Spinorial relativistic rotator: The transformation from quasi-Newtonian to Minkowski coordinates

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There exists a remarkably close relationship between the operator algebra of the Dirac equation and the corresponding operators of the spinorial relativistic rotator (an indecomposable object lying on a mass-spin Regge trajectory). The analog of the Foldy-Wouthuysen transformation (more generally, the transformation between quasi-Newtonian and Minkowski coordinates) is constructed and explicit results are discussed for the spin and position operators. *Zitterbewegung* is shown to exist for a system having only positive energies.

I. INTRODUCTION AND SUMMARY

In both classical and quantal mechanics the relativistic rotator is one of the most important physical constructs for detailed study, yielding precedence only to the study of the motion of the relativistic mass point. The physical interest in the study of such structures at the present time lies in the fact that hadrons are now considered to be indecomposable composite objects lying on Regge trajectories, and must accordingly behave, under some circumstances, as relativistic rotators. Of particular interest is the influence of the internal (rotational) structure on the (external) interactions of these composite objects, taking account of relativistic effects.

Relativistic rotators are an abstraction (similar to the abstraction underlying the concept of mass point) in which a composite object is described by its position in space-time with the remaining structure truncated to a description of the internal state by spin angular momenta. For the spinorial relativistic rotator there are two internal variables, ξ_1 and ξ_2 (which form a spinor), in addition to the Minkowski (position) variables x_μ (a four-vector). The variables describing the spin carry no linear momentum, and each internal state of angular momentum s has in the rest frame exactly $(2s+1)$ components [which contrasts with nonrelativistic rotators (symmetric tops) having $(2s+1)^2$ components]. The Hamiltonian for such a system is a trajectory function uniting the Poincaré mass label M with the Poincaré spin label s, that is, $M=M(s)$. This structure has been discussed by Mukunda, van Dam, and Biedenharn.[1] Although the present paper is in a sense a sequel to this work, we shall not presuppose knowledge of Ref. 1, so that the present paper is self-contained.

The problem of the relativistic rotator is in no sense new: there are numerous treatments in the literature.[2–9] All the previous structures, however, have the difficulty that they involved *constraints* which make the problem of interactions more or less impossible.[9] By contrast the spinorial relativistic rotator (which may be extended to the limiting case of infinitely many spinors in such a way that it becomes a spinorial string model) does not have such constraints.[1]

The feature which underlies the spinorial model and accounts for its success is the remarkable connection between the algebraic structure of this model and the Lie algebra $B_2 \sim C_2$ [which integrates to the Lie group SO(3,2)]; the importance of this group structure was emphasized in the work of Bohm.[10] As discussed in Refs. 1 and 10 both the familiar Dirac equation (for the electron/positron) and the spinorial relativistic rotator share this same underlying algebraic structure. Indeed there exists a (commutation-relation preserving) mapping (which we call the "Dirac mapping") by which the ten symmetric elements of the 4×4 Hermitian matrix algebra of the Dirac matrices (for the Dirac equation) map into the ten SO(3,2) generators for the spinorial relativistic rotator. (This map is the analog of the famous Jordan-Schwinger map for the rotation group which takes the spin-$\frac{1}{2}$ defining example into all spins.[11]) Similarly, the Dirac map carries the Dirac four-component structure into the set of all Poincaré $[M=M(s),s]$ irreducible representations (irreps); the trajectory constraint $M=M(s)$ functions as the Hamiltonian in Dirac's constrained mechanics.

It is the existence of this mapping which (as we shall show below) allows one to understand, intuitively, the structure of the quasi-Newtonian↔Minkowski transformation (abbreviated QN↔M) of the spinorial relativistic rotator from the structure of the analogous transformation on the Dirac equation itself, a transformation known most widely as the Foldy-Wouthuysen (FW) transformation.[12] (See also Ref. 30.)

The introduction of quasi-Newtonian coordinates was implicit in the fundamental work of Wigner in which all irreps of the Poincaré group (all possible elementary-particle models) were first determined.[13] Wigner constructed these irreps by induction from the stability group of the momentum vector; no explicit determination of the generators of the group was given in his paper, but it is easy to do so and one obtains the ten Poincaré generators in a momentum-space realization. Although it is not completely straightforward,[1] one can Fourier transform these momentum-space generators, thereby obtaining ten generators in configuration space, realizing the *Poincaré group in quasi-Newtonian form* (as we shall discuss in Sec. II; see also Refs. 14—16).

It is a historical accident that the Dirac equation—which embodies Minkowski coordinates *ab initio*—preceded the construction by Wigner of the irreps of the Poincaré group (which, as noted above, realizes the Poincaré group in quasi-Newtonian coordinates). The generators of the Poincaré group in Minkowski form follow, of course, immediately from the Dirac equation itself.

This situation posed an interesting conceptual problem: to go backwards and determine the Dirac equation starting from Wigner's quasi-Newtonian form. To our knowledge this was first carried out by Thomas,[17] who showed that for the Dirac equation the quasi-Newtonian Poincaré generators (of the Wigner realization) went over into the Minkowski generators (of the Dirac formulation) under the Foldy-Wouthuysen transformation.

The approach of Foldy and Wouthuysen (which was preceded by the work of Pryce,[3] Becker,[18] and Schrödinger,[19] among others) was very different conceptually from that of Thomas and was very closely tied in outlook, as well as technically, to the particularities of the Dirac equation. (The Poincaré structure was in fact not considered at all.) The underlying idea of Foldy and Wouthuysen was to reduce the Dirac equation from four to two components. This they accomplished, for the free particle, by the transformation

$$U=\exp\left[i\rho_2\tan^{-1}\left[\frac{\vec{\sigma}\cdot\vec{\mathrm{P}}}{m_0c}\right]\right]$$

(which, it should be noted, is a transformation whose parameters are operator valued). The result of this transformation is to take the Dirac equation $H=\rho_1\vec{\sigma}\cdot\vec{\mathrm{P}}c+\rho_3m_0c^2$ into the form $H=\rho_3(\vec{\mathrm{P}}^2c^2+m^2c^4)^{1/2}$—that is (aside from the sign doubling given by $\rho_3\rightarrow\pm1$), one arrives at just the equation Dirac rejected. Each of these two forms for Dirac's Hamiltonian describes the same physics. For the incorporation of external interactions, however, these two forms are profoundly different; *in the Minkowski form* (Dirac's equation) interaction with the electromagnetic field is easily accomplished; by contrast, a *local* electromagnetic interaction in the *quasi-Newtonian form* is impossible.[20] This is the physical background, for the fundamental problem of interactions, which motivates the study of the analogous transformation for the spinorial relativistic rotator.

There is another set of phenomena for the Dirac equation which must be discussed in connection with these ideas: the so-called *Zitterbewegung* and *Spinzitterbewegung* found by Schrödinger in his study of the Dirac equation.[19]

For the free Dirac electron, the Heisenberg equation of motion for the Minkowski position operators $\vec{\mathrm{X}}^{\mathrm{Dirac}}$ can be integrated to yield $\vec{\mathrm{X}}^{\mathrm{Dirac}}(t)$, which is conveniently split into two parts:

$$\vec{\mathrm{X}}^{\mathrm{Dirac}}(t)\equiv\vec{\mathrm{X}}^{\mathrm{Dirac}}_{\mathrm{ave}}(t)+\vec{\mathrm{X}}^{\mathrm{Dirac}}_{\mathrm{ZBW}}(t)\ , \tag{1.1}$$

where

(a) an average position is defined

$$\vec{\mathrm{X}}^{\mathrm{Dirac}}_{\mathrm{ave}}(t)=[\vec{\mathrm{X}}^{\mathrm{Dirac}}(0)-\tfrac{1}{2}i\vec{\alpha}(0)H^{-1}+\tfrac{1}{2}iH^{-2}\vec{\mathrm{P}}]+H^{-1}\vec{\mathrm{P}}t\ , \tag{1.2}$$

and (b) a fluctuating position (*Zitterbewegung*)

$$\vec{\mathrm{X}}^{\mathrm{Dirac}}_{\mathrm{ZBW}}(t)=\tfrac{1}{2}i[\vec{\alpha}(0)-H^{-1}\vec{\mathrm{P}}]H^{-1}e^{-2iHt}\ , \tag{1.3}$$

whose time average is zero. Thus, the Minkowski coordinate $\vec{\mathrm{X}}^{\mathrm{Dirac}}$—which we shall say *denotes the position of the charge*—does not follow a straight line but moves in a complicated way, which as is well known, lies at the origin of the spin, and the spin magnetic moment.[2]

Now let us consider further the FW transformation. For the quasi-Newtonian coordinates $\vec{\mathrm{X}}^{\mathrm{QN}}$, the Hamiltonian is $H^{\mathrm{QN}}=\rho_3(P^2c^2+m^2c^4)^{1/2}$ and it is easily seen that

$$\frac{d}{dt}\vec{\mathrm{X}}^{\mathrm{QN}}\equiv i[H^{\mathrm{QN}},\vec{\mathrm{X}}^{\mathrm{QN}}]=\vec{\mathrm{P}}(H^{\mathrm{QN}})^{-1}\ ,$$

which implies that

$$\vec{\mathrm{X}}^{\mathrm{QN}}(t)=\vec{\mathrm{X}}^{\mathrm{QN}}(0)+(H^{\mathrm{QN}})^{-1}\vec{\mathrm{P}}t\ . \tag{1.4}$$

