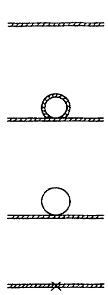
$$Z_\phi=1, \tag{2.22}$$

$$Z_m=1+\frac{\lambda_{1R}}{\epsilon}+\frac{M_R^2}{m_R^2}\frac{\lambda_{3R}}{\epsilon}-\frac{1}{2}\frac{M_R^2}{m_R^2}\lambda_{3R}\left[\ln\frac{M_R^2}{\mu^2}-1\right], \tag{2.23}$$

to achieve our goal.

Next we consider $\Gamma_R^{(2\sigma)}$ and find

$$\Gamma_R^{(2\sigma)}=-i\Big[p^2+M_R^2$$
$$+\tfrac{1}{2}\lambda_{2R}M_R^2\left[-\frac{2}{\epsilon}+\ln\frac{M_R^2}{\mu^2}-1\right]$$
$$+\tfrac{1}{2}\lambda_{3R}m_R^2\left[-\frac{2}{\epsilon}+\ln\frac{m_R^2}{\mu^2}-1\right]$$
$$+(Z_\sigma-1)p^2+(Z_MZ_\sigma-1)M_R^2\Big]. \tag{2.24}$$



Again we may subtract away the contribution due to the heavy graph by defining (accurate to first order in $\lambda_R$)

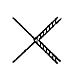
$$Z_\sigma=1\ ,$$
$$Z_M=1+\frac{\lambda_{2R}}{\epsilon}+\frac{m_R{}^2}{M_R{}^2}\frac{\lambda_{3R}}{\epsilon}-\tfrac{1}{2}\lambda_{2R}\left[\ln\frac{M_R{}^2}{\mu^2}-1\right]. \tag{2.25}$$

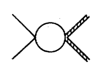
To determine the renormalization constants for $\lambda_i$, we consider in turn the four-point Green's functions. It is by now fairly obvious what is needed to subtract away the contribution due to the heavy graphs for both $\Gamma_R^{(4\phi)}$ and $\Gamma_R^{(4\sigma)}$. Define

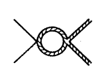
$$Z_1\lambda_{1R}=\lambda_{1R}+\frac{3(\lambda_{1R}{}^2+\lambda_{3R}{}^2)}{\epsilon}-\tfrac{3}{2}\lambda_{3R}{}^2\ln\frac{M_R{}^2}{\mu^2}\ , \tag{2.26}$$

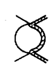
$$Z_2\lambda_{2R}=\lambda_{2R}+\frac{3(\lambda_{2R}{}^2+\lambda_{3R}{}^2)}{\epsilon}-\tfrac{3}{2}\lambda_{2R}{}^2\ln\frac{M_R{}^2}{\mu^2}\ . \tag{2.27}$$

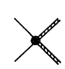
In $\Gamma_R^{(2\phi,2\sigma)}$ we encounter the following ($p^2 \ll M_R{}^2$):

$$\Gamma_R^{(2\phi,2\sigma)}=-i(16\pi^2)\Big\{\lambda_{3R}$$


$$+\tfrac{1}{2}\lambda_{1R}\lambda_{3R}\left[-\frac{2}{\epsilon}+\ln\frac{m_R{}^2}{\mu^2}\right]$$


$$+\tfrac{1}{2}\lambda_{2R}\lambda_{3R}\left[-\frac{2}{\epsilon}+\ln\frac{M_R{}^2}{\mu^2}\right]$$


$$+2\lambda_{3R}{}^2\left[-\frac{2}{\epsilon}+\ln\frac{M_R{}^2}{\mu^2}-1+O\left[\frac{m_R{}^2}{M_R{}^2}\right]\right]$$


$$+(Z_3-1)\lambda_{3R}$$

$$\Big\}\ . \tag{2.28}$$

Here, in the fourth term of (2.28) we have assumed the hierarchy condition and dropped the $m_R{}^2/M_R{}^2$ term. We now can again define $Z_3$ such that the graphs with heavy internal lines decouple, viz.,

$$Z_3\lambda_{3R}=\lambda_{3R}+\frac{4\lambda_{3R}{}^2+(\lambda_{1R}+\lambda_{2R})\lambda_{3R}}{\epsilon}-\tfrac{1}{2}\lambda_{2R}\lambda_{3R}\ln\frac{M_R{}^2}{\mu^2}-2\lambda_{3R}{}^2\left[\ln\frac{M_R{}^2}{\mu^2}-1\right]. \tag{2.29}$$

With these definitions of $Z$'s, we have achieved the desired decoupling, to one-loop accuracy, for *all* Green's functions, whatever the nature and number of external legs. The contribution due to heavy internal lines are suppressed by powers of $m_R{}^2/M_R{}^2$ relative to those graphs with light internal lines. The one-loop renormalization-group equations for the DS scheme can be obtained again from the observation that the bare parameters do not depend on $\mu$. They are, to one-loop order,

$$\mu\frac{\partial}{\partial\mu}\lambda_{1R}=-\epsilon\left[\lambda_{1R}-\tfrac{3}{2}\lambda_{3R}{}^2\ln\frac{M_R{}^2}{\mu^2}\right]+3\lambda_{1R}{}^2\ , \tag{2.30}$$

