Nontopological solitons, Green's functions, and bound states

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The connection between nontopological solitons, Green's functions, and bound states is demonstrated using the functional-integral representation of quantum field theory. In the process we put into clearer perspective recent work relating nontopological solitons and Green's functions. We consider the familiar system of a self-interacting scalar field coupled to a set of independent fermion fields, although the arguments can be extended to other systems with conserved charge(s).

I. INTRODUCTION

It is important to search for and study extended objects in quantum field theory (QFT), since hadrons appear to be extended systems of bound quarks. The quarks seem confined to the inside of hadrons. In the quantum chromodynamics (QCD) theory of strong interactions the gluon self-interaction is thought to give rise to this confinement mechanism.

A simple system which can reasonably be regarded as a primitive form of QCD is that of a (quartically) self-interacting scalar field coupled to a set of independent fermion fields. The scalar field replaces the gluon fields, with a scalar self-interaction instead of a vector-gluon self-interaction. The different fermion fields are analogous to quark fields of different flavors and colors. For this reason the system has received considerable attention in the literature. In particular, nontopological-soliton equations for this system have been obtained by independent approximation schemes.[1,2] These equations have been studied analytically[1] and numerically,[3] and it has been shown that for appropriate choices of parameters they have localized solutions, referred to as either nontopological solitons or soliton bags. Both the MIT[4] and SLAC[5] bag models of hadron structure can be obtained as limiting cases of these nontopological solitons (soliton bags).

We recently showed that these nontopological-soliton equations can be obtained from the functional-integral (FI) formulation of QFT when chemical potentials are added to select the fermion numbers for the system.[6] Subsequently, we found that these equations also arise when we ask about certain properties of the fermion Green's functions.[7] It is the purpose of this present paper to clarify and further develop the connection between nontopological solitons and fermion Green's functions, and to demonstrate the role played in this by the fermion bound states of the system.

In Sec. II we introduce the necessary FI formalism and obtain a general stationary-configuration equation for fermion Green's functions. In Sec. III we expand the fermion Green's functions in terms of a complete set of momentum eigenstates and show how bound-state contributions can be isolated. Finally, in Sec. IV we show that the nontopological-soliton equations arise as time-independent stationary-configuration equations in a FI representation of these contributions to the fermion Green's functions.

II. FUNCTIONAL-INTEGRAL FORMALISM

We work in Minkowski space and, as before for this system, appeal to the Euclidean formulation when it is convenient. It is well known that the Green's functions of a system are just the vacuum expectation values (VEV's) of time-ordered products of field operators and that they can be obtained by successively differentiating the generating functional with respect to the sources and then setting the sources to zero. Let n be the number of fermion fields. The generating functional is denoted $W[j,\bar\eta^k,\eta^k]$ (where $\bar\eta^k$ is to imply $\bar\eta^1,\ldots,\bar\eta^n$, etc.) and is given by

$$W[j,\bar\eta^k,\eta^k]=\frac{\int \mathscr{D}\phi \int \prod_{k=1}^{n} \mathscr{D}\bar\psi^k\mathscr{D}\psi^k \exp(iS[\phi,\bar\psi^k,\psi^k,j,\bar\eta^k,\eta^k])}{\int \mathscr{D}\phi \int \prod_{k=1}^{n} \mathscr{D}\bar\psi^k\mathscr{D}\psi^k \exp(iS[\phi,\bar\psi^k,\psi^k])}\,, \tag{2.1}$$

where the action $S[\cdots]$ for this system is

$$S[\phi,\bar\psi^k,\psi^k,j,\bar\eta^k,\eta^k]=\int d^4x \left[\tfrac{1}{2}\partial_\mu\phi\partial^\mu\phi-U(\phi)+j\phi+\sum_{k=1}^{n}[\bar\psi^k(i\gamma^\mu\partial_\mu-m-g\phi)\psi^k+\bar\eta^k\psi^k+\bar\psi^k\eta^k]+\text{counterterms}\right]. \tag{2.2}$$