That is, the quasi-Newtonian position operator moves in a straight line and, hence, behaves exactly as one expects for a classical free particle. If we now take the FW transformation of these equations, one finds that

$$(\mathrm{a})\ \vec{\mathrm{P}}\rightarrow\vec{\mathrm{P}}\ , \tag{1.5}$$

$$(\mathrm{b})\ H^{\mathrm{QN}}\rightarrow H^{\mathrm{Dirac}}\ , \tag{1.6}$$

$$(\mathrm{c})\ \vec{\mathrm{X}}^{\mathrm{QN}}\rightarrow\vec{\mathrm{X}}^{\mathrm{mean}}(t)=\vec{\mathrm{X}}^{\mathrm{mean}}(0)+(H^{\mathrm{Dirac}})^{-1}\vec{\mathrm{P}}t\ . \tag{1.7}$$

Thus we have obtained *another* position operator,[12] in Minkowski variables, which is free of ZBW: this is the "mean" position operator $\vec{\mathrm{X}}^{\mathrm{mean}}$ which we could more properly call the center-of-mass coordinate. Although both the mean position operator $\vec{\mathrm{X}}^{\mathrm{mean}}$ and the average position operator $\vec{\mathrm{X}}^{\mathrm{Dirac}}_{\mathrm{ave}}$ move uniformly in a straight line, they are very different operators physically.[21] This is most easily seen from the fact (cf. Sec. V) that the three components of $\vec{\mathrm{X}}^{\mathrm{mean}}$ commute with each other (since this is true of $\vec{\mathrm{X}}^{\mathrm{QN}}$), whereas the three components of $\vec{\mathrm{X}}^{\mathrm{Dirac}}_{\mathrm{ave}}$ do *not* commute.

Remark. It is worth noting that one can go backward and carry the Minkowski coordinates into the quasi-Newtonian frame. Using these coordinates as the position of the charge in, say, the Coulomb interaction and then taking the time average over the ZBW leads to the spin-orbit, Darwin, etc., terms. The important point is that all numerical factors come out precisely correctly. In several textbooks (e.g., Ref. 22, p. 71 and Ref. 23, p. 116) in-correct numerical factors are obtained in the discussion of ZBW effects, because the electron velocity magnitude is taken as c instead of the correct Dirac result $\sqrt{3}c$.

The work of Foldy and Wouthuysen was of basic importance, very clearly and persuasively written, but because of the point of view used by these authors, their work has reinforced several not wholly correct impressions. Foremost among these is the impression that ZBW is an artifact of the Dirac equation, and more particularly, of the existence of positive and negative energies in the (one-body) Dirac equation. It is our view that if one considers these phenomena, not in the special context of the Dirac equation, but in the context of the structure of Poincaré irreps (with nonvanishing mass), then it becomes clear that there exist two unitarily-equivalent formulations for space-time operators: the *quasi-Newtonian frame* whose position observable is a *three-vector* (the mass center) with commuting components and the *Minkowski frame* whose position observable is a *four-vector* (the charge center), again with commuting components. From this more general viewpoint, these phenomena are physical consequences of relativity.

These phenomena all exist for the spinorial relativistic rotator, as we shall show, and owing to the remarkable structural connection (the mapping mentioned earlier) between the Dirac equation and the Hamiltonian for the spinorial relativistic rotator, it is technically feasible to carry out the (quasi-Newtonian)↔(Minkowski) transformation in complete detail. The existence of the same physical phenomena, discussed above, for the spinorial relativistic rotator can then be shown. Since the spinorial relativistic rotator has only *positive*-energy states, it is then clear, by construction, that these phenomena are indeed more general than the particularities of the Dirac equation and its attendant negative-energy states.

The plan of the paper is as follows. In Sec. II we develop, in some detail, the quasi-Newtonian form of Poincaré irreps with fixed mass M and fixed spin s. In this section we comment on the significant features of the quasi-Newtonian form and develop an alternative form which has advantages for the work to follow. In Sec. III we construct (using the quasi-Newtonian form of Sec. II) the reducible Poincaré trajectory representation that constitutes the spinorial relativistic rotator. In Sec. IV we derive the explicit transformation that carries the quasi-Newtonian form of the spinorial relativistic rotator into the Minkowski form. Section V is devoted to applications of the transformation developed in Sec. IV. In addition to developing the mean position and mean spin operators, we also give a direct proof that the Hamiltonian of the quasi-Newtonian form goes over into the covariant Hamiltonian for the Minkowski form.

II. THE QUASI-NEWTONIAN FORM OF THE POINCARÉ IRREPS

A. Notational conventions

It is useful to define all notational conventions fully and explicitly.

Metric:

$$g_{00}=-1\ ,\quad g_{11}=g_{22}=g_{33}=+1\ ,$$

$$g_{55}=1\quad [\text{for SO}(3,2)]\ .$$

Poincaré group $\mathscr{P}$:

$$\text{elements } (\Lambda^{\mu}{}_{\nu},a^{\mu})\ ,\quad \Lambda\in\text{SO}(3,1). \tag{2.1}$$

The product law is

$$(\Lambda',a')(\Lambda,a)=(\Lambda'\Lambda,a'+\Lambda'a)\ , \tag{2.2a}$$

or, in index notation,

$$(\Lambda'^{\mu}{}_{\nu},a'^{\mu})(\Lambda^{\mu}{}_{\nu},a^{\mu})\equiv(\Lambda'^{\mu}{}_{\rho}\Lambda^{\rho}{}_{\nu},a'^{\mu}+\Lambda'^{\mu}{}_{\rho}a^{\rho})\ . \tag{2.2b}$$

It is useful to note that every element may be written as the product of a translation and a Lorentz transformation, $(\Lambda,a)=(\mathbb{1},a)(\Lambda,0)$.

Covering group $\overline{\mathscr{P}}$ of P: elements (A,a), $A\in\text{SL}(2,\mathbf{C})$, $a=a^{\mu}$=translation. $A\in\text{SL}(2,\mathbf{C})$ determines $\Lambda(A)\in\text{SO}(3,1)$ which acts on vectors. For simplicity we can imagine that A itself acts directly on a vector x to give a new vector Ax by using the realization

$$Ax^{\mu}\sigma_{\mu}A^{\dagger}\equiv A(x^0\mathbb{1}+\vec{x}\cdot\vec{\sigma})A^{\dagger}\equiv(Ax)^{\mu}\sigma_{\mu}\ . \tag{2.3}$$

All representations discussed below will be considered unitary without explicit mention of this fact. To give a representation of $\overline{\mathscr{P}}$, we must (1) set up some Hilbert space $\mathscr{H}$, (2) define on $\mathscr{H}$ a set of unitary operators $U(A,a)$ such that

$$U(A',a')U(A,a)=U(A'A,a'+A'a)\ . \tag{2.4}$$

Instead of giving the $U(A,a)$, we could equally well give the (Hermitian) generators identified as

$$U(\mathbb{1},a)=e^{-ia^{\mu}P_{\mu}}\ ; \tag{2.5}$$

for

$$\Lambda^{\mu}{}_{\nu}=\delta^{\mu}{}_{\nu}-\omega^{\mu}{}_{\nu}\ ,\quad \omega_{\mu\nu}=-\omega_{\nu\mu},\quad |\omega|\ll1\ ,$$
$$U(\Lambda,0)\simeq\mathbb{1}-\frac{i}{2}\omega^{\mu\nu}M_{\mu\nu}\ . \tag{2.6}$$

P_{μ}, $M_{\mu\nu}$ must be Hermitian in $\mathscr{H}$ and obey the commutation relations

$$-i[M_{\mu\nu},M_{\rho\sigma}]=g_{\mu\rho}M_{\nu\sigma}-g_{\nu\rho}M_{\mu\sigma}+g_{\mu\sigma}M_{\rho\nu}-g_{\nu\sigma}M_{\rho\mu}\ ,$$
$$-i[M_{\mu\nu},P_{\rho}]=g_{\mu\rho}P_{\nu}-g_{\nu\rho}P_{\mu},\quad [P_{\mu},P_{\nu}]=0\ . \tag{2.7}$$

We will often specify a representation by giving P_{μ}, $U(A,0)\equiv U(A)$, and $\vec{J}=(M_{23},M_{31},M_{12})$, $\vec{K}=(M_{10},M_{20},M_{30})$ explicitly. Space parts of four-vectors $x,p,p',\ldots$ are denoted as $\vec{x},\vec{p},\vec{p}\,',\ldots$. Note that we shall generally denote numerical quantities by lower case letters, and operators by capitals. We deal with positive timelike representations of $\overline{\mathscr{P}}$ alone.

B. Quasi-Newtonian Poincaré irreps

Let us take an irreducible representation of nine Hermitian (three-vector) operators $\vec{X},\vec{P},\vec{S}$ whose nonvanishing commutators are

$$[X_j,P_k]=i\delta_{jk}, \quad [S_j,S_k]=i\epsilon_{jkl}S_l \; . \tag{2.8}$$

The Hilbert space $\mathscr{H}$ carrying the representation will be the space of an irrep of $\overline{\mathscr{P}}$. Let the spin operators $\vec{S}$ correspond to the spin-s representation of the SU(2) algebra. Take a real positive number M. Then the mass M, spin-s irrep $[M,s]$ of $\overline{\mathscr{P}}$ is realized on $\mathscr{H}$ by the construction

$$\vec{P}\rightarrow\vec{p} \; ; \tag{2.9a}$$

$$P^0=(\vec{P}\cdot\vec{P}+M^2)^{1/2} \; ; \tag{2.9b}$$

$$\vec{J}=\vec{X}\times\vec{P}+\vec{S} \; ; \tag{2.9c}$$

$$\vec{K}=\tfrac{1}{2}\{\vec{X},P^0\}+\vec{P}\times\vec{S}/(P^0+M) \; . \tag{2.9d}$$

A realization of $\overline{\mathscr{P}}$ by operators of the form of Eqs. (2.8) and (2.9a)–(2.9d) defines what we shall call the *quasi-Newtonian form for Poincaré* irreps$[M,s]$.