$$\mu\frac{\partial}{\partial\mu}\lambda_{2R}=-\epsilon\left[\lambda_{2R}-\tfrac{3}{2}\lambda_{2R}{}^2\ln\frac{M_R{}^2}{\mu^2}\right]+3\lambda_{3R}{}^2\ , \tag{2.31}$$

$$\mu\frac{\partial}{\partial\mu}\lambda_{3R}=-\epsilon\left[\lambda_{3R}-(2\lambda_{3R}{}^2+\tfrac{1}{2}\lambda_{2R}\lambda_{3R})\ln\frac{M_R{}^2}{\mu^2}+2\lambda_{3R}{}^2\right]+\lambda_{1R}\lambda_{3R}\ , \tag{2.32}$$

$$\mu\frac{\partial}{\partial\mu}m_R{}^2=\lambda_1 m_R{}^2\ , \tag{2.33}$$

$$\mu\frac{\partial}{\partial\mu}M_R{}^2=\lambda_{3R}M_R{}^2\ . \tag{2.34}$$

To higher loops, the procedure is a straightforward generalization of the BPHZ formalism[11] as adapted to dimensional regularization. Consider a graph $G$ of the full theory. Recall that the usual BPHZ procedure consists in first introducing a $K$ operator[12] such that it picks out only the pole pieces of a Laurent series,

$$K\left[\sum_{n=-\infty}^{\infty} b_n\epsilon^n\right]=\sum_{n<0} b_n\epsilon^n \tag{2.35}$$

and then defining $R'G$ through the recursive relation

$$R'G=G+\sum(-KR'\gamma_1)\cdots(-KR'\gamma_m)\frac{G}{\gamma_1+\cdots+\gamma_m}\,, \tag{2.36}$$

where $\gamma_1,\ldots,\gamma_m$ are disjoint one-particle-irreducible divergent subgraphs and $G/(\gamma_1+\cdots+\gamma_m)$ is the new reduced graph obtained from $G$ by shrinking the disjoint subgraphs $\gamma_1,\ldots,\gamma_m$ to a point. The amplitude $R'G$ contains still overall $1/\epsilon$ singularities which can be removed by performing one more $K$ operation, i.e.,

$$RG=(1-K)R'G\,. \tag{2.37}$$

For the DS scheme, our modification consists in introducing a $K_\gamma$ operator which depends on the subgraph $\gamma$. We distinguish between two types of $\gamma$:

(I) $\gamma$ has only light internal lines.

(II) $\gamma$ heavy internal lines and reduces

(a) to a point (with two, three, or four legs) upon shrinking topologically the heavy internal lines, or

(b) to a $\gamma'$ with light internal lines.

For type I, the $K_\gamma$ operation is as before, while for type II,

$$K_\gamma\mathscr{R}'\gamma(\mathrm{II})=K_\gamma\left[\sum_{n=-\infty}^{\infty} b_n\epsilon^n\right]=\sum_{n\le 0} b_n\epsilon^n\,. \tag{2.38}$$

The $b_0$ coefficient is to be calculated

(i) with zero momentum coming into the subgraph $\gamma(\mathrm{II})$, and

(ii) in an expansion in the parameter $m^2/M^2$, only those terms in $b_0$ that would remain as $M^2\to\infty$ are to be kept.

Just as in BPHZ, $\mathscr{R}'G$ is defined recursively

$$\mathscr{R}'G=G+\sum(-K_\gamma\mathscr{R}'\gamma_1)\cdots(-K_\gamma\mathscr{R}'\gamma_m)\times\frac{G}{\gamma_1+\cdots+\gamma_m} \tag{2.39}$$

with finally

$$\mathscr{R}G=(1-K_\gamma)\mathscr{R}'G\,. \tag{2.40}$$

The net result, for a theory without spontaneous symmetry breaking, is very simple. To leading order as $M^2\to\infty$, and for $p\ll M$, only *light* lines propagate inside all one-particle-irreducible Green's functions, regardless of number and nature of external legs. This does not mean that graphs with heavy internal lines totally disappear from the full theory. For $p\gtrsim M$, they again contribute. It is only for $p\ll M$ that they decouple.

For completeness, we record here the two-loop RGE's for the Lagrangian (2.2) in the DS scheme ($L\equiv\ln M_R{}^2/\mu^2$)

$$\mu\frac{\partial}{\partial\mu}\lambda_{1R}=-\epsilon[\lambda_{1R}-\tfrac{3}{2}\lambda_{3R}{}^2L+(3\lambda_{3R}{}^3+\tfrac{3}{2}\lambda_{1R}\lambda_{3R}{}^2+\tfrac{3}{4}\lambda_{2R}\lambda_{3R}{}^2)L^2+\tfrac{5}{2}\lambda_{1R}\lambda_{3R}{}^2L]+3\lambda_{1R}{}^2-\tfrac{17}{3}\lambda_{3R}{}^2\,, \tag{2.41}$$

$$\mu\frac{\partial}{\partial\mu}\lambda_{2R}=-\epsilon[\lambda_{2R}-\tfrac{3}{2}\lambda_{2R}{}^2L+(3\lambda_{3R}{}^2-\tfrac{3}{2}\lambda_{2R}\lambda_{3R}{}^2+\tfrac{9}{4}\lambda_{2R}{}^3)L^2+(\tfrac{17}{6}\lambda_{2R}{}^3+\tfrac{5}{2}\lambda_{2R}\lambda_{3R}{}^2)L]+3\lambda_{3R}{}^2\,, \tag{2.42}$$

