Classical relativistic constituent particles and composite-particle scattering

Marcia J. King
228 Buckingham Avenue, Syracuse, New York 13210
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A nonlocal Lagrangian formalism is developed to describe a classical many-particle system. The nonstandard Lagrangian is a function of a single parameter s which is not, in general, associated with the physical clock. The particles are constrained to be constituents of composite systems, which in turn can decompose into asymptotic composite states representing free observable particles. To demonstrate this, explicit models of composite-composite particle scattering are constructed. Space-time conservation laws are not imposed separately on the system, but follow upon requiring the constituents to "pair up" into free composites at $s=+\infty,-\infty$. One model is characterized by the appearance of an "external" zero-mass composite particle which participates in the scattering process without affecting the space-time conservation laws of the two-composite system. Initial conditions on the two incoming composite particles and the zero-mass participant determine the scattering angle and the final states of the two outgoing composite particles. Although the formalism is classical, the model displays some features usually associated with quantum field theory, such as particle scattering by means of constituent exchange, creation and annihilation of particles, and restriction of values of angular momentum.

INTRODUCTION

There is now a long history of investigation of the problem of describing a classical relativistic N-body system by means of action at a distance.[1] In recent years, considerable effort has been concentrated on the development of Hamiltonian formulations based upon Dirac's theory of constraints,[2,3] since this formalism provides a direct prescription for quantization. (For a review of the progress and the comparison of several models, see Lusanna.[1]) Here, we take an alternative approach to the application of constraints, working within the Lagrangian formalism. In order to set forth the differences in the two approaches, we first briefly review the Hamiltonian approach.

In the Hamiltonian formulation, the initial description is based on an $8N$-dimensional phase space with covariant canonical coordinates, $(x(a),p(a))$, $a=1,\ldots,N$ (four-vector indices are suppressed throughout the paper). Not all these coordinates are independent, however. For example, if there is a Lagrangian present, it is nonstandard, and the canonical Hamiltonian vanishes. Thus, one or more constraints arise. Further constraints are introduced, if necessary, to reduce the number of independent degrees of freedom to $6N$. The constraint approach corresponds to eliminating the "extra" variables $(x^0(a),p^0(a))$. Manifest invariance is lost, but the no-interaction theorem of Currie, Jordan, and Sudarshan[4-6] is avoided.

If there is a Lagrangian, evolution of the coordinates $(x(a),p(a))$ is described by a scalar parameter, denoted by, say, s. The Lagrangian is parametrically invariant, and the "gauge-fixing" constraint has generally been used to associate s with some physical clock.[1]

Obstacles remain in the above Hamiltonian approach. In particular, there exists the problem of separability or cluster decomposition for the case $N>2$, such decomposition being necessary for scattering theory. In addition, because of the loss of manifest invariance, the particle coordinates do not transform as Lorentz four-vectors. The question then arises as to how to interpret the world lines of these particles.

In this study, we work within the Lagrangian formulation and adopt an approach which deals with the above two problems at the outset. One point of departure with the above Hamiltonian approach is to assume that the *evolution parameter s is not associated with the physical clock.* Although the parameter s is not measurable, we adopt the point of view that s describes the evolution of the system in space-time in a manner analogous to time t describing the evolution of a system in space.[7] A second difference is to retain all four components of the constituent four-vectors and their momenta. *To reduce the number of arbitrary constants, we replace the constraint approach of the Hamiltonian formulation with certain asymptotic boundary conditions. This is equivalent to reducing the dimension of the available phase space.* Namely, we demand that as s tends to $+\infty,-\infty$ the constituent particles must cluster together into one or more composite states which form representations of the Poincaré group, and whose internal motion is independent of the kinetic variables of other composite particles. These composite states are identified as observable particles.

The formalism is introduced by consideration of a Lagrangian in the form of the square root of the product of a scalar potential and the kinetic energy terms.[8,9] In the limit when the scalar potential goes to unity, the Lagrangian describes N "free" particles. However, the choice of a single parameter s to describe the Lagrangian constrains the trajectories of the particles even though they describe straight-line world lines characteristic of free particles. The particles are not in fact independent. They are also not on the mass shell, in general.

This might be reason enough to reject such a formalism,[10] but we choose to regard the particles as *constituents* of observable composite systems. We do not address the question of the observability of the constituents, but *we shall require that they can never exist as free particles.*

The Lorentz-invariant Lagrangian is nonstandard and gives rise to one arbitrary constraint, which we choose to apply to the fourth component of the center-of-mass vector. We put $X^0=cs$. This choice of constraint simplifies the generally complicated equations of motion and leads to Newtonian-looking equations. *It is worth stressing that although the equations take on the appearance of a simple generalization of Newton's equations to four-vector form, they are not so.*

In Sec. II, the equations are solved for the two-constituent attractive harmonic-oscillator force. The equations of motion, unlike the nonrelativistic case, are nonlinear. The solution shows that the frequency of oscillation for the system depends not only on the force constant (as it does nonrelativistically), but also on the amplitude of oscillation.

The two-constituent system is on the mass shell, i.e.,

$$P^2=M^2c^2 , \tag{1}$$

where P is the total momentum four-vector. The constant M, related to one Casimir operator of the Poincaré group, is identified as the mass of the system and the corresponding solution as the composite "M."

It is not enough to construct a formalism in which the total system corresponds to observable free composite particles; we must have a theory to describe the interactions between these composite particles. It is to this end that we have constructed a model of composite-composite scattering in Sec. III, where a four-constituent harmonic-oscillator system is introduced. The constituents are assumed to come in (at least) two varieties and may attract or repel. Asymptotic boundary conditions in the parameter s are applied which force the constituents to pair up into free composite particles. The resulting constraints on the arbitrary constants yield a solution in which the observer sees two free composites at the observer's time $t_{\rm OB}=-\infty$, and two free composites at $t_{\rm OB}=+\infty$, and verifies that energy, momentum, and angular momentum are conserved. The scattering takes place by means of constituent exchange as shown schematically in Fig. 1. It turns out, however, that the scattering in this model takes place only at the forward and backward angles.

Examination of Fig. 1 indicates that we are using a classical formalism in which constituent particles can go backward and forward in time. It will be seen that this does not give rise to causality problems. We suggest that the diagram can be interpreted as representing particle-antiparticle annihilation and creation.[11] In fact, as will be discussed later, this is just one aspect of this "classical" theory which causes it to resemble quantum field theory.