For convenience we define the absolute minimum of the scalar self-interaction $U(\phi)$ to be at $\phi=0$. We will be restricting our attention to Green's functions involving fermions only and so can set the scalar source $j(x)$ to zero. The fermion functional integrations can be carried out to yield

$$W[\bar{\eta}^k,\eta^k]=\frac{\int \mathscr{D}\phi \exp\left[i\left[S[\phi]-in\,\mathrm{Tr}\,\mathrm{Ln}[(-G\gamma^0)^{-1}]-\sum_{k=1}^{n}\int d^4x\int d^4y\,\bar{\eta}^k G\eta^k\right]\right]}{\int \mathscr{D}\phi \exp(i\{S[\phi]-in\,\mathrm{Tr}\,\mathrm{Ln}[(-G\gamma^0)^{-1}]\})}\,, \tag{2.3}$$

where $S[\phi]$ is just the scalar part of the action,

$$S[\phi]=\int d^4x[\tfrac{1}{2}\partial_\mu\phi\partial^\mu\phi-U(\phi)]\,, \tag{2.4}$$

and $G[\phi]$, actually a functional of ϕ, is defined by

$$G[\phi](y,x)=[(i\gamma^\mu\partial_\mu-m-g\phi)\delta^4(u-v)]^{-1}(y,x)\,. \tag{2.5}$$

The spinor indices on $G[\phi]$ have been suppressed and for convenience spinor indices will be suppressed in all that follows. We have given the above arguments in more detail elsewhere.[7] The form of Eq. (2.3) implies that only VEV's with equal numbers of $\bar{\psi}^k$ and ψ^k operators for all $k=1,\ldots,n$ will be nonzero.[8] This is a manifestation of fermion-number conservation. To illustrate this consider the single-fermion-field case ($n=1$),

$$\langle 0|\,T(\psi(y_1)\cdots\psi(y_j)\bar{\psi}(x_1)\cdots\bar{\psi}(x_k))\,|0\rangle=\left[\frac{1}{i}\right]^j\left[\frac{-1}{i}\right]^k\left[\frac{\delta^{j+k}}{\delta\bar{\eta}(y_1)\cdots\delta\bar{\eta}(y_j)\delta\eta(x_1)\cdots\delta\eta(x_k)}W[\bar{\eta},\eta]\right]\Bigg|_{\bar{\eta}=\eta=0}\,, \tag{2.6}$$

again suppressing spinor indices. Using Eq. (2.3) it is clear that the right-hand side (RHS) of Eq. (2.6) is zero unless $j=k$. Thus only m-fermion-to-m-fermion Green's functions are nonzero and are given by

$$\langle 0|\,T(\psi(y_1)\cdots\psi(y_m)\bar{\psi}(x_m)\cdots\bar{\psi}(x_1))\,|0\rangle$$
$$=\frac{\int \mathscr{D}\phi(iG)^m(y_1,\ldots,y_m,x_1,\ldots,x_m)\exp(i\{S[\phi]-i\,\mathrm{Tr}\,\mathrm{Ln}[(-G\gamma^0)^{-1}]\})}{\int \mathscr{D}\phi\exp(i\{S[\phi]-i\,\mathrm{Tr}\,\mathrm{Ln}[(-G\gamma^0)^{-1}]\})} \tag{2.7}$$

where we have defined

$$(iG)^m(y_1,\ldots,y_m,x_1,\ldots,x_m)=\sum_{j_1=1}^{m}\cdots\sum_{j_m=1}^{m}\epsilon_{j_1,\ldots,j_m}(iG(y_1,x_{j_1}))\cdots(iG(y_m,x_{j_m}))\,, \tag{2.8}$$

and where $\epsilon_{j_1,\ldots,j_m}$ is the antisymmetric tensor. There are of course no summations over the suppressed spinor indices in any of the above expressions.