$$\begin{aligned}\mu\frac{\partial}{\partial\mu}\lambda_{3R}=&-\epsilon[\lambda_{3R}-(2\lambda_{3R}{}^2+\tfrac{1}{2}\lambda_{2R}\lambda_{3R})L+2\lambda_{3R}{}^2+(4\lambda_{3R}{}^3-\tfrac{1}{2}\lambda_{1R}\lambda_{3R}{}^2+\tfrac{3}{2}\lambda_{2R}\lambda_{3R}{}^2+\tfrac{1}{2}\lambda_{2R}{}^2\lambda_{3R})L^2\\&+(-\tfrac{7}{2}\lambda_{3R}{}^3+4\lambda_{1R}\lambda_{3R}{}^2+\tfrac{23}{12}\lambda_{2R}\lambda_{3R}{}^2+\tfrac{1}{2}\lambda_{2R}{}^2\lambda_{3R})L]+\lambda_{1R}\lambda_{3R}-\tfrac{5}{6}\lambda_{1R}{}^2\lambda_{3R}\,,\end{aligned} \tag{2.43}$$

$$\mu\frac{\partial}{\partial\mu}m_R{}^2=(\lambda_{1R}-\tfrac{5}{6}\lambda_{1R}{}^2)m_R{}^2\,, \tag{2.44}$$

$$\mu\frac{\partial}{\partial\mu}M_R{}^2=\lambda_{3R}m_R{}^2\,. \tag{2.45}$$

## III. HIERARCHY WITH SPONTANEOUS BREAKING

In this section we turn to the physically more interesting case of a spontaneous symmetry breaking with mass hierarchy. Because of the presence of anomalous vertices that arise from the shift in the scalar field, the analysis is technically more involved. The key will still be that the renormalization constants $Z$ include an $\epsilon^0$ piece designed to make the heavy-particle virtual effects disappear.

Consider the Lagrangian (1.1) but now fully dressed with renormalization scale factors and shifted with respect to $\sigma$,

$$\sigma\to\sigma+\tilde{\mu}^{-\epsilon/2}V\,, \tag{3.1}$$

where $\tilde{\mu}^2\equiv\mu^2e^\gamma/4\pi$. For convenience we shall refer to $16\pi^2\lambda_i$ as $\kappa_i$. Thus

$$\begin{aligned}\mathscr{L}_t=&-\tfrac{1}{2}(\partial_\nu\phi)^2-\tfrac{1}{2}(\partial_\nu\sigma)^2+\tfrac{1}{2}\left[M^2-\frac{\kappa_2V^2}{2}\right]\sigma^2-\tfrac{1}{2}\left[M'^2+\frac{\kappa_3V^2}{2}\right]\phi^2+\left[M^2-\frac{\kappa_2V^2}{6}\right]V\sigma\tilde{\mu}^{-\epsilon/2}\\&-\tilde{\mu}^{\epsilon}\left[\frac{\kappa_1}{24}\phi^4+\frac{\kappa_2}{24}\sigma^4+\frac{\kappa_3}{4}\phi^2\sigma^2+\frac{\kappa_2}{6}\tilde{\mu}^{-\epsilon/2}V\sigma^3+\frac{\kappa_3}{2}\tilde{\mu}^{-\epsilon/2}V\phi^2\sigma\right].\end{aligned} \tag{3.2}$$

At the tree level, we shall choose $V$ to be given by

$$V^2=\frac{6M^2}{\kappa_2} \tag{3.3}$$

so that the term linear in $\sigma$ vanishes in the tree Lagrangian.

The counterterms will be chosen as

$$\begin{aligned}\mathscr{L}_c=&-\tfrac{1}{2}(Z_\phi-1)(\partial_\nu\phi)^2-\tfrac{1}{2}(Z_\sigma-1)(\partial_\nu\sigma)^2+(Z_M Z_\sigma-1)M^2V\sigma\tilde{\mu}^{-\epsilon/2}-\frac{V^3}{6}(Z_2-1)\kappa_2\sigma\tilde{\mu}^{-\epsilon/2}\\&+\left[M^2-\frac{\kappa_2V^2}{2}\right]T\sigma\tilde{\mu}^{-\epsilon/2}+\tfrac{1}{2}(Z_MZ_\sigma-1)M^2\sigma^2-\frac{V^2}{4}(Z_2-1)\kappa_2\sigma^2-\tfrac{1}{2}\kappa_2VT\sigma^2-\tfrac{1}{2}(Z_{M'}Z_\phi-1)M'^2\phi^2\\&-\frac{V^2}{4}(Z_3-1)\kappa_3\phi^2-\tfrac{1}{2}\kappa_3VT\phi^2-\tilde{\mu}^{\epsilon}\left[(Z_1-1)\frac{\kappa_1}{24}\phi^4+(Z_2-1)\frac{\kappa_2}{24}\sigma^4+(Z_3-1)\frac{\kappa_3}{4}\phi^2\sigma^2\right]\\&-\tilde{\mu}^{\epsilon/2}\left[(Z_2-1)\frac{\kappa_2V}{6}\sigma^3+\frac{\kappa_2}{6}T\sigma^3+(Z_3-1)\frac{\kappa_3V}{2}\phi^2\sigma+\frac{\kappa_3}{2}T\phi^2\sigma\right]\end{aligned} \tag{3.4}$$