Take an idealized basis for $\mathscr{H}$ labeled by sharp momentum eigenvalues $\vec{p},m$ for $\vec{P},S_3$, respectively:

$$\langle \vec{p}\,'',m'' \mid \vec{p}\,',m' \rangle=\delta^{(3)}(\vec{p}\,''-\vec{p}\,')\delta_{m''m'} \; ;$$
$$m=s,s-1,\ldots,-s \; ; \tag{2.10a}$$

$$\vec{P}\mid\vec{p},m\rangle=\vec{p}\mid\vec{p},m\rangle \; , \quad \vec{X}\mid\vec{p},m\rangle=i\frac{\partial}{\partial\vec{p}}\mid\vec{p},m\rangle \; , \tag{2.10b}$$

$$S_3\mid\vec{p},m\rangle=m\mid\vec{p},m\rangle \; ; \tag{2.10c}$$

$$(S_1\pm iS_2)\mid\vec{p},m\rangle=[(s\mp m)(s\pm m+1)]^{1/2}\mid\vec{p},m\pm1\rangle \; . \tag{2.10d}$$

Here and in the following we will use $\vec{X}\mid\vec{p}m\rangle = i(\partial/\partial\vec{p})\mid\vec{p},m\rangle$ as a symbolic way of writing $\langle\psi\mid\vec{X}\mid\vec{p}m\rangle=i(\partial/\partial\vec{p})\langle\psi\mid\vec{p}m\rangle$ for every well-behaved vector ψ.

For each $\vec{p}$, define a four-vector p by

$$p(\vec{p})\equiv(\vec{p},(\vec{p}\cdot\vec{p}+M^2)^{1/2}) \; . \tag{2.11}$$

For each such p, define the SL$(2,c)$ element $B(p)$ by

$$B(p)\equiv[2M(M+p^0)]^{-1/2}(M+p^\mu\sigma_\mu) \; .$$

Then the quasi-Newtonian form of the irrep $[M,s]$ becomes

$$P_\mu\mid\vec{p},m\rangle=p_\mu\mid\vec{p},m\rangle \; , \tag{2.12}$$

$$U(A)\mid\vec{p},m\rangle = \left[\frac{(Ap)^0}{p^o}\right]^{1/2}\sum_{m'}D^{(s)}_{m'm}(B(Ap)^{-1}AB(p))\mid A\vec{p},m'\rangle \; . \tag{2.13}$$

[The argument of the rotation matrix in (2.13) is called the "Wigner rotation."]

C. Remarks on the quasi-Newtonian form

The quasi-Newtonian form of the Poincaré irreps, in Eqs. (2.13), has a number of significant features that deserve mention. First of all, one notes that the momentum operator P^0 plays a distinguished role (as is clear, for example, in the boost operator), and moreover, the position operator $\vec{X}$ is inherently a three-vector; the covariance of the structure is not manifest. The name "quasi-Newtonian" is chosen to call attention to this (seemingly noncovariant) three-dimensional aspect of this particular realization.

A technically more troublesome feature of the QN form is the *explicit appearance* of the mass parameter M in the realization of the four generators P^0 and $\vec{K}$; it is absent in the six generators $\vec{J}$ and $\vec{P}$. To see that this poses difficulties let us recall that Lorentz transformations that involve space-time coordinate rotation (called in the following "boosts" and denoted by B) are more properly to be considered as *velocity* (and not momentum) transformations. Although a given momentum does indeed transform under a boost, the change in momentum depends on the particular length (mass) of the given momentum four-vector; the change in the velocity is, however, uniform. [*A* passive (coordinate frame) Lorentz boost imparts to particles of different mass the *same* velocity increment, but *different* momentum increments.] Intuitive considerations based on velocity are, however, more or less tied to classical concepts, since (unlike momentum) velocity is not an *a priori* concept in quantum mechanics, but a derived concept based on a particular Hamiltonian. We can achieve the same goal, using momentum concepts, by recognizing that action of a boost is mass independent and the three parameters of a boost $\exp(-i\vec{B}\cdot\vec{K})$ are determined by the four-vector momentum *scaled by its length* (mass). This motivates an alternative formulation of the QN Poincaré irreps, which we now discuss.

Up to unitary equivalence, we already have in hand the QN irreps $[M,s]$. Let us express the generators (2.9a)–(2.9d) in another form. In terms of the basic irreducible set of operators $\vec{X},\vec{P},\vec{S}$ define now

$$\vec{N}=\vec{P}/M, \; \vec{Z}=M\vec{X}, \; [Z_j,N_k]=i\delta_{jk} \; . \tag{2.14}$$

The $\vec{Z},\vec{N},\vec{S}$ obey the same algebra (2.8) as $\vec{X},\vec{P},\vec{S}$, and s is the same, so the two sets must unitarily be related. In fact,

$$(\vec{Z},\vec{N},\vec{S})=e^{iD\ln M}(\vec{X},\vec{P},\vec{S})e^{-iD\ln M} \; , \tag{2.15a}$$

where

$$D=\tfrac{1}{2}(\vec{X}\vec{P}+\vec{P}\vec{X})=\tfrac{1}{2}(\vec{Z}\vec{N}+\vec{N}\vec{Z}) \; ,$$
$$[D,\vec{X}]=-i\vec{X}[D,\vec{P}]=i\vec{P} \; . \tag{2.15b}$$

We also now introduce basis kets in $\mathscr{H}$ that are labeled by the eigenvalues of P^μ/M and S_3.[22]

Expressed in terms of $\vec{Z}$, $\vec{N}$, and $\vec{S}$, the generators $\vec{P}^\mu$, $\vec{J}$, $\vec{K}$, (2.9a)–(2.9d), take the forms

$$P_\mu=MN_\mu \; , \quad \vec{J}=\vec{Z}\times\vec{N}+\vec{S}, \quad \vec{K}=\tfrac{1}{2}\{\vec{Z},N^0\}+\frac{\vec{N}\times\vec{S}}{1+N^0} \; , \tag{2.16}$$
$$N^0\equiv(N\cdot N+1)^{1/2} \; .$$

In this alternative QN form, *M appears explicitly only in* P_μ, *and is absent in* $M_{\mu\nu}$—this will prove to be useful.

Let us write eigenkets of $\vec{N}$, S_3 as $(\vec{n},m)$. For any three-vector $\vec{n}$, we can associate the corresponding four-vector: $n=[\vec{n},(\vec{n}\cdot\vec{n}+1)^{1/2}]$. The two bases for $\mathscr{H}$ are related as (note the use of a round ket symbol)

$$
\begin{aligned}
&|\vec{n},m)=M^{3/2}|M\vec{n},m\rangle\ ,\quad |\vec{p},m\rangle=M^{-3/2}|\vec{p}/M,m)\ ,\\
&(\vec{n}'',m''|\vec{n}',m')=\delta^{(3)}(\vec{n}''-\vec{n}')\delta_{m''m'}\ ,\\
&\vec{N}|\vec{n},m)=\vec{n}|\vec{n},m),\ \vec{Z}|\vec{n},m)=i\frac{\partial}{\partial\vec{n}}|\vec{n},m)\ , \qquad (2.17)\\
&S_3|\vec{n},m)=m(\vec{n},m)\ ,\\
&(S_1\pm iS_2)|\vec{n},m)=[(s\mp m)(s\pm m+1)]^{1/2}|\vec{n},m\pm 1)\ .
\end{aligned}
$$

Then, in this alternative QN formulation, the irrep $[M,s]$ has the form

$$
\begin{aligned}
&P_\mu|n,m)=(Mn)_\mu|\vec{n},m)\ ;\\
&U(A)|\vec{n},m) \qquad (2.18)\\
&\quad=\left[\frac{(An)^0}{n^0}\right]^{1/2}\sum_{m'}D^{(s)}_{m'm}(B(An)^{-1}AB(n))|\vec{An},m')\ .
\end{aligned}
$$

The advantage of this reformulation is that now the mass M appears only in the action of P_μ and not in $U(A)$ at all.

III. THE REDUCIBLE UNITARY REPRESENTATION FOR A REGGE TRAJECTORY: THE SPINORIAL RELATIVISTIC ROTATOR

We will now consider a set of Poincaré irreps taken in a quasi-Newtonian realization (Sec. II), *in which the mass M is a specified function of the spin s.* That is, we take the trajectory function $M=M(s)$, $s=0,\ \frac{1}{2},\ 1,\ldots$, and the desired Poincaré representation for this trajectory is then a discrete direct sum of quasi-Newtonian irreps. This defines the spinorial relativistic rotator by direct construction.