We now generalize Eq. (2.7) to the n-fermion-field case and exponentiate all the (iG) terms, which gives

$$\langle 0|\,T(\cdots)\,|0\rangle=\frac{\int \mathscr{D}\phi\exp(i\bar{S}_{\mathrm{eff}}[\phi])}{\int \mathscr{D}\phi\exp(iS_{\mathrm{eff}}[\phi])}\,, \tag{2.9}$$

where

$$\bar{S}_{\mathrm{eff}}[\phi]=S[\phi]-in\,\mathrm{Tr}\,\mathrm{Ln}[(-G\gamma^0)^{-1}]-i\ln\left[\prod_{k=1}^{n}(iG)^{m_k}\right] \tag{2.10a}$$

and

$$S_{\mathrm{eff}}[\phi]=S[\phi]-in\,\mathrm{Tr}\,\mathrm{Ln}[(-G\gamma^0)^{-1}]\,. \tag{2.10b}$$

For brevity we have omitted writing in the fermion field operators and their respective space-time indices. We cannot evaluate the remaining scalar FI. However, we can inquire about the stationary configurations of $\bar{S}_{\mathrm{eff}}[\phi]$. These are configurations ϕ, which satisfy

$$\frac{\delta\bar{S}_{\mathrm{eff}}[\phi]}{\delta\phi(z)}=0\,.$$

These are expected to be important configurations in the FI in the sense that we expand about these configurations when making a semiclassical approximation to the FI. Up to the present we have ignored renormalization problems and the associated counterterms. We follow our earlier treatments[6,7] and assume that the divergent $\mathrm{Tr}\,\mathrm{Ln}[\cdots]$ terms, referred to as the fermion-loop contributions, are a negligible effect after any necessary renormalizations.[9] We thus arrive at the general stationary-configuration equation for the (nonzero) fermion Green's functions (neglecting loop effects),

$$-\partial_\mu\partial^\mu\phi(z)-U'(\phi(z))-i\sum_{k=1}^{n}[(iG)^{m_k}(\cdots)]^{-1}\times\left[\frac{\delta}{\delta\phi(z)}(iG)^{m_k}(\cdots)\right]=0\,, \tag{2.11}$$

where the $(iG)^{m_k}(\cdots)$ terms written more fully are

$$(iG)^{m_k}(y_1^k,\ldots,y_{m_k}^k,x_1^k,\ldots,x_{m_k}^k)\ .$$

Using definition (2.5) we note that

$$\frac{\delta}{\delta\phi(z)}\{iG[\phi](y,x)\}=-ig(iG[\phi](y,z))(iG[\phi](z,x))\ , \tag{2.12}$$

where the RHS of Eq. (2.12) is a matrix multiplication with respect to spinor indices.

We shall restrict our attention to a study of time-independent nontopological stationary configurations, since only then are we able to give $G[\phi]$ a usable form. Write $\phi_0(\vec{x})$ for time-independent $\phi(x)$ and consider the eigenvalue equation

$$\{-i\vec{\alpha}\cdot\vec{\nabla}+\beta[m+g\phi_0(\vec{x})]\}\psi_\alpha(\vec{x})=\epsilon_\alpha\psi_\alpha(\vec{x})\ . \tag{2.13}$$