so that

$$M_B{}^2=Z_MM^2\ ,\quad M_B'{}^2=Z_{M'}M'^2\ , \tag{3.5}$$

$$V_B=\tilde{\mu}^{-\epsilon/2}\sqrt{Z_\sigma}(V+T)\ , \tag{3.6}$$

$$\lambda_{1B}=\tilde{\mu}^{\epsilon}Z_1Z_\phi{}^{-2}\lambda_1\ , \tag{3.7}$$

$$\lambda_{2B}=\tilde{\mu}^{\epsilon}Z_2Z_\sigma{}^{-2}\lambda_2\ , \tag{3.8}$$

$$\lambda_{3B}=\tilde{\mu}^{\epsilon}Z_3Z_\phi{}^{-1}Z_\sigma{}^{-1}\lambda_3\ . \tag{3.9}$$

In (3.4), $T$ is to be fixed by the requirement that

$$\langle\sigma\rangle=0 \tag{3.10}$$

including radiative corrections.

Together, (3.3) with (3.4) gives the finite renormalized Green's functions for the broken theory. The counterterm Lagrangian (3.4) includes in it the needed infinite tadpole counterterms; or, put in another way, the infinite tadpole counterterms have been successfully related to $Z_M$, $Z$, and $Z_2$ renormalization constants. In our notation, $T$ is the remaining finite counterterm, determined by (3.10).

If we define

$$m^2\equiv M'^2+3\kappa_3\frac{M^2}{\kappa_2}\ , \tag{3.11}$$

then in $\overline{\text{MS}}$, the one-loop RGE's read

$$\mu\frac{\partial}{\partial\mu}\lambda_1=-\epsilon\lambda_1+3(\lambda_1{}^2+\lambda_3{}^2)\ , \tag{3.12}$$

$$\mu\frac{\partial}{\partial\mu}\lambda_2=-\epsilon\lambda_2+3(\lambda_2{}^2+\lambda_3{}^2)\ , \tag{3.13}$$

$$\mu\frac{\partial}{\partial\mu}\lambda_3=-\epsilon\lambda_3+4\lambda_3{}^2+(\lambda_1+\lambda_2)\lambda_3\ , \tag{3.14}$$

$$\mu\frac{\partial}{\partial\mu}m^2=\left[\lambda_1-\frac{3\lambda_3{}^2}{\lambda_2}\right]m^2-4\left[\lambda_3-\frac{3\lambda_3{}^2}{\lambda_2}\right]M^2\ , \tag{3.15}$$

$$\mu\frac{\partial}{\partial\mu}M^2=-\lambda_3m^2+\left[\lambda_2+\frac{3\lambda_3{}^2}{\lambda_2}\right]M^2\ . \tag{3.16}$$

For the discussion of decoupling renormalization in the presence of the anomalous (three-legged) vertices, we rely on the insight gained from all the earlier work on decoupling, especially the notion of light-particle-irreducible light Green's functions.

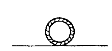
We consider in turn the one-particle-irreducible Green's functions and use the decoupling criterion to determine the renormalization constants. We begin with $\Gamma_R^{(2)}$. For convenience of notation, we leave off the subscript $R$ on all the quantities below as understood. Then for $p^2\ll M^2$, and $m^2\ll M^2$ ($L\equiv\ln 2M^2/\mu^2$, $l\equiv\ln m^2/\mu^2$)

$$\Gamma^{(2\phi)}=-i\left[p^2+m^2\right.$$

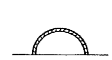


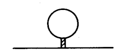
$$+\tfrac{1}{2}\lambda_1m^2(l-1)-\frac{\lambda_1m^2}{\epsilon}$$

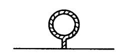
$$+\lambda_3M^2(L-1)-\frac{2\lambda_3M^2}{\epsilon}$$



$$+\frac{3}{2}\frac{\lambda_3{}^2}{\lambda_2}p^2+\frac{6\lambda_3{}^2}{\lambda_2}M^2(L-1)+\frac{3\lambda_3{}^2}{\lambda_2}m^2(L-l)-\frac{12\lambda_3{}^2M^2}{\lambda_2\epsilon}$$



$$-\frac{3}{2}\frac{\lambda_3{}^2}{\lambda_2}m^2(l-1)+\frac{3\lambda_3{}^2m^2}{\lambda_2\epsilon}$$


$$-3\lambda_3M^2(L-1)+\frac{6\lambda_3M^2}{\epsilon}$$


$$+(Z_\phi-1)p^2+(Z_{M'}Z_\phi-1)M'^2+\frac{3\lambda_3}{\lambda_2}M^2(Z_MZ_\sigma-1-Z_2+Z_3)\Bigg] . \tag{3.17}$$

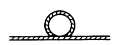
In DS renormalization, we demand that

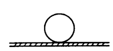
$$\Gamma_R^{(2\phi)}=-i\left[p^2+m^2+\frac{1}{2}\left[\lambda_1-\frac{9\lambda_3{}^2}{\lambda_2}\right]m^2(l-1)\right] , \tag{3.18}$$

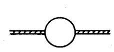
where $\lambda^*\equiv\lambda_1-9\lambda_3{}^2/\lambda_2$ is the effective coupling constant of the decoupled theory. In terms of the diagrams shown above, the $\lambda^*$ is the light-particle-irreducible $4\phi$ coupling and its contribution can be traced by shrinking the heavy internal lines to a point.