To realize this trajectory representation explicitly, we take the primitive algebra of operators:

$$
[X_j,P_k]=i\delta_{jk},\quad [a_r,a_{r'}^\dagger]=\delta_{rr'}\ ,
$$
$$
\text{where } (j,k)\equiv(1,2,3) \text{ and } (r,r')\equiv(1,2)\ . \qquad (3.1)
$$

This algebra consists therefore of five canonical pairs, and, up to equivalence, there is just one unique irreducible representation. $\mathscr{H}$ will be the space of this representation. As a basis for $\mathscr{H}$, let us take P, $a_1^\dagger a_1$, and $a_2^\dagger a_2$ to be diagonal. The basis is then the set of (idealized) kets $\{|\vec{p};s,m\rangle\}$ with $\vec{p}\,'$ any numerical three-vector, $s=0,\ \frac{1}{2}$, $1,\ldots,\ m=s,\ s-1,\ldots,\ -s$. We normalize

$$
\langle\vec{p}\,'';s''m''|\vec{p}\,';s'm'\rangle=\delta^{(3)}(\vec{p}\,''-\vec{p}\,')\delta_{s''s'}\delta_{m''m'};
$$

the defining relations are

$$
\begin{aligned}
&\vec{P}|\vec{p};sm\rangle=\vec{p}|\vec{p};sm\rangle,\\
&a_1^\dagger a_1|\vec{p};sm\rangle=(s+m)|\vec{p};sm\rangle,\\
&a_2^\dagger a_2|\vec{p};sm\rangle=(s-m)|\vec{p};sm\rangle\ ,\\
&\vec{X}|\vec{p};sm\rangle=i\frac{\partial}{\partial\vec{p}}|\vec{p};sm\rangle\ , \qquad (3.2)\\
&a_{1,2}|\vec{p};sm\rangle=(s\pm m)^{1/2}\vec{p};s-\tfrac{1}{2},m\mp\tfrac{1}{2}\rangle\ ,\\
&a_{1,2}^\dagger|\vec{p};sm\rangle=(s\pm m+1)^{1/2}|\vec{p};s+\tfrac{1}{2},m\pm\tfrac{1}{2}\rangle,
\end{aligned}
$$

and

$$
|\vec{p};sm\rangle=[(s+m)!(s-m)!]^{-1/2}\times(a_1^\dagger)^{s+m}(a_2^\dagger)^{s-m}|\vec{p};00\rangle\ .
$$

The pair of boson operators $(a_1^\dagger,a_2^\dagger)$ and their conjugates define ten independent, bilinear quadratic expressions which may be chosen to be the generators of an SO(3,2) Lie algebra. This is a special case of the general result that n sets of canonical boson operators span the (symplectic group) Lie algebra C_n of the symplectic group Sp$(2n)$. There is a general construction[11] which maps the lowest-dimensional $2n\times 2n$ matrix realization of C_n into the bilinear boson operators, preserving commutation relations. As a special case we have the Dirac mapping, $\mathscr{D}$, which maps the ten Dirac matrices γ_μ and $[\gamma_\mu,\gamma_\nu]$ into the ten boson operators $\{V_\mu,S_{\mu\nu}\}$ of the spinorial relativistic rotator. That is,

$$
\begin{aligned}
&\mathscr{D}:\frac{i}{2}\gamma_\mu\to V_\mu\ ,\\
&-\frac{i}{4}[\gamma_\mu,\gamma_\nu]\to S_{\mu\nu}\ .
\end{aligned}
$$

(Note that the map preserves commutators, but not Hermiticity.)

It is now easily checked, directly, that the operators $\{V_\mu,S_{\mu\nu}\}$ obey the commutation relations

$$
\begin{aligned}
&-i[S_{\mu\nu},S_{\rho\sigma}]=g_{\mu\rho}S_{\nu\sigma}-g_{\nu\rho}S_{\mu\sigma}+g_{\mu\sigma}S_{\rho\nu}-g_{\nu\sigma}S_{\rho\mu}\ ,\\
&-i[S_{\mu\nu},V_\rho]=g_{\mu\rho}V_\nu-g_{\nu\rho}V_\mu\ , \qquad (3.3)\\
&i[V_\mu,V_\nu]=S_{\mu\nu}\ .
\end{aligned}
$$

The existence of the Dirac map sets up a far-reaching correspondence between the operators of the Dirac (electron/positron) equation and those of the spinorial relativistic rotator.

In order to be fully explicit let us note that—expressing the boson operators $a_i^\dagger$ and a_i in terms of canonical (harmonic oscillator) position and momenta (ξ_i,π_i)—the ten generators V_μ and $S_{\mu\nu}$ take the form

$$
\begin{aligned}
&S_{12}=\tfrac{1}{2}(\xi_2\pi_1-\xi_1\pi_2)\ ,\ S_{23}=\tfrac{1}{2}(\xi_1\xi_2+\pi_1\pi_2)\ ,\\
&S_{31}=\tfrac{1}{4}(\xi_1{}^2+\pi_1{}^2-\xi_2{}^2-\pi_2{}^2)\ ,\\
&S_{01}=\tfrac{1}{4}(\xi_1{}^2-\pi_1{}^2-\xi_2{}^2+\pi_2{}^2)\ ,\\
&S_{02}=\tfrac{1}{2}(\pi_1\pi_2-\xi_1\xi_2)\ ,\ S_{03}=\tfrac{1}{2}(\xi_1\pi_1+\pi_2\xi_2)\ , \qquad (3.4)\\
&V_1=\tfrac{1}{2}(\xi_2\pi_2-\xi_1\pi_1)\ ,\ V_2=\tfrac{1}{2}(\xi_1\pi_2+\xi_2\pi_1)\ ,\\
&V_3=\tfrac{1}{4}(\xi_1{}^2-\pi^2+\xi_2{}^2-\pi_2{}^2)\ ,\\
&V_0=\tfrac{1}{4}(\xi_1{}^2+\pi_1{}^2+\xi_2{}^2+\pi_2{}^2)\ .
\end{aligned}
$$

All ten of the operators $\{V_\mu,S_{\mu\nu}\}$ in (3.3) are Hermitian. It is useful to note that *operator V_0 is positive definite.*

The *mass operator* is now taken to be an explicit function of the spin, using the trajectory function $M(s)$, in the following way:

$$
\text{Mass operator: } M_{\rm op}\equiv M(V_0)\ , \qquad (3.5)
$$
$$
\text{with the action: } M_{\rm op}|\vec{p};sm\rangle=M(s+\tfrac{1}{2})|\vec{p};sm\rangle\ .
$$

It is useful to note that the mass operator commutes with the operators $\vec{X}$, $\vec{P}$, and $\vec{S}$, that is,

$$[M_{op},\vec{X}]=[M_{op},\vec{P}]=[M_{op},\vec{S}]=0 , \tag{3.6}$$

but that M_{op} does not, in general, commute with the boson operators a_i or $a_i^\dagger$ or with the operator $\vec{V}$.

We take the normalized vectors $|s,m\rangle$, as given in Eq. (3.2), to be a basis for the representation space $\mathscr{H}$ of SO(3,2), where the generators V_μ, $S_{\mu\nu}$ are defined in Eq. (3.4) and satisfy Eq. (3.3). The (infinite-dimensional) representation of SO(3,2) generated in this way has been termed the Majorana representation,[10] or the "remarkable representation" by Dirac,[24] because of its many interesting properties. This representation consists of two irreducible pieces (the integer- and half-integer-spin parts of the remarkable representation); these two parts remain irreducible when restricted to the Lorentz subgroup [the irreps $(k_0=\frac{1}{2},c=0)$ and $(k_0=0,c=\frac{1}{2})$ of SO(3,1)] in Naimark's notation.[25] The remarkable representation was first discussed by Majorana, accordingly we denote this representation by $D^{\rm Maj}$, and define this representation explicitly by

$$\begin{aligned} &g\in SO(3,2)\to \mathscr{U}(g)\equiv\exp\left[-\frac{i}{2}\omega^{AB}S_{AB}\right] ,\\ &\mathscr{U}(g)\,|\,\vec{p};sm\rangle=\sum_{s'm'}D^{\rm Maj}_{s'm';sm}(g)\,|\,\vec{p};s'm'\rangle . \end{aligned} \tag{3.7}$$

[Note that we use the notation $(AB)=0,1,2,3,5$, with $V_\mu=S_{\mu5}$, $\mu=0,1,2,3$.]

To complete the construction of the spinorial relativistic rotator, it is only necessary to note that for each spin s, and hence each mass $M=M(s)$, there is a corresponding Poincaré irrep $[M(s),s]$ defined in the quasi-Newtonian form by the construction in Sec. II, above. The basis for $\mathscr{H}$ on which this representation is defined is the basis $|\,\vec{p};sm\rangle$ given in Eq. (3.2).

Under the action of the Poincaré generators, each of these $[M(s),s]$ irreps undergoes the transformations given by Eq. (2.13) of Sec. II. Thus, we have a well-defined (denumerably infinitely reducible) Poincaré representation that belongs to the entire Regge trajectory; it is this structure which constitutes the spinorial relativistic rotator.

In this construction, the action of the SO(3,2) generators, aside from the spin operator $\vec{S}$, is not in a convenient form. This is important, since the technical problem in carrying this trajectory representation from the QN form (in which it was defined) into the Minkowski form is the problem of factorizing the Wigner rotation $B(Ap)^{-1}AB(p)$ into its components, three Lorentz rotations. One would like to replace the explicit spin matrices that appear in the generators and treat all spins as a whole by going over to the spin operators as, say, differential operators on the (ξ_1,ξ_2) variables. Such a step immediately runs into trouble; the mass operator (and hence the operator P_0) cannot have a sharp value unless the spin magnitude is itself sharp. It follows that the QN irreps, which for the discrete parameters $[M(s),s]$ have a sharp formal four-momentum would lose this property if the continuous (ξ_1,ξ_2) variables are used.