In Sec. IV, we construct a model to describe composite-composite scattering at arbitrary angles. A six-constituent harmonic-oscillator system is introduced, but the *same asymptotic boundary conditions are applied as in the previous four-constituent model,* i.e., constituents 1, 2, 3, and 4 are required to pair up into composites in the asymptotic regions of s. Surprisingly, this *forces* constituents 5 and 6 to form a zero-mass composite for all s. Even more surprisingly, the following description obtains: Two composites, of mass $M(1,2)$ and $M(2,3)$, scatter via exchange of constituent particles, yielding a final state consisting of two composites $M(3,4)$ and $M(4,1)$. Energy, momentum, and angular momentum are conserved; a third composite, of zero mass, participates but remains unchanged in the final state. It is this zero-mass composite which determines (along with other initial conditions) the two-particle scattering angle and the masses of the final-state composite particles.

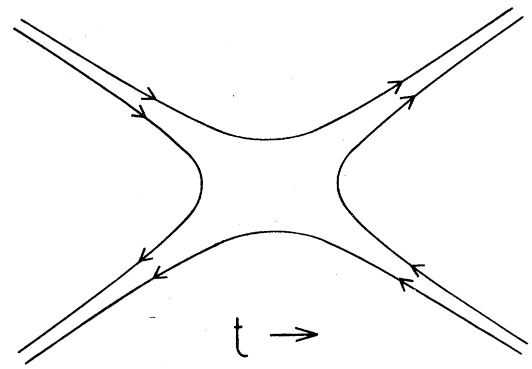

FIG. 1. Schematic illustration of two composite particles scattering via exchange of constituents. The horizontal axis represents the observer's time; arrows on the constituents indicate increasing s. The constituents which turn around in time can also be interpreted as constituent-anticonstituent pair annihilation.

In Sec. V, we apply a different set of asymptotic boundary conditions to the solution of the six-constituent harmonic-oscillator system to describe the decay of a particle.

An appendix is included which contains a discussion of a particular set of solutions to the general equations of motion which reduce to the usual Newtonian ones in the limit when c tends to infinity. We have called these the "equal-time" solutions because of additional constraints imposed on the time components of the constituent coordinates. Two example problems are discussed corresponding to the harmonic-oscillator and the inverse-square law. We stress, however, that it is the existence of solutions other than ones with a Newtonian limit which allow the following model of relativistic particle scattering.[12]

I. CENTER-OF-MASS LAGRANGIAN FORMULATION FOR N INTERACTING PARTICLES

The discussion is limited to scalar potentials. The Lagrangian is postulated to be[13]

$$L(s)=-\left\{Mc^2V(x(1,s),\ldots,x(N,s))\times\sum_{a=1}^{N}m(a)[\dot{x}(a,s)]^2\right\}^{1/2} \tag{2}$$

with

$$V(x(1,s),\ldots,x(N,s))=1+(1/Mc^2)\times\sum_{a\neq b}^{N}V_{ab}([x(a,s)-x(b,s)]^2)\ ,\tag{3}$$

$$M=\sum_{a=1}^{N}m(a)\ ,$$

and

$$V_{ab}=V_{ba}\ .$$

The equations of motion (with suppressed indices) resulting from application of the variational principle are

$$m(a)\ddot{x}(a,s)=-\left[\sum_{b=1}^{N}m(b)\dot{x}^2(b)/Mc^2V\right]\sum_{b=1}^{N}[\partial V_{ab}/\partial x(a)]-m(a)\dot{x}(a)\left[\sum_{b=1}^{N}m(b)\dot{x}^2(b)/V\right]\times\frac{d}{ds}\left[V\Big/\sum_{b=1}^{N}m(b)\dot{x}^2(b)\right]^{1/2}.\tag{4}$$

The Lagrangian is nonstandard and the action $I=\int L\,ds$ is parametrically invariant. We choose the constraint

$$X^0(s)=cs\ ,\tag{5}$$

where X^0 is the fourth component of the center-of-mass vector

$$X(s)=\left[\sum_{a=1}^{N}m(a)x(a,s)\right]\Big/M\ .\tag{6}$$

This constraint considerably simplifies the equations of motion. To see this, consider the constants in s resulting from the Lorentz invariance of L:

$$P^{\mu}=\sum_{a=1}^{N}p^{\mu}(a,s)\tag{7}$$

and

$$J^{\mu\nu}=\sum_{a=1}^{N}[x^{\mu}(a,s)p^{\nu}(a,s)-x^{\nu}(a,s)p^{\mu}(a,s)]\ ,\tag{8}$$

where the conjugate momenta $p(a)$ are

$$p(a)\equiv\partial L/\partial\dot{x}(a)=m(a)\dot{x}(a)\left[Mc^2V\Big/\sum_{b=1}^{N}m(b)\dot{x}^2(b)\right]^{1/2}.\tag{9}$$

Then Eqs. (7) and (9), for the fourth components, yield the constraint in the form

$$\frac{d}{ds}\left[V\Big/\sum_{b=1}^{N}m(b)\dot{x}^2(b)\right]^{1/2}=0\ .\tag{10}$$

The equations of motion (4) become

$$\ddot{x}(a,s)=-[1/m(a)h^2]\sum_{b=1}^{N}[\partial V_{ab}/\partial x(a)]\ ,\tag{11}$$

where

$$h^2=Mc^2V\Big/\sum_{a=1}^{N}m(a)\dot{x}^2(a)=\text{const in } s\ .\tag{12}$$

Relativistic invariance holds. The P^{μ} and the $J^{\mu\nu}$ obey the Poisson brackets (PB) of the Poincaré group, and the constraint arising from Eq. (5), namely,

$$\phi[p(1),\ldots,p(N),V]\equiv\sum_{a=1}^{N}[p^2(a)/m(a)]-Mc^2V\ ,\qquad \phi\approx 0\tag{13}$$

has vanishing PB with P^{μ} and $J^{\mu\nu}$.

In the Appendix, additional constraints $x^0(a,s)=cs$ are imposed. For these solutions, the conservation laws (7) and (8) become the familiar conservation laws of energy, momentum, and angular momentum in time. It is these equal-time solutions which reduce to solutions of Newton's equations in the limit c going to infinity.