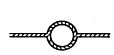
Next we consider $\Gamma^{(2\sigma)}$, again for $p^2\ll M^2$, $m^2\ll M^2$:

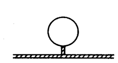
$$\Gamma^{(2\sigma)}=-i\Bigg[p^2+2M^2$$

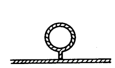

$$+\lambda_2M^2(L-1)-\frac{2\lambda_2M^2}{\epsilon}$$


$$+\tfrac{1}{2}\lambda_3m^2(l-1)-\frac{\lambda_3m^2}{\epsilon}$$


$$+\frac{1}{2}\frac{\lambda_3{}^2}{\lambda_2}\frac{M^2}{m^2}p^2+\frac{3\lambda_3{}^2}{\lambda_2}M^2l-\frac{6\lambda_3{}^2}{\lambda_2}\frac{M^2}{\epsilon}$$


$$+\tfrac{1}{4}\lambda_2p^2+3\lambda_2M^2L-\frac{6\lambda_2M^2}{\epsilon}$$


$$-\tfrac{3}{2}\lambda_3m^2(l-1)+\frac{3\lambda_3m^2}{\epsilon}$$


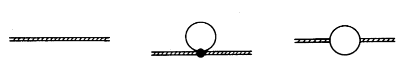
$$-3\lambda_2M^2(L-1)+\frac{6\lambda_2M^2}{\epsilon}$$


$$+(Z_\sigma-1)p^2+2M^2(Z_MZ_\sigma-1)\Bigg] . \tag{3.19}$$

In the DS renormalization, we now demand that

$$\Gamma_R^{(2\sigma)}=-i\left[p^2+2M^2-\lambda_3m^2(l-1)+\frac{1}{2}\frac{\lambda_3{}^2}{\lambda_2}\frac{M^2}{m^2}p^2+\frac{3\lambda_3{}^2}{\lambda_2}M^2l\right] . \tag{3.20}$$

This reflects the fact that upon topologically shrinking the heavy internal lines, the DS-renormalized $\Gamma^{(2\sigma)}$ becomes given by the simple graphs

 ,

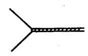
where the black box is the light-particle-irreducible $2\phi2\sigma$-coupling vertex. For $p^2\ll M^2$, this lPI vertex is given, to lowest order, by

$$\lambda_3-\frac{V^2\lambda_3\kappa_2}{2M^2}=-2\lambda_3 \,. \tag{3.21}$$

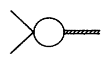
Having studiously, up to this point, included the tadpole contributions together with the tadpole counterterms, we note that Eq. (3.10) implies that, actually, the tadpole graphs will always cancel against all tadpole counterterms. It was only for pedagogical reasons that we included them in our discussion so far. For our discussion of higher-point functions we will for simplicity not even list them although we shall include them in our results.

For the one-particle-irreducible $\Gamma^{(2\phi,1\sigma)}$, we have (as $m^2/M^2 \to 0$)

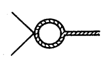
$$\Gamma^{(2\phi,1\sigma)}=-i(16\pi^2)V\Big[\lambda_3$$



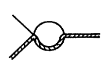
$$+\tfrac{1}{2}\lambda_1\lambda_3 l-\frac{\lambda_1\lambda_3}{\epsilon}$$



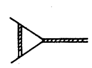
$$+\tfrac{1}{2}\lambda_2\lambda_3 L-\frac{\lambda_2\lambda_3}{\epsilon}$$



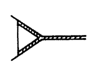
$$+2\lambda_3{}^2(L-1)-\frac{4\lambda_3{}^2}{\epsilon}$$



$$+\frac{3\lambda_3{}^3}{\lambda_2}(L-l-1)$$

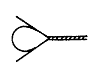


$$+3\lambda_3{}^2$$

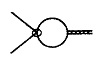


$$+(Z_3-1)\lambda_3+\frac{T}{V}\lambda_3\Big] \,. \tag{3.22}$$

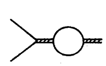
Upon shrinking the heavy internal lines, we find a new reduced graph that cannot be generated by the original Lagrangian. It is



which we will subtract away in our DS renormalization. The other reduced graph is of the type



which can be generated by the original Lagrangian. It, together with the original graph (that gave rise to $\frac{1}{2}\lambda_1\lambda_3 l$), looks *almost* like the lPI $4\phi$ coupling, but not quite. The thing that is missing is the graph

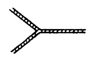
 .

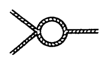
Such a graph is not allowed in our $\Gamma^{(2\phi,1\sigma)}$ since it is to be one-particle-irreducible. Thus our DS-renormalization requirement can only be

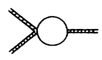
$$\Gamma_R^{(2\phi,1\sigma)}=-i(16\pi^2)V\left[\lambda_3+\frac{1}{2}\left[\lambda_1-\frac{6\lambda_3{}^2}{\lambda_2}\right]\lambda_3 l\right] \,. \tag{3.23}$$

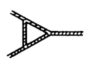
Note that it is by adhering to the one-particle irreducibility that we will be able to interpolate easily between $\Gamma^{(2\phi,1\sigma)}$ for $p^2 \ll M^2$ and $\Gamma^{(2\phi,1\sigma)}$ for $p^2 \gg M^2$.