On the other hand, it appears essential to introduce the variables (ξ_1,ξ_2) in the ket vectors, because in this way one can factorize the Wigner rotation into well-defined Lorentz boost operators.

It is at this point that the alternative formulation of the QN irreps is so valuable. By going over to scaled momenta, in this alternative formulation, we may remove the explicit mass dependence of the boost operation, and thereby facilitate extending the spin rotations to the full SO(3,1) action defined in the SO(3,2) structure. In this way we shall "disentangle the Wigner rotation" arriving at another term of the trajectory representation, the manifestly covariant form of the spinorial relativistic rotator.

IV. DERIVING THE TRANSFORMATION: QUASI-NEWTONIAN↔MINKOWSKI FORM

Stage 1. Passage to alternative QN form.

Let us introduce a new basis for $\mathscr{H}$, along the lines of (2.17), for each s. The new vectors will be written $|\,\vec{n};sm)$—note the round ket symbol—with $\vec{n}$ any three-vector and s,m as before. The connection to $|\,\vec{p};sm\rangle$ is

$$\begin{aligned} &|\,\vec{n};sm)=M(s)^{3/2}\,|\,M(s)\vec{n};sm\rangle ,\\ &|\,\vec{p};sm\rangle=M(s)^{-3/2}\left|\frac{1}{M(s)}\vec{p};sm\right) ,\\ &(\vec{n}\,'';s''m''\,|\,\vec{n};s'm')=\delta^{(3)}(\vec{n}\,''-\vec{n}\,')\delta_{s''s'}\delta_{m''m'} . \end{aligned} \tag{4.1}$$

Having set up this new basis in $\mathscr{H}$, it is natural to define new operators $\vec{N},\vec{Z},b,b^\dagger$ whose actions on the new basis states looks exactly like that of $\vec{P},\vec{X},a,a^\dagger$ on the old ones. So, we define $\vec{N},\vec{Z},b,b^\dagger$ on $\mathscr{H}$ as

$$\begin{aligned} &\vec{N}\,|\,\vec{n};sm)=\vec{n}\,|\,\vec{n};sm),\quad \vec{Z}\,|\,\vec{n};sm)=i\frac{\partial}{\partial\vec{n}}\,|\,\vec{n};sm) ;\\ &b_{1,2}\,|\,\vec{n};sm)=(s\pm m)^{1/2}\,|\,\vec{n};s-\tfrac{1}{2},m\mp\tfrac{1}{2}) ,\\ &b^\dagger_{1,2}\,|\,\vec{n};sm)=(s\pm m+1)^{1/2}\,|\,\vec{n};s+\tfrac{1}{2},m\pm\tfrac{1}{2}) . \end{aligned} \tag{4.2}$$

It is then a consequence of these definitions that $\vec{Z},\vec{N},b_i,b_i^\dagger$ form an irreducible representation, on $\mathscr{H}$, of the same algebra (3.1) as before. Also one has, easily,

$$\begin{aligned} |\,\vec{n};sm)=&[(s+m)!(s-m)!]^{-1/2}\\ &\times(b_1^\dagger)^{s+m}(b_2^\dagger)^{s-m}\,|\,\vec{n};00) . \end{aligned} \tag{4.3}$$

The set $(\vec{Z},\vec{N},b_i,b_i^\dagger)$ is unitarily related to the set $(\vec{X},\vec{P},a_i,a_i^\dagger)$. It is not hard to show that, with $D=\frac{1}{2}(\vec{X}\cdot\vec{P}+\vec{P}\cdot\vec{X})$, and since M_{op} commutes with $\vec{X},\vec{P}$ on the one hand while D commutes with $a_i,a_i^\dagger$, on the other,

$$(\vec{Z},\vec{N},b_i,b_i^\dagger)=e^{iD\ln M_{op}}(\vec{X},\vec{P},a_i,a_i^\dagger)e^{-iD\ln M_{op}} . \tag{4.4}$$

So we have, on $\mathscr{H}$, two unitarily equivalent irreducible-operator sets $(\vec{X},\vec{P},a_i,a_i^\dagger)$ and $(\vec{Z},\vec{N},b_i,b_i^\dagger)$ with *corresponding* bases $|\,\vec{p};sm\rangle$ and $|\,\vec{n};sm)$. In fact,

$$|\,\vec{n};sm)=e^{iD\ln M_{op}}\,|\,\vec{p};sm\rangle . \tag{4.5}$$

Since the $D^{\mathrm{Maj}}_{\cdots}$ are *numerical,* they commute with everything and we can use (4.10) to write the above RHS as

$$\mathrm{RHS}=\left[\frac{(An)^0}{n^0}\right]^{1/2}\sum_{\substack{s_1m_1\\ s_2m_2}} D^{\mathrm{Maj}}_{s_1m_1;s_2m_2}(A)D^{\mathrm{Maj}}_{s_2m_2;sm}(B(n)) \times \mathscr{U}'(B(An)^{-1})\,|\,\vec{\mathrm{An}};s_1m_1) \ . \tag{4.19}$$

[Recall that we use the notation $\mathscr{U}'$ to denote the (Majorana) representation generated by the $S_{\mu\nu}$ alone, as distinguished from U' generated by $M_{\mu\nu}$.] Next we take the $\mathscr{U}'$ operator *outside the summation* (it commutes with the numerical $D^{\mathrm{Maj}}_{\cdots}$) and then use (4.14) to rewrite the kets:

$$\mathrm{RHS}=\mathscr{U}'(B(An)^{-1})\sum_{s_2m_2} D_{s_2m_2;sm}(B(n))U'(A)\vec{\mathrm{n}};s_2m_2) \ . \tag{4.20}$$

Using Eq. (4.10) once again we obtain finally

$$\mathrm{RHS}=\mathscr{U}'(B(An)^{-1})U'(A)\mathscr{U}'(B(n))\,|\,\vec{\mathrm{n}};sm) \ . \tag{4.21}$$

Equation (4.21) is not quite the desired result since the two boosts on the right-hand side do no appear to be the inverse of each other. We can remedy this, however, by using the concept of *operator-valued boosts.* If we interpret $\mathscr{U}'_{\mathrm{op}}(B(n))$ on the right in (4.21) as an operator-valued boost $\mathscr{U}'(B(N))$, then on the ket $|\,\vec{\mathrm{n}},sm)$ it takes on the numerical value n. After the action by $U'(A)$ the vector n becomes An, so that the operator-valued boost $\mathscr{U}'(B(N))^{-1}$ correctly becomes the numerical-valued transformation $\mathscr{U}'(B(An)^{-1})$.

Using then operator-valued boosts, we may write Eq. (4.21) in the desired form:

$$U(A)\,|\,\vec{\mathrm{n}};sm)=\mathscr{U}'(B(N)^{-1})U'(A)\mathscr{U}'(B(N))\,|\,\vec{\mathrm{n}},sm) \ . \tag{4.22}$$

Thus we have achieved our goal of relating $U(A)$ to $U'(A)$ by an (operator-valued) unitary transformation. To be explicit, one has the transformation

$$\mathscr{U}'(B(N))=\exp\left[iS'_{0j}\frac{N_j}{|\vec{\mathrm{N}}|}\ln(N_0+|\vec{\mathrm{N}}|)\right] \ . \tag{4.23}$$

This is a well-defined unitary operator on $\mathscr{H}$ since S'_{0j} formed from b_i and $b_i^\dagger$ commute with the operators N_μ, which in turn commute among themselves, so that $\mathscr{U}'(B(N))$ is the exponential of an anti-Hermitian operator.

From Eq. (4.21) we find—the SL(2,C) part of $\mathscr{P}$—that the originally given unitary representation $U(A)$ is equivalent to the unitary representation $U'(A)$ in the form

$$U(A)=\mathscr{U}'(B(N)^{-1})U'(A)\mathscr{U}'(B(N)) \ . \tag{4.24}$$

Let us note that the transformation $\mathscr{U}'(B(N))$ preserves $\vec{\mathrm{J}}$ but decisively alters $\vec{\mathrm{K}}$. Thus, to be completely detailed, we have

$$\vec{\mathrm{J}}=\vec{\mathrm{X}}\times\vec{\mathrm{P}}+\vec{\mathrm{S}}=\vec{\mathrm{J}}\,'=\vec{\mathrm{Z}}\times\vec{\mathrm{N}}+\vec{\mathrm{S}}\,' \ ,$$

$$[\vec{\mathrm{J}},\mathscr{U}'(B(N))]=0 \ , \tag{4.25}$$

$$\vec{\mathrm{K}}=\tfrac{1}{2}\{\vec{\mathrm{X}},P^0\}+\frac{\vec{\mathrm{P}}\times\vec{\mathrm{S}}}{P^0+M_{\mathrm{op}}}=\tfrac{1}{2}\{\vec{\mathrm{Z}},N^0\}+\frac{\vec{\mathrm{N}}\times\vec{\mathrm{S}}\,'}{N^0+\mathbb{1}}$$

$$=\mathscr{U}'(B(N)^{-1})(\tfrac{1}{2}\{Z,N^0\}+(S'_{10},S'_{20},S'_{30}))\mathscr{U}'(B(N))$$

$$\equiv\mathscr{U}'(B(N))^{-1}(\vec{\mathrm{K}}\,')\mathscr{U}'(B(N)) \ .$$

Now, P_μ combines with $U(A)$ to yield the representation $U(A,a)$ of $\mathscr{P}$ we started with. From Eq. (4.24), it is clear how to set up a P'_μ which, along with $U'(A)$, gives a representation of $\mathscr{P}$ unitarily equivalent to $U(A,a)$. We must define

$$P'_\mu=\mathscr{U}'(B(N))P_\mu\mathscr{U}'(B(N))^{-1} \ . \tag{4.26}$$

From Eq. (4.13) we have $P_\mu=M_{\mathrm{op}}N_\mu$. The transformation $\mathscr{U}'$ commutes with N so that we obtain

$$P'_\mu=M'_{\mathrm{op}}N_\mu \ , \tag{4.27}$$

where we have defined

$$M'_{\mathrm{op}}\equiv\mathscr{U}'(B(N))M_{\mathrm{op}}\mathscr{U}'(B(N))^{-1} \ . \tag{4.28}$$

(We will shortly put this in a more accessible form.)