II. HARMONIC-OSCILLATOR POTENTIAL

We have derived the general equations of motion for N particles based on the square-root Lagrangian (2) and the choice of constraint (5). These equations (11) resemble Newtonian equations with the difference that the mass $m(a)$ is replaced with the quantity $m(a)h$, where h is defined in Eq. (12). As a result, the solutions are significantly different from the corresponding Newtonian ones.

As a first step to constructing a model of interacting composite particles, we consider the case when the N particles interact pairwise by means of harmonic-oscillator potentials. The potential V in the Lagrangian (2) is taken to be

$$V=1-\tfrac{1}{2}\sum_{a\neq b}^{N}[\omega(a,b,0)/c]^2[x(a)-x(b)]^2\ ,\tag{14}$$

where $\omega(a,b,0)=\omega(b,a,0)$ is the force constant for particles a and b. We shall assume equal masses for all the particles $m(a)=m$. We further define a new constant $\omega(a,b)$,

$$\omega^2(a,b)=N\left[\sum_{a'=1}^{N}\dot{x}^2(a')/c^2V\right]\omega^2(a,b,0)=\text{const}\ .\tag{15}$$

The equations of motion then become

$$\ddot{x}(a,s)=-(1/N)\sum_{b=1}^{N}\omega^2(a,b)[x(a)-x(b)]\ ,\qquad a=1,\ldots,N\ .\tag{16}$$

Now consider the case of a two-body harmonic-oscillator system. Solutions to that system will be classified by means of the values of one of the Casimir operators of the associated Poincaré group. This will lead us to definitions of what we call composite particles.

For a pair of constituents interacting by means of $\omega(0)=\omega(a,b,0)$, we obtain solutions

$$x(1)=As+B+(a\cos\omega s+b\sin\omega s)\ , \tag{17}$$

$$x(2)=As+B-(a\cos\omega s+b\sin\omega s)\ , \tag{18}$$

where $\vec{A}$, $\vec{B}$, a, and b are arbitrary constants. The frequency is given by

$$\omega^2=[2\omega(0)/c]^2[dx^0(1,2)/ds]^2\{1-[\vec{v}(1,2)/c]^2\}/\{1-[2\omega(0)/c]^2[a^2+b^2]\}\ , \tag{19}$$

where $x^0(1,2)$ and $\vec{v}(1,2)$ are c.m. variables,

$$x^0(1,2)=\tfrac{1}{2}[x^0(1)+x^0(2)]$$
$$=A^0s+B^0 \tag{20}$$

and

$$\vec{v}(1,2)/c=d\vec{x}(1,2)/dx^0(1,2)\ . \tag{21}$$

The momenta conjugate to $x(a)$ are

$$p(a)=mc[dx(a)/dx^0(1,2)]\{1-[2\omega(0)/c]^2[a^2+b^2]\}^{1/2}/\{1-[\vec{v}(1,2)/c]^2\}^{1/2}\ , \tag{22}$$

so that the total momentum $P=p(1)+p(2)$ becomes

$$\vec{P}=M(1,2)\vec{v}(1,2)/\{1-[\vec{v}(1,2)/c]^2\}^{1/2}\ , \tag{23}$$

$$P^0=M(1,2)c/\{1-[\vec{v}(1,2)/c]^2\}^{1/2}\ , \tag{24}$$

$$P^2=M^2(1,2)c^2\ , \tag{25}$$

where

$$M(1,2)=M\{1-[2\omega(0)/c]^2[a^2+b^2]\}^{1/2}\ . \tag{26}$$

The invariant $M(1,2)$ corresponds to one Casimir operator of the Poincaré group.

Finally, let us express the angular momentum $\vec{J}$ as $\vec{J}(\text{c.m.})+\vec{j}$. We have

$$\vec{J}(\text{c.m.})=\vec{X}\times\vec{P},\quad \vec{j}=2\omega(0)M\vec{a}\times\vec{b} \tag{27}$$

[for a zero-mass composite, replace $2\omega(0)$ by $c/(a^2+b^2)^{1/2}$].

A. Classification of solutions

For these two-body solutions we assume $\omega^2>0$, i.e., an attractive harmonic-oscillator force. We choose the gauge

$$X^0=A^0s\ ,$$

take s to be the observer's time, and define $\vec{V}=\vec{v}(1,2)$.

Case (i): M(1,2) real and >0. For this case, $\vec{V}^2/c^2<1$. We denote this two-constituent system a "composite" of mass $M(1,2)$.

Case (ii): M(1,2) imaginary and | M(1,2) | >0. It follows that $\vec{V}^2/c^2>1$, and the system corresponds to a "tachyon."

Case (iii): M(1,2)=0. Then $[2\omega(0)/c]^2[a^2+b^2]=1$. Upon solving the two-composite equations we find $\vec{V}^2/c^2=1$, and the frequency ω in Eqs. (17) and (18) is indeterminate.

There is another solution worth mentioning, which comes from a different choice of constraint:

$$A^0=0,\quad B^0=0\ .$$

For $\omega^2>0$, the world lines of the constituents in this case are confined to

$$|\,x^0(a,s)\,|<(\,|\,a^0\,|+|\,b^0\,|\,)\ .$$

III. FOUR-CONSTITUENT HARMONIC OSCILLATOR: FORWARD/BACKWARD SCATTERING OF TWO COMPOSITES

In the previous sections we laid the groundwork for relativistic particle mechanics capable of modeling scattering interactions of classical composite particles. The inherent structure of the formalism is based on constituent particles which are, in general, off the mass shell. It is this feature which suggests the possibility of modeling classical hadron scattering through the exchange of constituent quarks.