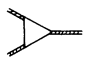
Now we turn to $\Gamma^{(3\sigma)}$, for $p^2$, $m^2 \ll M^2$:

$$\Gamma^{(3\sigma)} = -i(16\pi^2)V\Big[\lambda_2$$


$$+\tfrac{3}{2}\lambda_2{}^2 L - \frac{3\lambda_2{}^2}{\epsilon}$$


$$+\tfrac{3}{2}\lambda_3{}^2 l - \frac{3\lambda_3{}^2}{\epsilon}$$


$$+\tfrac{3}{2}\lambda_2{}^2$$


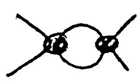
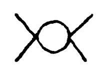
$$+\frac{3\lambda_3{}^3}{\lambda_2}\frac{M^2}{m^2}$$


$$+(Z_2-1)\lambda_2 + \frac{T}{V}\lambda_2\Big] . \tag{3.24}$$

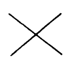
For this, it is fairly obvious that our requirement for the DS renormalization is to subtract away the graphs with heavy internal lines. Thus we must have

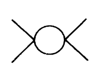
$$\Gamma_R^{(3\sigma)} = -i(16\pi^2)V\left[\lambda_2 + \tfrac{3}{2}\lambda_3{}^2 l + \frac{3\lambda_3{}^3}{\lambda_2}\frac{M^2}{m^2}\right] \tag{3.25}$$

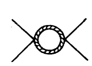
corresponding to the graphs, respectively,

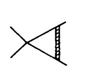
 $=$  $+2$  $+$  .

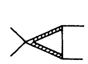
Finally we come to $\Gamma^{(4\phi)}$. For $p^2$, $m^2 \ll M^2$, it is given by

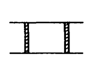
$$\Gamma^{(4\phi)} = -i(16\pi^2)\Big[\lambda_1$$


$$+\tfrac{3}{2}\lambda_1{}^2 l - \frac{3\lambda_1{}^2}{\epsilon}$$
 +permutations

$$+\tfrac{3}{2}\lambda_3{}^2 L - \frac{3\lambda_3{}^2}{\epsilon}$$
 +permutations

$$+\frac{18\lambda_1\lambda_3{}^2}{\lambda_2}(L-l-1)$$
 +permutations

$$+\frac{18\lambda_3{}^3}{\lambda_2}$$
 +permutations

$$-54\frac{\lambda_3{}^4}{\lambda_2{}^2}(L-l-2)$$
 +permutations

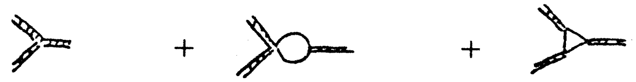
$$+(Z_1-1)\lambda_1\Big] . \tag{3.26}$$

By now, it should be fairly clear that our DS-renormalization requirement is to be

$$\Gamma_R^{(4\phi)} = -i(16\pi^2)\left[\lambda_1 + \frac{3}{2}\left[\lambda_1 - \frac{6\lambda_3{}^2}{\lambda_2}\right]^2 l\right] \tag{3.27}$$

corresponding to the graphs

where the round black box is the partially light-particle-irreducible $4\phi$ vertex, consisting of



Equations (3.18), (3.20), (3.23), (3.25), and (3.27) together with (3.10) completely fix the counterterms, to one-loop order. They are ($m^2 \equiv M'^2 + 3\lambda_3 M^2/\lambda_2$)

$$Z_1\lambda_1=\lambda_1+\frac{3\lambda_1^2+3\lambda_3^2}{\epsilon}-\left[\tfrac{3}{2}\lambda_3^2+\frac{18\lambda_1\lambda_3^2}{\lambda_2}-\frac{54\lambda_3^4}{\lambda_2^2}\right]L+\frac{18\lambda_1\lambda_3^2}{\lambda_2}-\frac{18\lambda_3^3}{\lambda_2}-\frac{108\lambda_3^4}{\lambda_2^2}\,, \tag{3.28}$$

$$Z_2\lambda_2=\lambda_2+\frac{3\lambda_2^2+3\lambda_3^2}{\epsilon}-\tfrac{3}{2}\lambda_2^2(L+3)\,, \tag{3.29}$$

$$Z_3\lambda_3=\lambda_3+\frac{4\lambda_3^2+(\lambda_1+\lambda_2)\lambda_3}{\epsilon}-2\lambda_3^2(L+\tfrac{1}{2})-\tfrac{1}{2}\lambda_2\lambda_3(L+6)-\frac{3\lambda_3^3}{\lambda_2}(L-1)\,, \tag{3.30}$$

$$Z_M M^2=M^2+\frac{\lambda_2 M^2}{\epsilon}+\frac{3\lambda_3^2}{\lambda_2}\frac{M^2}{\epsilon}-\frac{\lambda_3 m^2}{\epsilon}-\tfrac{1}{2}\lambda_2 M^2(L+\tfrac{3}{2})\,, \tag{3.31}$$