The net result, as expressed by Eq. (4.24) is that we have factored the Wigner rotation and in so doing expressed the QN Poincaré irrep by a unitarily transformed irrep $U'(A,a)$, where the operator $\mathscr{U}'(B(N))$ effects the transformation. This is very close to the desired final result, but not quite, since the QN irrep has been defined on the alternative (rescaled momentum) basis.

To complete the transformation we put Eq. (4.24) in the inverse form

$$U'(A,a)=\mathscr{U}'(B(N))U(A,a)\mathscr{U}'(B(N))^{-1} \ , \tag{4.29}$$

and *then carry out the inverse transformation to Eq. (4.8) taking the scaled momentum back into the unscaled form.* This inverse transformation has the effect of replacing the operators $(\vec{\mathrm{Z}},\vec{\mathrm{N}},b_i,b_i^\dagger)$ by the operators $(\vec{\mathrm{X}},\vec{\mathrm{P}},a_ia_i^\dagger)$ and carries the SO(3,2) operators S'_{AB} into S_{AB} [see Eqs. (4.8)]. As a result, the transformation $\mathscr{U}'(B(N))$ generated by $S'_{\mu\nu}$ with operator-valued parameters (determined by N) becomes a transformation generated by $S_{\mu\nu}$ which is operator valued as determined by P. Thus we find

$$\text{inverse scaling (4.8): } \mathscr{U}'(B(N))\rightarrow\mathscr{U}(B(P)) \ , \tag{4.30}$$

where $\mathscr{U}$ is generated by $S_{\mu\nu}$ [Eq. (3.4)].

Let us also *define*

$$\text{inverse scaling: } U'(A,a)\rightarrow U^M(A,a) \ , \tag{4.31}$$

where U^M will be shown to be the Poincaré representation for the trajectory in Minkowski form.

As a result of the inverse-scaling transformation, Eq. (4.29) takes the form

$$\mathscr{U}(B(P))U^{\rm QN}(A,a)\mathscr{U}(B(P))^{-1}\equiv U^{M}(A,a)\;, \qquad (4.32)$$

where we have denoted by $U^{\rm QN}$ the (trajectory) Poincaré representation in the original (unscaled) quasi-Newtonian form.

To establish that the right-hand side of (4.30) is indeed the trajectory representation *in Minkowski form* we need only look at the generators [Eq. (4.25)] after the inverse-scaling transformation. Clearly, from Eq. (4.25) the Lorentz generators take the form

$$\vec{\rm J}=\vec{\rm X}\times\vec{\rm P}+\vec{\rm S}\;, \qquad (4.33)$$

and

$$K_i=\tfrac{1}{2}\{X_i,P^0\}+S_{i0}\;.$$

It is also clear (from the form of $\vec{\rm K}$) that the generators are in the standard (manifestly covariant) form.

It remains to determine the form of the momentum operators. After the inverse scaling, Eqs. (4.27) and (4.28) show that P_μ is a four-vector operator whose length is given by

$$P\cdot P=(M''_{\rm op})^2 \qquad (4.34)$$

with $M''_{\rm op}$ [the inverse-scaled operator $M'_{\rm op}$ of Eq. (4.28)] being given by

$$M''_{\rm op}=M(\hat{P}\cdot V)\;, \qquad (4.35)$$

where $\hat{P}_\mu\equiv P_\mu(P\cdot P)^{-1/2}$ and $M(\cdot)$ is the trajectory function. [V_μ is defined in (3.4).] An independent direct proof of Eq. (4.35) will be given in Sec. V.

V. APPLICATIONS OF THE TRANSFORMATION

We have shown in previous sections that the model can be developed from two distinct but unitarily equivalent viewpoints: the quasi-Newtonian form and the Minkowski form. Each form possesses advantages and disadvantages. Both forms realize the symmetry of the Poincaré group, so that, accordingly, there exists a unitary transformation, $\mathscr{U}$, carrying one form into the other. This transformation was developed in Secs. III and IV.

For each of the two forms there exists a position operator; it is the purpose of the present section to determine these operators explicitly, and to develop their properties.

It is important to distinguish the various operators in the two distinct forms, and a suitable notation for this is required. Let us denote the quasi-Newtonian frame (synonyms: "Newton-Wigner" or "Thomas form") by QN and the Minkowski frame (synonym: "manifestly covariant form") by M. Operators appropriate to the two frames will be distinguished by a superscript, for example, $\vec{\rm X}^{\rm QN}$, denotes the quasi-Newtonian position operator (to be defined below) and $\vec{\rm X}^{M}$ the Minkowski position operator. Throughout this section $\mathscr{U}$ will denote the transformation *from* the QN frame to the M frame. [As developed in the previous section, $\mathscr{U}=U(B_{\hat{P}})^{\rm spin}$, where the boost parameters (determined by $\hat{P}$) are operator valued.] Under the action of the operator $\mathscr{U}$ the Poincaré generators in the QN frame $\{P_0^{\rm QN},\vec{\rm P}^{\rm QN},\vec{\rm J}^{\rm QN},\vec{\rm K}^{\rm QN}\}$ transform into their M counterparts. Thus, for example,

$$\mathscr{U}P_0^{\rm QN}\mathscr{U}^{-1}=P_0^{M}\;, \qquad (5.1)$$

and similarly. We note that the three-momenta $\vec{\rm P}$ and the angular momenta $\vec{\rm J}$ commute with $\mathscr{U}$ so that $\vec{\rm P}^{\rm QN}=\vec{\rm P}^{M}$ and $\vec{\rm J}^{\rm QN}=\vec{\rm J}^{M}$.

Let us consider now the question of defining a suitable position operator in the QN frame. To serve as a position operator it is first of all essential that the three components of the operator be simultaneously observable (so as to "locate" the particle). Accordingly, we must have

$$[\vec{\rm X}^{\rm QN},\vec{\rm X}^{\rm QN}]=0\;. \qquad (5.2)$$

Moreover, the operator $\vec{\rm X}^{\rm QN}$ must obey the Heisenberg commutation rule with the momentum operator $\vec{\rm P}^{\rm QN}$ (which is itself determined *ab initio* from the existence of the Poincaré generators of the QN realization). That is,

$$-i[\vec{\rm P}^{\rm QN},\vec{\rm X}^{\rm QN}]=\mathbb{1}\;. \qquad (5.3)$$

Using the fact that the operators $P_0^{\rm QN}$ has the form

$$P_0^{\rm QN}=[(\vec{\rm P}^{\rm QN})^2+M^2(V_0^{\rm QN})]^{1/2}\;, \qquad (5.4)$$

it follows, that, if we assume that the spin operator $\vec{\rm S}^{\rm QN}$ (and hence $V_0^{\rm QN}$) is not only translationally invariant but also commutes with $\vec{\rm X}^{\rm QN}$, then

$$\dot{\vec{\rm X}}^{\rm QN}=i[P_0^{\rm QN},\vec{\rm X}^{\rm QN}]=\vec{\rm P}^{\rm QN}(P_0^{\rm QN})^{-1}\;. \qquad (5.5)$$

Since $\dot{\vec{\rm P}}^{\rm QN}=0$, it follows that the quasi-Newtonian position (and hence the "particle") moves uniformly in a straight line. *This elementary property is characteristic of the quasi-Newtonian position operator.*

This property is of critical importance, since, as we will see, the Minkowski position operator does *not* define a particle position that moves uniformly, even for a free particle. This fact alone implies that the quasi-Newtonian position operator $\vec{\rm X}^{\rm QN}$ and the Minkowski position operator $\vec{\rm X}^{M}$, *cannot be transforms of each other.* [This follows since the transformation $\mathscr{U}$ preserves commutation relations, and hence a Minkowski version of Eq. (5.5) would then exist, contrary to fact.] Accordingly, there must exist *yet another position operator:* the transform of $\vec{\rm X}^{\rm QN}$ (expressed in terms of Minkowski frame operators),

$$\mathscr{U}\vec{\rm X}^{\rm QN}\mathscr{U}^{-1}\equiv\vec{\rm X}^{\rm mean}\;. \qquad (5.6)$$

Determining the properties of this operator will be an essential part of the work below.