The operator in Eq. (2.13) is Hermitian and so the ϵ_α are real and the $\psi_\alpha(\vec{x})$ are orthonormal. Note that if ϵ_α is an eigenvalue, then so is $-\epsilon_\alpha$ and so we define $\epsilon_{-\alpha}=-\epsilon_\alpha$. Friedberg and Lee[1] and Nishimura[2] have shown that nontopological ϕ_0 cannot give rise to any zero eigenvalues in Eq. (2.13), i.e. $\epsilon_\alpha\neq 0$ for any α. By nontopological we mean

$$\lim_{|\vec{x}|\to\infty}\phi_0(\vec{x})=0$$

[recall $\phi=0$ is the minimum of $U(\phi)$]. Using spectral decomposition it can be shown that we can write for nontopological ϕ_0[7]

$$iG[\phi_0](y,x)=\sum_\alpha \mathrm{sgn}(\epsilon_\alpha)\theta(\epsilon_\alpha(y^0-x^0))\times\psi_\alpha(\vec{y})\bar{\psi}_\alpha(\vec{x})e^{-i\epsilon_\alpha(y^0-x^0)}\ , \tag{2.14}$$

where $\mathrm{sgn}(x)=+1,0$ or -1 when x is positive, zero, or negative, respectively, and where the step function θ is defined

$$\theta(x)=\tfrac{1}{2}[1+\mathrm{sgn}(x)]\ .$$

Consider the simple case of one fermion field $(n=1)$ and say $m=1$. We then have for Eq. (2.11), for time-independent (nontopological) ϕ,

$$-\nabla^2\phi_0(\vec{z})+U'(\phi_0(\vec{z})) = \left[-g\sum_\alpha\sum_\beta \mathrm{sgn}(\epsilon_\alpha)\mathrm{sgn}(\epsilon_\beta)\theta(\epsilon_\alpha(y^0-z^0))\theta(\epsilon_\beta(z^0-x^0)) \times e^{iz^0(\epsilon_\alpha-\epsilon_\beta)}e^{-i\epsilon_\alpha y^0+i\epsilon_\beta x^0}(\bar{\psi}_\alpha(\vec{z})\psi_\beta(\vec{z}))\psi_\alpha(\vec{y})\bar{\psi}_\beta(\vec{x})\right]\Bigg/\left[\sum_\alpha \mathrm{sgn}(\epsilon_\alpha)\theta(\epsilon_\alpha(y^0-x^0)) \times e^{-i\epsilon_\alpha(y^0-x^0)}\psi_\alpha(\vec{y})\bar{\psi}_\alpha(\vec{x})\right]\ , \tag{2.15}$$

where spinor summation is implied in the $(\bar{\psi}_\alpha(\vec{z})\psi_\beta(\vec{z}))$ term in the numerator. Clearly the RHS of Eq. (2.15) is time dependent (z^0 dependent) and so there can be no solutions to this equation. This, of course, is also true for all choices of n and m. Thus there are *no time-independent nontopological* stationary configurations for the fermion Green's functions. In fact we expect no time-independent stationary configurations at all, since explicit times (by definition) appear in all Green's functions.

In our previous work relating Green's functions and nontopological solitons[7] we found that certain amplitudes formed from the fermion Green's functions had time-independent stationary configurations and that these were in fact given by the nontopological-soliton equations. These amplitudes were formed by projecting out parts of the Green's functions (using only spatial and spinor labels) and then taking the explicit times to $\pm\infty$. With regard to the simple example of Eq. (2.15) the effect of the former operation was to select out a single $\bar{\psi}_\alpha(\vec{z})\psi_\alpha(\vec{z})$ term and the effect of the latter was to remove the step functions, θ.

The remainder of this work will be devoted to demonstrating that nontopological solitons arise from looking at bound-state contributions to multifermion Green's functions.

III. BOUND-STATE CONTRIBUTIONS

We now need to expand the fermion Green's functions in terms of a complete set of normalized momentum eigenstates.[10] We illustrate the expansion for a particular example. Consider the Green's function given by

$$\langle 0|\,T(\psi^i(y_i)\psi^j(y_j)\psi^k(y_k)\bar{\psi}^k(x_k)\bar{\psi}^j(x_j)\bar{\psi}^i(x_i))\,|0\rangle\ , \tag{3.1}$$