To demonstrate this, we shall construct a model of composite-composite scattering based on the assumption that two kinds of constituents are participating: constituents with "like" and "opposite charges" interacting via repulsive and attractive harmonic-oscillator forces, respectively. This mechanism will allow us to apply boundary conditions which force the constituents to pair up into observable composites in the asymptotic regions of the observer's time $t_{\text{OB}}=\pm\infty$. It is encouraging to find that as a result of these boundary conditions, the space-time conservation laws are enforced at $t_{\text{OB}}=\pm\infty$ (we recall that Lorentz invariance implies constants in s, which as we shall see, is not in general related to the observer's time). We will find that at the observer's time $t_{\text{OB}}=-\infty$, there exists two free composites. The scattering proceeds by means of constituent exchange as depicted schematically in Fig. 1. Note that in the diagram, the exchanged constituents have world lines which turn around in time as s goes from $-\infty$ to $+\infty$. The observer can interpret one world line as constituent-anticonstituent pair annihilation and the other world line as pair creation.[11]

Assigning "plus charges" to constituents 1 and 3 and "minus charges" to constituents 2 and 4, we write

$$\omega(1,2,0)=\omega(2,3,0)=\omega(1,4,0)=\omega(3,4,0)=\omega(0)$$

and (28)

$$\omega(1,3,0)=\omega(2,4,0)\equiv\omega(R,0)\ ,$$

where $\omega(0)$ is real and $\omega(R,0)$ is imaginary. We further assume

$$|\omega(R,0)|=|\omega(0)|+|\epsilon|\ , \tag{29}$$

where ϵ is an infinitesimal quantity. With these assumptions and Eqs. (15) and (16), the solutions for the coordinates of the four constituents become

$$\begin{aligned} x(1,s)&=y(1,s)+y(2,s)+y(3,s)+y(4,s)\ ,\\ x(2,s)&=y(1,s)-y(2,s)+y(3,s)-y(4,s)\ ,\\ x(3,s)&=y(1,s)+y(2,s)-y(3,s)-y(4,s)\ ,\\ x(4,s)&=y(1,s)-y(2,s)-y(3,s)+y(4,s) \end{aligned} \tag{30}$$

with

$$\begin{aligned} y(1,s)&=As+B\ ,\\ y(2,s)&=a\cos\omega s+b\sin\omega s\ ,\\ y(r,s)&=a(r)\cosh(us)+b(r)\sinh(us),\quad r=3,4 \end{aligned} \tag{31}$$

and with

$$\omega^2=[4\omega(0)]^2(A/c)^2/\{1-[4\omega(0)/c]^2[a^2+b^2]+[4u(0)/c]^2\times[a^2(3)+a^2(4)-b^2(3)-b^2(4)]\}\ , \tag{32}$$

$$u=[\omega(0)/u(0)]\omega\ , \tag{33}$$

and

$$u(0)=\{-\tfrac{1}{2}[\omega^2(R,0)+\omega^2(0)]\}^{1/2}\ . \tag{34}$$

A. Application of the boundary conditions (selection rules)

The equations (30) contain $8N-1=31$ arbitrary constants which, mathematically, can be determined by specifying initial conditions in the parameter s. For example, the values of $y(j,s)$ and $dy(j,s)/ds$ at $s=0$ uniquely determine the constants A, B, a, b, $a(r)$, $b(r)$, and ω (even though the equations of motion are nonlinear). We call attention once more to the interpretation of s as the evolution parameter in space-time analogous to time t as the evolution parameter in space. Thus, specification of these initial conditions prevents causality paradoxes, even though particles can go backward and forward in time.

However, initial conditions in the parameter s do not correspond to physical "initial conditions" measured by the observer in the laboratory. The problem of determining values of the arbitrary constants from physical measurements and/or physical assumptions must be approached differently from the standard "initial-value problem."

We approach the problem by imposing the following boundary conditions which take the form of selection rules.

(1) At $s=-\infty$, constituents 1 and 2 form a free composite particle characterized by the two-body frequency ω given by Eq. (19). Constituents 3 and 4 form a free composite with the same frequency.

(2) At $s=+\infty$, constituents 1 and 4 and 2 and 3, form free composites, respectively, characterized by the same frequency.

These boundary conditions are met by putting

$$a(3)=-b(3) \tag{35}$$

and

$$a(4)=b(4)\ . \tag{36}$$

We also choose the constraint $A^0=c$, $B^0=0$. Checking that the above relations do satisfy conditions (1) and (2) above, we calculate the asymptotic forms of relative coordinates and find

$$\tfrac{1}{2}[x(1)-x(2)]\to\tfrac{1}{2}[x(3)-x(4)]\ \to a\cos\omega s+b\sin\omega s$$

as $s\to-\infty$,

$$\tfrac{1}{2}[x(3)-x(2)]\to\tfrac{1}{2}[x(1)-x(4)]\ \to a\cos\omega s+b\sin\omega s\ ,$$

as $s\to+\infty$. (37)

The frequency ω is given by

$$\omega=[4\omega(0)][1-(\vec{A}/c)^2]^{1/2}/\{1-[4\omega(0)/c]^2\times[a^2+b^2]\}^{1/2}\ . \tag{38}$$

The separability of the composites follows from the elimination of the arbitrary constants $a(r)$ and $b(r)$ from the frequency, which describes the internal motion of each composite.

We now define the asymptotic composite four-vectors, $x(i,j,s)$:

$$\tfrac{1}{2}[x(1)+x(2)]\to a(3)\exp(-us)\equiv x(1,2,s)$$

as $s\to-\infty$,

$$\tfrac{1}{2}[x(3)+x(2)]\to -a(4)\exp(us)\equiv x(2,3,s)$$

as $s\to+\infty$, (39)

$$\tfrac{1}{2}[x(3)+x(4)]\to -a(3)\exp(-us)\equiv x(3,4,s)$$

as $s\to-\infty$,

$$\tfrac{1}{2}[x(1)+x(4)]\to a(4)\exp(us)\equiv x(1,4,s)$$

as $s\to+\infty$.

From this it follows that

$$[dx(1,2,s)/ds+dx(2,3-s)/ds]\,|_{s=-\infty}$$
$$=[dx(3,4,-s)/ds+dx(1,4,s)/ds]\,|_{s=+\infty}\,. \quad (40)$$

Consider a two-body harmonic-oscillator system composed of constituents i and j which have a relative vector equal to $a\cos\omega s+b\sin\omega s$. Assume each constituent has mass $m(\text{eff})$ and that they interact via coupling $\omega(0,\text{eff})$. Then, from Eqs. (19) and (26), the frequency ω associated with this system is given by

$$\omega=[2\omega(0,\text{eff})/c][2m(\text{eff})][dx^0(i,j)/ds]$$
$$\times\{1-[\vec{v}(i,j)/c]^2\}^{1/2}/M(i,j,\text{eff})\,, \quad (41)$$

where

$$M(i,j,\text{eff})=2m(\text{eff})\{1-[2\omega(0,\text{eff})/c]^2$$
$$\times[a^2+b^2]\}^{1/2}\,. \quad (42)$$

$M(i,j,\text{eff})$ is interpreted as the mass of the ij composite.