How is the operator $\vec{\rm X}^{\rm QN}$ to be determined explicitly from the properties we have assumed above? There is an elegant answer[26–28] to this question: *the existence of the Poincaré generators* (with translationally invariant spin) *in the quasi-Newtonian form uniquely determines the QN-position operator $\vec{X}^{QN}$ and the QN-spin operator $\vec{S}^{QN}$, in terms of the Poincaré generators,* for nonvanishing mass. This construction carries over at once to the spinorial relativistic rotator (since zero mass is excluded in this model).

The result for the QN-position operator is

Naturally, $a_a(a_i^\dagger)$ have messy actions on $|\vec{n};sm)$, as B_i $(b_i^\dagger)$ do on $|\vec{p};sm\rangle$. Indeed, for example,

$$b_{1,2}|\vec{p};sm\rangle=\left[\frac{M(s-\frac{1}{2})}{M(s)}\right]^{3/2}(s\pm m)^{1/2}\times\frac{M(s-\frac{1}{2})}{M(s)}|\vec{p};s-\tfrac{1}{2},m\mp\tfrac{1}{2}\rangle\ . \quad (4.6)$$

Before completing the task of this section [fitting the Poincaré $U(A,a)$ into the alternative QN form] we use the new algebra $\vec{Z},\vec{N},b_i,b_i^\dagger$ to set up new SO(3,2) operators, denoted S'_{AB}:

$S'_{AB}\equiv$bilinear expressions formed from $b_i,b_i^\dagger$ in exactly the same way that S_{AB} are formed from $a_i,a_i^\dagger$ [cf. (3.3)] . (4.7)

Thus

$$\text{(i) } S'_{AB}=e^{iD\ln M_{\rm op}}S_{AB}e^{-iD\ln M_{\rm op}}\ , \quad (4.8)$$

$$\text{(ii) } S'_{jk}=S_{jk},\ \text{that is,}\vec{S}'=\vec{S}; \quad (4.9)$$

and $V'_0\equiv S'_{05}=V_0\equiv S_{05}$,

(iii) S'_{AB} obey the relations(3.4) of SO(3,2), hence generate a representation $\mathscr{U}'(g)$ of SO(3,2) on$\mathscr{H}$,

$$\text{(iv) } \mathscr{U}'(g)|\vec{n};sm)=\sum_{s'm'}D^{\rm Maj}_{s'm';sm}(g)|\vec{n};s'm')\ . \quad (4.10)$$

To compare the two sets further we note

$$\sum_i a_i^\dagger a_i=\sum_i b_i^\dagger b_i,\ \text{hence } M_{\rm op}=M(V_0)=M(V'_0)\ ; \quad (4.11)$$

$$[a_i \text{ or } a_i^\dagger,\vec{Z} \text{ or } \vec{N}]\neq 0$$

just as

$$[b_i \text{ or } b_i^\dagger,\vec{X} \text{ or } \vec{P}]\neq 0\ . \quad (4.12)$$

We can now put $U(A,a)$ into the alternative QN form: introduce the operator $N^0=(\mathbb{1}+\vec{N}\cdot\vec{N})^{1/2}$ and define the unit timelike four-vector $n=(\vec{n},(1+\vec{n}\cdot\vec{n})^{1/2})$ for any three-vector $\vec{n}$; then from (2.16) and the equations of transformation we obtain

$$P^\mu=M_{\rm op}N^\mu,\ \vec{J}=\vec{Z}\times\vec{N}+\vec{S}',\ \vec{K}=\tfrac{1}{2}\{\vec{Z},N^0\}+\frac{\vec{N}\times\vec{S}'}{N^0+\mathbb{1}}\ ,$$

$$U(A)|\vec{n};sm)=\left[\frac{(An)^0}{n^0}\right]^{1/2}\sum_{m'}D^{(s)}_{m'm}(B(An)^{-1}AB(n))|\vec{An};sm')\ , \quad (4.13)$$

$$P^\mu|\vec{n};sm)=M(s)n^\mu|\vec{n};sm)\ .$$

Repeating the obvious, the $P_\mu,\vec{J},\vec{K}[U(A)]$ are *the same* as set up previously, only now expressed as functions of a new operator basis and acting on a new (Hilbert space) vector basis. Equation (4.13) gives the representation $U(A,a)$ of$\mathscr{P}$ in the alternative QN form.

Stage 2. Disentangling the Wigner rotation. The crucial step in disentangling the Wigner rotation has been accomplished by going over to the alternative QN form; this we recognize as the content of Eq. (4.10) which shows *that the action of the entire group* SO(3,2) [and not just the spin (SU2) component] *commutes with the rescaled momentum operator* N (Werle relation),

$$[N_\mu,S'_{AB}]=0\ .$$

Restricting g to $A\in SL(2,\mathbb{C})$, $D'^{\rm Maj}$ gives us a representation of SL(2,C) acting on $\mathscr{H}$, again leaving N invariant. [The generators of $D'^{\rm Maj}(A)$ are of course $\vec{S}_{\mu\nu}$ formed bilinearly from $b_i,b_i^\dagger$.] We now define a representation $U'(A)$ of SL(2,C) [presently to be extended to a representation $U'(A,a)$ of $\overline{\mathscr{P}}$] as

$$U'(A)|\vec{n};sm)\equiv\left[\frac{(An)^0}{n^0}\right]^{1/2}\sum_{s'm'}D^{\rm Maj}_{s'm';sm}(A)|\vec{An};s'm'). \quad (4.14)$$

The generators of this new representation $U'(A)$ clearly are

$$\vec{J}'=\vec{J}=\vec{Z}\times\vec{N}+\vec{S}'\ ,$$
$$\vec{K}'=\tfrac{1}{2}\{\vec{Z},N^0\}+(S'_{10},S'_{20},S'_{30})\ . \quad (4.15)$$

$U'(A)$ is therefore recognized as the kinematic product of an "orbital" action of A on N and an "internal" SO(3,2) action on the space of b_i and $b_i^\dagger$. $U'(A)$ and $U(A)$ share the same SU(2) generators *but the boost generators differ greatly.*

Stage 3. The unitary transformation connecting $U(A)$ to $U'(A)$: We develop now a unitary transformation connecting $U(A)$ to $U'(A)$. Since for $A\in SU(2)$,

$$D^{\rm Maj}_{s'm';sm}(A)=\delta_{s's}D^{(s)}_{m'm}(A)\ , \quad (4.16)$$

and also $B(An)^{-1}AB(n)\in SU(2)$ for any n and A, we can transform the action of $U(A)$ on $|\vec{n};sm)$ given in Eq. (4.13) in this way. First we introduce (4.16) into (4.13) to obtain

$$U(A)|\vec{n};sm)=\left[\frac{(An)^0}{n^0}\right]^{1/2}\times\sum_{s'm'}D^{\rm Maj}_{s'm';sm}(B(An)^{-1}AB(n))|\vec{An};s'm')\ . \quad (4.17)$$

Now we use the fact that SO(3,2) transformations are well defined to rewrite the right-hand side of (4.17) as

$$\text{RHS}=\left[\frac{(An)^0}{n^0}\right]^{1/2}\sum_{\substack{s'm'\\s_1m_1\\s_2m_2}}D^{\rm Maj}_{s'm';s_1m_1}(B(An)^{-1})D^{\rm Maj}_{s_1m_1;s_2m_2}(A)\times D^{\rm Maj}_{s_2m_2;sm}(B(n))|\vec{An};s'm')\ . \quad (4.18)$$

$$\vec{X}^{QN}=\tfrac{1}{2}M^{-1}\{\vec{K}^{QN}-[P_0^{QN}(P_o^{QN}+M)]^{-1}\vec{P}\vec{P}\cdot\vec{K}^{QN}+(P_0^{QN}+M)^{-1}(\vec{J}\times\vec{P})\}+\text{H.c.}\ , \quad (5.7)$$

and for the QN-spin

$$\vec{S}^{QN}=\tfrac{1}{2}M^{-1}[P_0^{QN}\vec{J}-(P_0^{QN}+M)^{-1}\vec{P}\vec{P}\cdot\vec{J}-\vec{K}^{QN}\times\vec{P}]+\text{H.c.}\ , \quad (5.8)$$

where we have used the abbreviation

$$M^2\equiv M^2(V_0^{QN})=\alpha(V_0^{QN})\ . \quad (5.9)$$

It should be noted that the explicit realizations given in Eqs. (5.7) and (5.8) are not covariant equations; accordingly that the QN-position operator $\vec{X}^{QN}$ is *not* part of a four-vector and the QN-spin is *not* part of a six-vector (antisymmetric tensor).

We can now easily obtain the mean-position and mean-spin operators, simply by taking the transform of Eq. (5.6) and (5.7), and noting that the right-hand sides transform into the corresponding Minkowski operators. Thus we find

$$\mathscr{U}\vec{X}^{QN}\mathscr{U}^{-1}\equiv\vec{X}^{\text{mean}}=\tfrac{1}{2}(M^M)^{-1}\{\vec{K}^M-[P_0^M(P_0^M+M^M)]^{-1}\vec{P}\vec{P}\cdot\vec{K}^M+(P_0^M+M^M)^{-1}(\vec{J}\times\vec{P})\}+\text{H.c.} \quad (5.10)$$

and

$$\mathscr{U}\vec{S}^{QN}\mathscr{U}^{-1}\equiv\vec{S}^{\text{mean}}=\tfrac{1}{2}(M^M)^{-1}[P_0^M\vec{J}-(P_0^M+M^M)^{-1}\vec{P}\vec{P}\cdot\vec{J}-\vec{K}^M\times\vec{P}]+\text{H.c.}\ , \quad (5.11)$$

where the transformed-mass operator (in the Minkowski frame) is denoted M^M.