$$Z_{M'}M'^2=M'^2+\frac{\lambda_1 m^2}{\epsilon}-\frac{\lambda_3 M^2}{\epsilon}-\frac{3\lambda_1\lambda_3}{\lambda_2}\frac{M^2}{\epsilon}+\tfrac{1}{2}\lambda_3 M^2(L-7)+\frac{9\lambda_3^2}{\lambda_2}M^2+\frac{9\lambda_3^3}{\lambda_2^2}M^2(L-\tfrac{3}{2})-\frac{3\lambda_3^2}{\lambda_2}m^2(L-\tfrac{3}{2})\,, \tag{3.32}$$

$$Z_\phi=1-\frac{3}{2}\frac{\lambda_3^2}{\lambda_2}\,, \tag{3.33}$$

$$Z_\sigma=1-\tfrac{1}{4}\lambda_2\,, \tag{3.34}$$

$$T=3\lambda_2 V-\frac{m^2}{M^2}\left[\frac{\lambda_3 V}{4}\right](l-1)\,. \tag{3.35}$$

The one-loop DS-scheme RGE's may thus be written down, in the $\epsilon \to 0$ limit, as

$$\mu\frac{\partial}{\partial\mu}\lambda_1=3\left[\lambda_1-\frac{6\lambda_3^2}{\lambda_2}\right]^2, \tag{3.36}$$

$$\mu\frac{\partial}{\partial\mu}\lambda_2=3\lambda_3^2\,, \tag{3.37}$$

$$\mu\frac{\partial}{\partial\mu}\lambda_3=\left[\lambda_1-\frac{6\lambda_3^2}{\lambda_2}\right]\lambda_3\,, \tag{3.38}$$

$$\mu\frac{\partial}{\partial\mu}m^2=\left[\lambda_1-\frac{9\lambda_3^2}{\lambda_2}\right]m^2\,, \tag{3.39}$$

$$\mu\frac{\partial}{\partial\mu}M^2=-\lambda_3 m^2+\frac{3\lambda_3^2}{\lambda_2}M^2\,. \tag{3.40}$$

As was already mentioned in the Introduction, this set of DS RGE's includes the earlier decoupling result. Namely, if we restrict our attention to the light sector, and consider the 1PI Green's functions ($\Gamma_l$). Then for $p^2 \ll M^2$, the Green's function, $\Gamma_l$, can all be generated by a decoupled Lagrangian with $\lambda^*$ and $m^*$ satisfying the RGE's as $\epsilon \to 0$,

$$\mu\frac{\partial}{\partial\mu}\lambda^*=3\lambda^{*2}\,, \tag{3.41}$$

$$\mu\frac{\partial}{\partial\mu}m^*=\lambda^* m^*\,. \tag{3.42}$$

If we consider the $\Gamma_l^{(4\phi)}$ to lowest order, we find for $p^2 \ll M^2$

$$\Gamma_l^{(4\phi)}=-i(16\pi^2)\left[\lambda_1-\frac{9\lambda_3^2}{\lambda_2}\right] \tag{3.43}$$

and our DS RGE's can be directly used to check that they reproduce Eqs. (3.41) and (3.42) with the identification $\lambda^*=\lambda_1-9\lambda_3^2/\lambda_2$, $m^*=m$.

At this point we comment on the difference between the DS renormalization of $M^2>0$ theory of Sec. II and the spontaneous symmetry-breaking theory of this section. In Sec. II we could, for $p^2 \ll M^2$, arrange to remove all graphs with heavy internal lines, so that in *all* Green's functions, to leading order in $m^2$, $p^2/M^2$ only light lines propagate internally. It makes for simple rules for calculating higher-order graphs in the DS scheme.

Here, however, there are new anomalous vertices, but the number of renormalization constants remains as before. Our choice of the DS scheme arranges to remove graphs with heavy internal lines from $\Gamma^{(2\phi)}$, $\Gamma^{(2\sigma)}$, $\Gamma^{(3\sigma)}$, $\Gamma^{(2\phi,1\sigma)}$, and $\Gamma^{(4\phi)}$. For $\Gamma^{(4\sigma)}$ it turns out to be also in the end "clean." In $\Gamma^{(2\phi,2\sigma)}$, however some graphs with heavy internal lines remain. They do not affect the one-loop RGE's satisfied by $\Gamma^{(2\phi,2\phi)}$ and much as one-loop constants properly belong in a treatment of two-loop renormalization.

Finally, we may rephrase our result in terms of its relation to the $\overline{\text{MS}}$ scheme. Let $\Gamma_r$ be the $\overline{\text{MS}}$ renormalized 1PI Green's functions, and $\Gamma_R$ be the $\overline{\text{DS}}$-renormalized 1PI Green's functions, then

$$\Gamma_r^{(n\phi,k\sigma)}(p,\lambda_r,M_r,m_r,V_r,\mu) = Z_\phi^{-n/2} Z_\sigma^{-k/2} \Gamma_R(p,\lambda_R,M_R,m_R,V_R,\mu) , \quad (3.44)$$