and let us initially restrict our attention to the region $y_i^0,y_j^0,x_k^0>x_i^0,x_j^0,y_k^0$. This is the region with which we associate the fermion numbers $N^i=N^j=1$, $N^k=-1$. We consider this example since it is sufficiently general to illustrate the essential features of our argument. The Green's function (3.1) in this region can be written

$$\langle 0|\,T(\psi^i(y_i)\psi^j(y_j)\psi^k(y_k)\bar{\psi}^k(x_k)\bar{\psi}^j(x_j)\bar{\psi}^i(x_i))\,|0\rangle\theta(\min[y_i^0,y_j^0,x_k^0]-\max[x_i^0,x_j^0,y_k^0]) = -\langle 0|\,T(\psi^i(y_i)\psi^j(y_j)\bar{\psi}^k(x_k))T(\psi^k(y_k)\bar{\psi}^j(x_j)\bar{\psi}^i(x_i))\,|0\rangle\theta(\min[y_i^0,y_j^0,x_k^0]-\max[x_i^0,x_j^0,y_k^0])\ . \tag{3.2}$$

Inserting a complete set of momentum eigenstates gives for (3.2)

$$-\sum_M \sum_{r=1}^{n_M} \int d^4P\, \theta(P^0)\delta(P^2-M^2)\chi_{Pr}(y_i,y_j,x_k) \times\bar{\chi}_{Pr}(x_i,x_j,y_k) \times\theta(\min[y_i^0,y_j^0,x_k^0]-\max[x_i^0,x_j^0,y_k^0]) , \quad (3.3)$$

where we have defined

$$\chi_{Pr}(y_i,y_j,x_k)=\langle 0 \mid T(\psi^i(y_i)\psi^j(y_j)\bar{\psi}^k(x_k)) \mid P,r\rangle , \quad (3.4a)$$

$$\bar{\chi}_{Pr}(x_i,x_j,y_k)=\langle P,r \mid T(\psi^k(y_k)\bar{\psi}^j(x_j)\bar{\psi}^i(x_i)) \mid 0\rangle . \quad (3.4b)$$

In (3.3) we sum over all masses M (i.e, sum over all eigenvalues of the P^2 operator), sum over the corresponding n_M degeneracies, and integrate over all (positive-energy) eigenstates on the appropriate mass shell. The P^0 integration can be carried out to yield for (3.2)

$$-\sum_M \int \frac{d^3P}{2\omega_P} \sum_r \chi_{Pr}(y_i,y_j,x_k)\bar{\chi}_{Pr}(x_i,x_j,y_k) \times\theta(\min[y_i^0,y_j^0,x_k^0]-\max[x_i^0,x_j^0,y_k^0]) , \quad (3.5)$$

where

$$\omega_P=+(\vec{P}^2+M^2)^{1/2} .$$

We now introduce a_i, a_j and a_k which satisfy $0<a_i,a_j,a_k<1$ and $a_i+a_j+a_k=1$, but are otherwise arbitrary. We define

$$Y=a_iy_i+a_jy_j+a_kx_k ,$$

$$X=a_ix_i+a_jx_j+a_ky_k ,$$

and

$$x_i'=x_i-X,\quad y_i'=y_i-Y ,$$

etc., where X and Y are analogous to "center-of-mass" coordinates and where x_i', y_i', etc., are the corresponding relative coordinates. Translational invariance allows the center-of-mass coordinates to be factored out. This means that we can make the definitions

$$\chi_{Pr}(y_i,y_j,x_k)=(2\pi)^{-3/2}e^{-iP\cdot Y}\chi_{Pr}(y_i',y_j',x_k') , \quad (3.6a)$$

$$\bar{\chi}_{Pr}(x_i,x_j,y_k)=(2\pi)^{-3/2}e^{iP\cdot X}\bar{\chi}_{Pr}(x_i',x_j',y_k') , \quad (3.6b)$$