If we introduce the Minkowski realization of the Poincaré generators, that is,

$$P_0^M=[(\vec{P})^2+\alpha(P\cdot V)]^{1/2}\ , \quad (5.12)$$

$$\vec{P}^M=\vec{P}\ , \quad (5.13)$$

$$\vec{J}^M=\vec{X}^M\times\vec{P}+\vec{S}^M=\vec{J}\ , \quad (5.14)$$

$$\vec{K}^M=\tfrac{1}{2}\{\vec{X}^M,P_0^M\}+\vec{N}^M \quad (5.15)$$

(where $N_i^M\equiv S_{i,0}^M$, and is not to be confused with $\vec{N}$ in Sec. IV), which introduces (implicitly) the Minkowski position operator $\vec{X}^M$ with the properties

$$[\vec{X}^M,\vec{X}^M]=0\ , \quad (5.16)$$

$$-i[\vec{P},\vec{X}^M]=\mathbb{1}\ , \quad (5.17)$$

we can simplify the expressions for the mean-position operator. One finds

$$\vec{X}^{\text{mean}}=\tfrac{1}{2}\vec{X}^M+\tfrac{1}{2}(M^M)^{-1}\{\vec{N}^M-[P_0^M(P_0^M+M^M)]^{-1}\vec{P}\vec{P}\cdot\vec{N}^M+(P_0^M M^M)^{-1}(\vec{S}^M\times\vec{P})\} \quad (5.18)$$
$$-\frac{1}{4M^M}\left[P_0^M,\vec{X}-\frac{\vec{P}(\vec{P}\cdot\vec{X})}{P_0^M(P_0^M+M^M)}\right]+\text{H.c.}\ ,$$

$$\vec{S}^{\text{mean}}=\vec{S}^M+(M^M)^{-1}[\vec{P}\times\vec{N}^M+(P_0^M+M^M)^{-1}(\vec{P}\times\vec{S}^M)\times\vec{P}]-\tfrac{1}{2}(M^M)^{-1}[P_0^M,\vec{S}^M]\ . \quad (5.19)$$

It follows, by construction, that the mean-position operator $\vec{X}^{\text{mean}}$ has the following properties;

(a) commuting components: $[\vec{X}^{\text{mean}},\vec{X}^{\text{mean}}]=0$;

(b) $\dot{\vec{X}}^{\text{mean}}=\vec{P}(P_0^M)^{-1}$, that is, a uniform (straight-line) trajectory.

Similarly, from the construction the mean-spin operator $\vec{S}^{\text{mean}}$ necessarily has the properties

(i) $\vec{S}^{\text{mean}}$ is a *translationally invariant spin operator,* that is $\vec{S}^{\text{mean}}\times\vec{S}^{\text{mean}}=i\vec{S}^{\text{mean}}$, and $[\vec{P},S^{\text{mean}}]=0$;

(ii) $\vec{S}^{\text{mean}}$ is *constant in time.* Since J itself is time independent, it follows that the mean orbital-angular momentum, $\vec{L}^{\text{mean}}\equiv\vec{X}^{\text{mean}}\times\vec{P}$, is also conserved.

(iii) $\vec{S}^{\text{mean}}\cdot\vec{S}^{\text{mean}}$ is *Poincaré invariant.* This is a very useful, even important, result since it provides an explicit operator realization of the Poincaré spin label $\vec{S}^{\text{mean}}\cdot\vec{S}^{\text{mean}}\to s(s+1)$ exactly analogous to the Poincaré mass label realized by $P^M\cdot P^M\to M^2$. It is of interest to note that these two invariant operators provide labels for each of the denumerably many states $(M(s),s)$ of the Regge band defined in our model.

(iv) Under a general Poincaré transformation it follows (again by construction) that the operator $\vec{S}^{\text{mean}}$ undergoes a spatial rotation. *Thus the operator $\vec{S}^{\text{mean}}$ is the proper operator to use for discussing the spin-polarization properties of the states of the model.* Note that the existence of the operator $\vec{S}^{\text{mean}}$ obviates the need for the usual (roundabout) tetrad definition of covariant spin-polarization operators.[29]

There is another way in which we can approach the problem of determining the mean-position operator $\vec{X}^{\text{mean}}$ and the mean-spin operator $\vec{S}^{\text{mean}}$, a way very different from the previous method, which offers additional insight into the QN$\to M$ transformation. This new approach is based on the concept of "aligned boson operators," and is more closely analogous to the concepts used for the Dirac electron equation.

We begin by noting that the ten bilinear boson operators defined over the four boson operators $(a_i^\dagger,a_2^\dagger,a_1,a_2)$ are properly to be taken as operators in the QN frame. Now let us consider the action of $\mathscr{U}=\mathscr{U}(B_{\hat{P}})^{\text{spin}}$ on the bosons $\{a_i^\dagger,a_i\}$; this action aligns the bosons and has the explicit form

$$\mathscr{U}(B_{\hat{P}})^{\rm spin}{:}Q\equiv\begin{bmatrix}a_1^\dagger\\ a_2^\dagger\\ a_2\\ -a_1\end{bmatrix}\to Q'=AQ\ , \qquad (5.20)$$

where

$$A=[2(1+\hat{P}_0^M)]^{-1/2}[(1+\hat{P}_0^M)\mathbb{1}+\rho_1\vec{\sigma}\cdot\hat{\vec{P}}^M]\ . \qquad (5.21)$$

[In this result, A is a 4×4 matrix which we express in terms of the Dirac matrices (ρ,σ), simply for ease of writing.] Note that the momenta in Eq. (5.21) are (unit length) Minkowski frame *four-momentum operators* (so that A is operator valued). The bosons defined by AQ are the aligned bosons. When expressed in terms of the original bosons, using Eq. (5.21), the aligned bosons are linear combinations of both creation and destruction operators, and the "aligned vacuum" for the aligned bosons thus has—in terms of the original bosons—indefinitely many excitations. [This curious state of affairs reminds one of the BCS wave functions (and the Bogoliubov-Valatin transformation), but unlike that theory the present results are not approximate but exact.]

We can use these results to obtain the transformation of the spin operator $\vec{S}^{\rm QN}$ under $\mathscr{U}=\mathscr{U}(B_{\hat{P}})^{\rm spin}$. The spin operator $\vec{S}^{\rm QN}$, defined by Eqs. (2.4) in terms of bilinear forms on the boson variables, transforms into the spin operator defined bilinearly on the aligned bosons. It is easily determined, with the use of (5.20) and (5.21), that $\vec{S}^{\rm mean}\equiv\mathscr{U}S^{\rm QN}\mathscr{U}^{-1}$ takes exactly the form as in (5.19), thereby verifying this result in a different way.

There is an advantage, however, in this alternative method, in that we can explicitly determine the transformation of the mass operator.

The mass operator M has been defined in Sec. III at Eq. (3.5). We seek now to determine the transform of this operator under $\mathscr{U}(B_{\hat{P}})^{\rm spin}$. To do so, note first that M is a function of the operator V_0, as defined by Eq. (3.4). Thus we seek the transform

$$\mathscr{U}(B_{\hat{P}})^{\rm spin}{:}\ V_0\to(V_0)' = \mathscr{U}(B_{\hat{P}})\left[\frac{a_1^\dagger a_1+a_2^\dagger a_2+1}{2}\right]\mathscr{U}(B_{\hat{P}}^{-1})\ . \qquad (5.22)$$

Using the result developed in Eqs. (5.20) and (5.21) for the boson operators, one finds that

$$(V_0)'=\hat{P}_\mu V^\mu\ . \qquad (5.23)$$

This result is very useful since it implies that the Hamiltonian of the quasi-Newtonian form

$$P_0^{\rm QN}=(\vec{\rm P}\cdot\vec{\rm P}+M^2)^{1/2} \qquad (5.24)$$

transforms into the Minkowski Hamiltonian

$$P_0^{\rm QN}\to(P_0^{\rm QN})'\equiv P_0^M=[\vec{\rm P}\cdot\vec{\rm P}+\alpha(\hat{P}\cdot V)]^{1/2}\ . \qquad (5.25)$$

This determination of the Minkowski-frame Hamiltonian, for the spinorial relativistic rotator, is the direct proof mentioned in the Introduction.

To conclude, let us recall that in the Introduction we discussed briefly the fact that for the Dirac equation in the QN frame (Foldy-Wouthuysen transformation) one cannot achieve local electromagnetic interactions, in contrast to the Minkowski frame (the original form of the Dirac equation) where local EM interactions are readily obtained by the minimal prescription $p\to p-eA/c$. Exactly the same situation occurs for the spinorial relativistic rotator: in the Minkowski frame, the same minimal prescription achieves local EM interactions. (This is verified in Ref. 1 by using a Lagrangian formalism.) By means of the M→QN transformation one can, in principle, achieve a nonlocal form of the same EM interaction in the QN frame, although in practice useful results are obtained only in the form of a series expansion. One obtains in this way analogs of the Darwin, spin-orbit,... terms familiar from the Dirac equation. Since the purpose of the present paper is to demonstrate, by actual construction, the QN↔M transformation we shall not develop further these effective electromagnetic interactions (in the QN frame); a discussion can be found in Ref. 31.

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