where

$$\begin{aligned} \lambda_{1R} &= \widetilde{Z}_1^{-1} Z_\phi^2 \lambda_{1r} , \\ \lambda_{2R} &= \widetilde{Z}_2^{-1} Z_\sigma^2 \lambda_{2r} , \\ \lambda_{3R} &= \widetilde{Z}_3^{-1} Z_\phi Z_\sigma \lambda_{3r} , \\ M_R^2 &= \widetilde{Z}_M^{-1} M_r^2 , \\ M_R'^2 &= \widetilde{Z}_{M'}^{-1} M_r'^2 , \\ m_R^2 &\equiv M_R'^2 + \tfrac{1}{2} V_R^2 \kappa_{3R} M_R^2 , \\ \sqrt{Z_\sigma}(V_R+T_R) &= V_r + T_r , \end{aligned} \quad (3.45)$$

with

$$\begin{aligned} Z_\sigma &= 1 - \tfrac{1}{4}\lambda_{2r} + O(\lambda_r^2) , \\ Z_\phi &= 1 - \frac{3}{2}\frac{\lambda_{3r}^2}{\lambda_{2r}} + O(\lambda_r^2) , \end{aligned} \quad (3.46)$$

and the other heavy renormalization constants, $\widetilde{Z}$, being functions of $\lambda_r$, $M_r/\mu$.

## IV. RGE SOLUTION

The major result of our previous section may be restated as follows. In the presence of Higgs breakdown, it is nevertheless possible to define to one-loop RG accuracy the decoupling renormalization of the one-particle irreducible Green's functions. As a result, in contrast with earlier treatments of hierarchy problem, it is now possible to follow the evolution of *separately* the $\lambda_1,\lambda_2,\lambda_3$.

Let us now recall the situation for the relation between, say, $g_{3R}$ of the SU(3) gauge group and the "parent" SU(5) unbroken gauge coupling $g_r$. $g_{3R}$ obeys the one-loop RGE

$$\frac{d}{dt} g_{3R} = -\tfrac{1}{2} b_3 g_{3R}^3 / 16\pi^2 \quad (4.1)$$

while

$$\frac{d}{dt} g_r = -\tfrac{1}{2} b g_r^3 / 16\pi^2 . \quad (4.2)$$

For any $\mu$, the $g_{3R}$ equation may be solved by

$$\frac{16\pi^2}{g_{3R}^2} = \frac{16\pi^2}{g_0^2} - \tfrac{1}{2} b_3 \ln \frac{M_x^2}{\mu^2} , \quad (4.3)$$

where we have fixed the boundary condition to be such that

$$g_{3R}(\mu = M_x) = g_0 . \quad (4.4)$$

Previous work with grand unified models suggest the boundary condition that $g_0$ be identified as the unified coupling constants.

Similarly, for our $\lambda_{1R},\lambda_{2R},\lambda_{3R}$ we may integrate them with the boundary condition that $\lambda_{iR}(\mu = M_x) = \lambda_{i0}$, where $\lambda_{i0}$ are the tree-level parameters of the grand-unified Higgs potential.

For our set of equations, we may recast it as

$$\lambda^* \equiv \lambda_1 - \frac{9\lambda_3^2}{\lambda_2} , \quad \rho \equiv \frac{\lambda_3}{\lambda_2} \quad (4.5)$$

$$\frac{d}{dt}\lambda^* = 3(\lambda^*)^2 , \quad (4.6)$$

$$\frac{d}{dt}\rho = \lambda^* \rho , \quad (4.7)$$

$$\frac{d}{dt}\lambda_2 = 3\rho^2 \lambda_2 , \quad (4.8)$$

and directly integrate for the result

$$\frac{1}{\lambda^*(\mu)} = \frac{1}{\lambda_0^*} + \tfrac{3}{2} \ln \frac{M_x^2}{\mu^2} , \quad (4.9)$$

$$\rho(\mu) = \frac{\lambda_{30}}{\lambda_{20}} \left[ \frac{\lambda^*}{\lambda_0^*} \right]^{1/3} , \quad (4.10)$$

$$\frac{1}{\lambda_2(\mu)} = \frac{1}{\lambda_{20}} + \frac{3\lambda_{30}^2}{\lambda_{20}^2} \frac{1}{\lambda_0^*} \left[ \left[ \frac{\lambda_0^*}{\lambda^*} \right]^{1/3} - 1 \right] . \quad (4.11)$$

Thus, for our very simple-minded Higgs-breaking Lagrangian, the behavior of the coupling parameters is summarized very simply by the qualitative description

$$0 < \lambda^*(\mu) < \lambda_0^* \text{ for } \mu < M_x , \quad (4.12)$$

$$0 < \lambda_2(\mu) < \lambda_{20} \text{ for } \mu < M_x , \quad (4.13)$$

and

$$0 < \frac{\rho(\mu)}{\rho_0} < 1 \text{ for } \mu < M_x . \quad (4.14)$$

## V. CONCLUSION

In this paper we have introduced the decoupling renormalization scheme that we claim should be used to study the field theory of mass hierarchy. We have here only studied two examples of Lagrangians with mass hierarchy. Further work on extending it to cover gauge hierarchy will be reported elsewhere.

*Note added in proof.* I have been informed by J. C. Collins that he has a similar scheme that has been discussed in his book *Renormalization* (to be published by Cambridge University Press, New York, 1984). O. Foda has also brought my attention to his work, Phys. Lett. **124B**, 192 (1983).

## ACKNOWLEDGMENTS

It is a pleasure to thank Professor H. Sugawara and Professor M. Yoshimura for their very kind invitation for a sabbatical leave at KEK and to thank them and members of the theory group, including Professor Y. Kazama, for warm hospitality.

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