We shall write

$$[2\omega(0,\text{eff})/c][2m(\text{eff})]=G\,, \quad (43)$$

and assume G to be a universal constant. We will further assume that after allowing s to tend to infinity, we let $\omega^2(R,0)$ tend to $-\omega^2(0)$ in such a way that the velocity $\vec{v}(i,j)$ is proportional to s. Solving for

$$dx^0(i,j)/ds$$

in Eq. (41), we obtain

$$dx^0(i,j)/ds=(\omega/G)M(i,j,\text{eff})/\{1-[\vec{v}(i,j)/c]^2\}^{1/2}\,. \quad (44)$$

Substitution of Eq. (44) into (40) yields the conservation laws for energy and momentum. Conservation of angular momentum is similarly confirmed.

Choosing $a^0(3)<0$ and $a^0(4)>0$, and letting $M(i,j)$ denote the ijth asymptotic two-constituent system, we arrive at the following picture: The initial state consists of two free composites, $M(1,2)$ and $M(2,3)$, which subsequently undergo either forward or backward scattering and yield a final state consisting of the free composites $M(3,4)$ and $M(1,4)$. The scattering takes place by means of the "exchange" of constituents 2 and 4, which "turn around" in time. Although the constituents can go forward and backward in time, causality is not violated, for we can apply initial conditions, based upon the observer's time, only on the composite (observable) states. As we discussed earlier, we can interpret trajectory 2, for example, as constituent-anticonstituent annihilation.

In the next section, we examine one possibility which allows two-composite inelastic scattering at arbitrary angle.

IV. MODEL OF COMPOSITE-COMPOSITE SCATTERING AT ARBITRARY ANGLES

It is clear from the analysis of the four-body problem of the last section that there are not enough degrees of freedom to give a description of composite-composite scattering at any angle. One way to introduce more arbitrary constants into the problem is to introduce more constituents into the system. In this section we consider a six-constituent harmonic-oscillator system, and find that application of the asymptotic boundary conditions of the last section *to four composites alone* is enough to yield a description of composite-composite scattering. The extra two constituents form a zero-mass composite which has the same initial and final state. It is the direction of the zero-mass composite's velocity which determines the scattering angle of the composite-composite collision.

Assign "positive charges" to constituents 1, 3, and 5, and "negative charges" to 2, 4, and 6. Put

$$\omega(i,j,0)=\omega(0),\quad i+j=\text{odd}$$

and (45)

$$\omega(i,j,0)=\omega(R,0),\quad i+j=\text{even}\,.$$

As before, we assume $|\omega(R,0)|>|\omega(0)|$. After solving the equations of motion (16), we can express the solutions as

$$\begin{aligned}
x(1)&=2y(1)+y(2)+y(5)+y(6)\,,\\
x(2)&=2y(3)+y(4)-y(5)+y(6)\,,\\
x(3)&=-y(1)+y(2)+y(5)+y(6)\,,\\
x(4)&=-y(3)+y(4)-y(5)+y(6)\,,\\
x(5)&=-y(1)-2y(2)+y(5)+y(6)\,,\\
x(6)&=-y(3)-2y(4)-y(5)+y(6)\,,
\end{aligned} \quad (46)$$

where

$$\begin{aligned}
y(r)&=a(r)\exp(us)+b(r)\exp(-us),\quad r=1,2,3,4\,,\\
y(5)&=a\cos\omega s+b\sin\omega s\,,\\
y(6)&=As+B\,,
\end{aligned} \quad (47)$$

$$\omega^2=[6\omega(0)/c]^2A^2/(1-[6\omega(0)/c]^2[a^2+b^2]+2[6u(0)/c]^2$$
$$\times\{2[a(2)b(1)+a(1)b(2)+a(4)b(3)+a(3)b(4)]$$
$$+[a(1)\cdot b(1)+a(2)\cdot b(2)+a(3)\cdot b(3)+a(4)\cdot b(4)]\})\,, \quad (48)$$

$$u=[u(0)/\omega(0)]\omega\,, \quad (49)$$

and

$$u(0)=-\tfrac{1}{3}[\omega^2(R,0)+\omega^2(0)]^{1/2} . \tag{50}$$

A. Application of asymptotic boundary conditions (selection rules)

We shall apply exactly the same boundary conditions as we did for the four-constituent system, namely, (1) and (2) of Sec. III. Constituents 1, 2, 3, and 4 pair up to form free composites $M(1,2)$ and $M(3,4)$ as s tends to $-\infty$, and free composites $M(2,3)$ and $M(1,4)$ as s tends to $+\infty$.

These conditions are met by putting

$$\begin{aligned} &a(3)=-a(1), \quad a(4)=a(1)+a(2) , \\ &b(3)=b(1), \quad b(4)=b(2) , \end{aligned} \tag{51}$$

and

$$[a(1)+2a(2)]\cdot[b(1)+2b(2)]=0 . \tag{52}$$

We then obtain the frequency

$$\omega^2=[6\omega(0)/c]^2A^2/\{1-[6\omega(0)/c]^2[a^2+b^2]\} . \tag{53}$$

The separability of the composites follows from the elimination of the arbitrary constants $a(r)$ and $b(r)$ from the frequency, which describes the internal motion of each composite.

Asymptotic relative vectors are now proportional to the oscillatory terms alone:

$$\begin{aligned} &\tfrac{1}{2}[x(1)-x(2)]\to\tfrac{1}{2}[x(3)-x(4)] \\ &\qquad\to a\cos\omega s+b\sin\omega s , \\ &\text{as } s\to-\infty, \\ &\tfrac{1}{2}[x(3)-x(2)]\to\tfrac{1}{2}[x(1)-x(4)] \\ &\qquad\to a\cos\omega s+b\sin\omega s , \end{aligned} \tag{54}$$

as $s\to+\infty$, while the relative vector for the 5-6 system is

$$\tfrac{1}{2}[x(5)-x(6)]=a\cos\omega s+b\sin\omega s , \tag{55}$$

for all s. This last relation follows from condition (52) alone.

We define the asymptotic four-vectors $x(i,j,s)$ for the composites

$$\begin{aligned} &x(1,2,s)\equiv[2b(1)+b(2)]\exp(-us) , \\ &x(2,3,s)\equiv[-a(1)+a(2)]\exp(us) , \\ &x(3,4,s)\equiv[-b(1)+b(2)]\exp(-us) , \\ &x(1,4,s)\equiv[2a(3)+a(4)]\exp(us) \end{aligned} \tag{56}$$

and

$$\begin{aligned} &x^{\text{in}}(5,6,s)\equiv-[b(1)+2b(2)]\exp(-us) , \\ &x^{\text{out}}(5,6,s)\equiv-[a(1)+2a(2)]\exp(us) . \end{aligned}$$

Condition (52) can be reexpressed as

$$[d\vec{x}^{\,\text{in}}(5,6)/dx^{\text{in}\,0}(5,6)]\cdot[d\vec{x}^{\,\text{out}}(5,6)/dx^{\text{out}\,0}(5,6)]=1 . \tag{57}$$

This is satisfied for a "physical" 5-6 system only if it is a zero-mass composite particle (defined in Sec. II) having the same velocity, equal in magnitude to c, in the initial and final states. Thus,

$$b(1)+2b(2)=-[a(1)+2a(2)] \tag{58}$$

and

$$|d\vec{x}^{\,\text{in}}(5,6)/dx^{\text{in}\,0}(5,6)|=1 . \tag{59}$$

We consider first the case when composite $M(1,2)$, $M(2,3)$, and a zero-mass composite exist as free particles at $t_{\text{OB}}=-\infty$, and $M(3,4)$, $M(1,4)$, and a zero-mass composite exist as free particles at $t_{\text{OB}}=+\infty$. Examining Eq. (56), we see that this implies

$$\begin{aligned} &2b^0(1)+b^0(2)<0, \quad -b^0(1)+b^0(2)>0 , \\ &-b^0(1)-2b^0(2)<0, \quad 2a^0(1)+a^0(2)>0 , \\ &-a^0(1)+a^0(2)<0, \quad -a^0(1)-2a^0(2)>0 . \end{aligned} \tag{60}$$

The asymptotic four-vectors satisfy

$$dx^{\text{in}}(5,6,s)/ds\,|_{s=-\infty}=dx^{\text{out}}(5,6,s)/ds\,|_{s=+\infty} \tag{61}$$

and

$$\begin{aligned} &[dx(1,2,s)/ds+dx(2,3-s)/ds]\,|_{s=-\infty} \\ &\qquad=[dx(3,4,-s)/ds+dx(1,4,s)/ds]\,|_{s=+\infty} . \end{aligned} \tag{62}$$

Identifying the composite states as two-body harmonic-oscillator systems, as in Sec. III, we write

$$dx^0(i,j)/ds=(\omega/G)M(i,j)/\{1-[\vec{v}(i,j)/c]^2\}^{1/2} . \tag{63}$$

From this and Eqs. (61) and (62), it follows that energy and momentum are separately conserved for the 1-2-3-4 system and the 5-6 system. One can verify similarly that angular momentum is conserved separately for the two systems. Note that the composites have equal spin, namely,

$$\vec{j}=cG\vec{a}\times\vec{b} . \tag{64}$$

Summarizing the solutions for the asymptotic velocities, we have

$$\begin{aligned} &\vec{v}(1,2)/c=[2\vec{b}(1)+\vec{b}(2)]/[2b^0(1)+b^0(2)] , \\ &\vec{v}(2,3)/c=[-\vec{a}(1)+\vec{a}(2)]/[-a^0(1)+a^0(2)] , \\ &\vec{v}(3,4)/c=[-\vec{b}(1)+\vec{b}(2)]/[-b^0(1)+b^0(2)] , \\ &\vec{v}(1,4)/c=[2\vec{a}(1)+\vec{a}(2)]/[2a^0(1)+a^0(2)] , \\ &\vec{v}^{\,\text{in}}(5,6)/c=\vec{v}^{\,\text{out}}(5,6)/c \\ &\qquad=[\vec{a}(1)+2\vec{a}(2)]/[a^0(1)+2a^0(2)] \end{aligned} \tag{65}$$

with

$$b(1)+2b(2)=-[a(1)+2a(2)] .$$

If we assume that the velocities $\vec{v}(1,2)$, $\vec{v}(2,3)$, $\vec{v}(5,6)$, and the energies $E(1,2)$, $E(2,3)$, $E(5,6)$ are given by the initial conditions, we may express the velocities of the outgoing composites as

$$\vec{v}(3,4)/c=\{[\vec{v}(1,2)/c]+[E(5,6)/E(1,2)][\vec{v}(5,6)/c)]\}/\{1+[E(5,6)/E(1,2)]\} , \quad (66)$$

and

$$\vec{v}(1,4)/c=\{[\vec{v}(2,3)/c]-[E(5,6)/E(2,3)][\vec{v}(5,6)/c]\}/\{1-[E(5,6)/E(2,3)]\} . \quad (67)$$

Their energies are given by

$$E(3,4)=E(1,2)+E(5,6) ,$$
$$E(1,4)=E(2,3)-E(5,6) . \quad (68)$$

Thus, as the energy of the 5-6 composite tends to zero, the model describes elastic forward scattering. Conversely, if the incoming and outgoing masses, respectively, are equal, i.e., $M(1,2)=M(2,3)$ and $M(3,4)=M(1,4)$, then Eqs. (66) and (67) imply $E(5,6)=0$, $M(1,2)=M(3,4)$, and therefore, once again forward elastic scattering.

On the other hand, if the energy of the zero-mass system tends to infinity, i.e., if

$$E(5,6)/E(1,2)\rightarrow E(5,6)/E(2,3)\rightarrow\infty ,$$

then (69)

$$M(3,4)\rightarrow M(1,4)\rightarrow 0 ,$$

or the model can be interpreted as describing the annihilation of the composites 1-2 and 2-3 and the production of two zero-mass composites.

In summary, the model of the previous section has been extended to describe inelastic scattering of two composites at arbitrary angle. This was accomplished by introducing two additional constituent particles which are forced by the selection rules imposed on the *original* system to pair up into a zero-mass composite particle. This composite plays a very peculiar role: The physical initial conditions on it and the two massive composites determine the final-state masses and scattering angle of the two-composite system, yet the space-time conservation laws continue to hold *separately* for the two systems at $t_{\text{OB}}=\pm\infty$. We should stress, however, that the zero-mass composite is not a "spectator" to the scattering process; it partakes of it but comes out "unscathed," so to speak.

It is interesting to note that according to our usual ways of interpreting scattering data, observation of the initial and final states of this two-composite scattering process alone would not suggest the existence of a third particle, since energy, momentum, and angular momentum are conserved (also let us recall that the impact parameter is not measured in high-energy scattering experiments).

It may be argued that the six-constituent model is an artificial one which does not accurately describe particle scattering, at least in our universe. However, the above result is nontrivial. Our experience with solutions to nonrelativistic scattering problems leads us to expect that the scattering angle is a function of the initial orbital angular momentum, but we see that that need not be the case.

The six-constituent model remains incomplete since it cannot, for example, explain elastic scattering at arbitrary angle. The question arises whether it is possible to construct a model which will provide for arbitrary scattering angles for fixed final-state masses. We "cured" one problem by going from the four- to the six-constituent system. It may be possible to obtain more realistic models by the introduction of larger numbers of constituents and couplings.

V. DECAY OF A COMPOSITE PARTICLE

In Sec. IV solutions were obtained for a six-constituent harmonic-oscillator system which described two-particle scattering. The same solution can be used to describe the decay of a single composite by replacing the positivity conditions (60) by the following set:

$$-2b^0(1)-b^0(2)>0,\quad -a^0(1)+2a^0(2)>0 ,$$
$$-b^0(1)+b^0(2)>0,\quad 2a^0(1)+a^0(2)>0 , \quad (70)$$
$$-b^0(1)-2b^0(2)>0,\quad a^0(1)+2a^0(2)>0 .$$

Now the energy-momentum conservation law (62) is replaced by

$$dx(1,2,s)/ds\,|_{s=-\infty}=[dx(2,3,s)/ds+dx(3,4,-s)/ds+dx(1,4,s)/ds]\,|_{s=+\infty} . \quad (71)$$

In this case, the initial conditions, i.e., $\vec{v}(1,2)$, $E(1,2)$, $\vec{v}(5,6)$, and $E(5,6)$, are not enough to uniquely determine the decay process.

The solutions still take the form (65). However, if we examine the solution in the rest frame of the 1-2 composite, we obtain

$$\vec{b}(2)=-2\vec{b}(1),\quad \vec{a}(1)+2\vec{a}(2)=3\vec{b}(1) , \quad (72)$$

and it follows that

$$\vec{v}(3,4)/c=[3\vec{b}(1)]/[b^0(1)-b^0(2)] , \quad (73)$$

$$\vec{v}(5,6)/c=[-3\vec{b}(1)]/[b^0(1)+2b^0(2)] . \quad (74)$$

That is, the 3-4 composite is emitted in the direction opposite to the velocity of the 5-6 zero-mass composite.

DISCUSSION

In the preceding sections, a formalism was developed to describe relativistic many-particle systems. This formalism is based on a Lagrangian which is a function of a single parameter s which is not in general identified with the physical clock. Therefore, the conservation laws which arise from Lorentz invariance yield constants in the parameter s, but not necessarily in terms of the physical clock. Thus, it is an interesting feature of the model constructed for composite scattering that the space-time conservation laws are not imposed separately, but follow automatically after requiring the constituents to pair up into free composites as s tends to $\pm\infty$. A second unusual feature is the appearance of a zero-mass composite which participates in the two-composite scattering without affecting the space-time conservation laws of either system. Along with the initial conditions on the incoming massive composites, it determines the scattering angle and the final-state masses. The internal motion of the various composite particles are determined by parameters independent of external kinematic variables.

The six-constituent model exhibits several interesting properties which suggest that further investigation of the formalism is warranted. It was pointed out in Sec. IV that although this is a classical model, it already has taken on features we usually associate with a quantum field theory, namely, (1) annihilation and creation of particles (both constituents and composites can be interpreted as undergoing such processes), (2) composite interactions describable in terms of exchange of constituents which are off the mass shell, (3) the fixed ratios of the internal angular momenta of the composite particles (the value being unity in this particular model). The model is not a "quantized" one, however, for although the ratios of the spins are fixed, there is nothing in the model which determines, for example, the angular momentum in units of $\hbar$.

Before attempting to quantize this particular model, we suggest further investigation of the formalism. For example, $N>6$ and more than two coupling constants might present interesting solutions. (One simple extension of the model consists of replacing the 5-6 system by another four-constituent system.) Examination of the frequency conditions (elimination of external kinematic parameters in the description of internal motion of composite particles) and application of asymptotic boundary conditions (selection rules) lead to constraints which determine the properties of the system.

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APPENDIX: EQUAL-TIME SOLUTIONS IN THE CENTER-OF-MASS FORMULATION

In Sec. I, the general c.m. formulation of classical particle dynamics was developed for an arbitrary scalar potential V. Parametric invariance allowed us to impose a single constraint on the system which we chose to be $X^0=cs$, where X^0 is the fourth component of the center-of-mass vector. In this appendix, we examine the case when we impose the *additional* $N-1$ constraints

$$x^0(a,s)=cs,\quad a=1,\ldots,N-1\ . \tag{A1}$$

We call the resulting solutions the equal-time solutions, and write $s=t$, where t is the observer's time.

The Lorentz generators (7) and (8) become constants in time, and we may express them as

$$H=cP^0=Mhc^2\ , \tag{A2}$$

$$\vec{\mathrm{P}}=\sum_{a=1}^{N}\vec{\mathrm{p}}(a)\ , \tag{A3}$$

$$\vec{\mathrm{J}}=\sum_{a=1}^{N}\vec{\mathrm{x}}(a)\times\vec{\mathrm{p}}(a)\ , \tag{A4}$$

$$\vec{\mathrm{K}}=h\sum_{a=1}^{N}m(a)\vec{\mathrm{x}}(a)-t\sum_{a=1}^{N}\vec{\mathrm{p}}(a)\ , \tag{A5}$$

where

$$\vec{\mathrm{p}}(a)=m(a)h\frac{d}{dt}\vec{\mathrm{x}}(a) \tag{A6}$$

and h in terms of the conjugate momenta $\vec{\mathrm{p}}(a)$ is given by

$$h=\left[V+(1/Mc^2)\sum_{a=1}^{N}\vec{\mathrm{p}}^{\,2}(a)/m(a)\right]^{1/2}\ . \tag{A7}$$

The three-vector PB relations for H, $\vec{\mathrm{P}}$, $\vec{\mathrm{J}}$, and $\vec{\mathrm{K}}$, correspond to the Lie algebra of the Poincaré group, and the center-of-mass coordinate X satisfies the canonical PB, i.e., it transforms as a Lorentz four-vector. However, the position vector $\vec{\mathrm{x}}(a)$ does not transform in the canonical manner under an arbitrary Lorentz boost.

The equations for $\vec{\mathrm{K}}$ and $\vec{\mathrm{P}}$ express the fact that the center-of-mass velocity is constant. Thus, the angular momentum associated with X is constant:

$$\vec{\mathrm{J}}(\mathrm{c.m.})=\vec{\mathrm{X}}\times\vec{\mathrm{P}}=\mathrm{const}\ . \tag{A8}$$

Let the angular momentum associated with the internal motion be denoted by $\vec{\mathrm{j}}$, where

$$\vec{\mathrm{j}}=\vec{\mathrm{J}}-\vec{\mathrm{J}}(\mathrm{c.m.})\ . \tag{A9}$$

The internal angular momentum $\vec{\mathrm{j}}$ has vanishing PB with H, $\vec{\mathrm{P}}$, $\vec{\mathrm{J}}$, and $\vec{\mathrm{K}}$, and obeys the PB for the rotation group.

Consider now the equations of motion (11) in Sec. I. For the equal-time solutions, they become

$$\ddot{x}(a,t)=-[1/m(a)h^2]\times\sum_{b=1}^{N}\{\partial V_{ab}[(x(a)-x(b))^2]/\partial x(a)\}\ , \tag{A10}$$

where

$$h^2=\left[Mc^2+\sum_{a\neq b}^{N}V_{ab}\right]\Big/\left[\sum_{a=1}^{N}m(a)c^2[1-\vec{\mathrm{v}}^{\,2}(a)/c^2]\right]\ . \tag{A11}$$

When c tends to infinity, h tends to unity and (A11) reduces to the nonrelativistic equations of motion.

We consider below two examples of equal-time solutions for a two-body system.

1. Two-body harmonic oscillator, unequal masses

Put

$$V=1-(\omega_0/c^2)(\vec{x}_1-\vec{x}_2)^2 \tag{A12}$$

in Eq. (2). The solutions are

$$\begin{aligned}\vec{x}_1&=\vec{V}t+\vec{B}+(m_2/M)(\vec{a}\cos\omega t+\vec{b}\sin\omega t)\ ,\\ \vec{x}_2&=\vec{V}t+\vec{B}-(m_1/M)(\vec{a}\cos\omega t+\vec{b}\sin\omega t)\ ,\end{aligned} \tag{A13}$$

with

$$\omega=\omega_0[(M/\mu)(1-\vec{V}^2/c^2)]^{1/2}/[1+(2\omega_0/c)^2 \times(\vec{a}^2+\vec{b}^2)]^{1/2}\ , \tag{A14}$$

where $\vec{V}=dX/dt$, and μ is the reduced mass. From Eqs. (7) and (9), the total mass of the system is found to be

$$M(1,2)=M[1+(2\omega_0/c)^2(\vec{a}^2+\vec{b}^2)]^{1/2}\ . \tag{A15}$$

2. Two-body inverse square law

Defining the relative vector $\vec{x}$ and center-of-mass vector $\vec{X}$,

$$\begin{aligned}\vec{x}&=\vec{x}_1-\vec{x}_2\ ,\\ \vec{X}&=(m_1\vec{x}_1+m_2\vec{x}_2)/M\ ,\end{aligned} \tag{A16}$$

we choose the potential

$$V_{12}=(\mu MG)/|\vec{x}|\ . \tag{A17}$$

The center-of-mass vector $\vec{X}$ obeys

$$\ddot{\vec{X}}=0\ . \tag{A18}$$

For motion in a plane, in cylindrical coordinates, the equations of motion for the relative coordinates become

$$\ddot{r}=r\dot{\theta}^2=-(GM)/(hr)^2\ , \tag{A19}$$

$$d(r^2\dot{\theta})/dt=0 \tag{A20}$$

with

$$h^2=(1+2\mu G/c^2r)/(1-\vec{V}^2/c^2-\mu\vec{v}^2/Mc^2) \tag{A21}$$

($\vec{V}=d\vec{X}/dt$, $\vec{v}=d\vec{x}/dt$). The solutions can be written

$$r=(h^2j^2/GM\mu^2)(1-\epsilon\cos\theta)^{-1}\ , \tag{A22}$$

$$j=\mu r^2\dot{\theta}=\text{const}\ , \tag{A23}$$

where ϵ is the eccentricity (a constant) and

$$h^2=\{1-(G\mu/c^2)(1-\epsilon^2)/[r(1-\epsilon\cos\theta)]\}/(1-\vec{V}^2/c^2)\ . \tag{A24}$$

In the center-of-mass system, $\vec{V}=0$ and the maximum velocity of the relative mass μ occurs at minimum radius $r=r_0$, where

$$r_0=(h^2j^2/GM\mu^2)(1+\epsilon)^{-1}\ . \tag{A25}$$

We can write

$$j=\mu r_0 v_{\max} \tag{A26}$$

and

$$h^2=[1-G\mu(1-\epsilon)/c^2r_0]\ . \tag{A27}$$

Combining these equations yields, for the maximum velocity $v_{\max}$,

$$v_{\max}{}^2=GM(1+\epsilon)/\{r_0[1-G\mu(1-\epsilon)/c^2r_0]\}\ . \tag{A28}$$

It is interesting to see at what radius r_0 the "particle" μ exceeds the velocity of light. Setting $v_{\max}{}^2=c^2$, and solving (A28) for r_0, we obtain, in units of the Schwarzschild radius R_s,

$$r_0=\tfrac{1}{2}[(1+\mu/M)+\epsilon(1-\mu/M)]R_s\quad(v_{\max}=c)\ , \tag{A29}$$

where

$$R_s=2GM/c^2\ . \tag{A30}$$

In other words, the particle μ will have velocities less than the speed of light for distances greater than, roughly, the Schwarzschild radius of the body of mass M.

The positivity of $v_{\max}{}^2$ also places a limit on the distance of closest approach:

$$r_0=(G\mu/c^2)(1-\epsilon),\quad v_{\max}=\infty\ . \tag{A31}$$

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$$x^2=-g_{\alpha\beta}x^\alpha x^\beta\ .$$