where P^0 here is to imply ω_P. Hence, we have

$$\langle 0 \mid T(\psi^i(y_i)\psi^j(y_j)\psi^k(y_k)\bar{\psi}^k(x_k)\bar{\psi}^j(x_j)\bar{\psi}^i(x_i)) \mid 0\rangle\theta(\min[y_i^0,y_j^0,x_k^0]-\max[x_i^0,x_j^0,y_k^0])$$
$$=-\sum_M \int \frac{d^3P}{(2\pi)^3}\frac{1}{2\omega_P}\left[\sum_{r=1}^{n_M}\chi_{Pr}(y_i',y_j',x_k')\bar{\chi}_{Pr}(x_i',x_j',y_k')\right]e^{-iP\cdot(Y-X)}$$
$$\times\theta(Y^0-X^0+\min[y_i'^0,y_j'^0,x_k'^0]-\max[x_i'^0,x_j'^0,y_k'^0]) . \quad (3.7)$$

Using

$$\theta(z)=\frac{-1}{2\pi i}\int dk\frac{1}{k+i\epsilon}e^{-ik\cdot z}$$

and changing variables $k\to P^0-\omega_P$ finally gives

$$\langle 0 \mid T(\psi^i(y_i)\psi^j(y_j)\psi^k(y_k)\bar{\psi}^k(x_k)\bar{\psi}^j(x_j)\bar{\psi}^i(x_i)) \mid 0\rangle\theta(\min[y_i^0,y_j^0,x_k^0]-\max[x_i^0,x_j^0,y_k^0])$$
$$=-\sum_M \int \frac{d^4P}{(2\pi)^4}i\left[\sum_{r=1}^{n_M}\chi_{Pr}^{P^0}(y_i',y_j',x_k')\bar{\chi}_{Pr}^{P^0}(x_i',x_j',y_k')\right]\frac{1}{2\omega_P(P^0-\omega_P+i\epsilon)}e^{-iP\cdot(Y-X)} , \quad (3.8)$$

where

$$\chi_{Pr}^{P^0}(y_i',y_j',x_k')=\exp\{-i(P^0-\omega_P)\min[y_i'^0,y_j'^0,x_k'^0]\}\chi_{Pr}(y_i',y_j',x_k') , \quad (3.9a)$$

$$\bar{\chi}_{Pr}^{P^0}(x_i',x_j',y_k')=\exp\{i(P^0-\omega_P)\max[x_i'^0,x_j'^0,y_k'^0]\}\bar{\chi}_{Pr}(x_i',x_j',y_k') . \quad (3.9b)$$

It is obvious from Eq. (3.8) that each mass M in the momentum spectrum has poles at $P^0=+(\vec{P}^2+M^2)^{1/2}$ in the same way as the free single-particle Green's functions. Had we instead begun with the region $x_i^0,x_j^0,y_k^0>y_i^0,y_j^0,x_k^0$, which we associate with $N^i=N^j=-1$ and $N^k=1$, we would have found the poles to be at

$P^0 = -(\vec{P}^2+M^2)^{1/2}$, which clearly is the antiparticle version of the previous case, as we would expect. Each momentum eigenstate $|P,r\rangle$ will lie in one or another fermion-number sector. These fermion-number sectors are labeled by the eigenvalues N^k $(k=1,\ldots,n)$ of the different fermion-number operators. Because of fermion-number conservation we clearly expect $\chi_{Pr}=\bar{\chi}_{Pr}=0$ [see definitions in (3.4)], unless $|P,r\rangle$ lies in the $N^i=N^j=1$, $N^k=-1$ fermion-number sector.

We will be interested in the minimum-energy eigenstates of each fermion-number sector, which we will refer to as the sector ground states. These will be the eigenstates of each sector for which $\vec{P}=0$ and $M=M_0$, where M_0 is to denote the smallest mass in the relevant sector. A sector ground state may not be a bound state of all the fermions. However, if a bound state of all the fermions can exist in a particular sector, then the sector ground state will be such a bound state at rest.

For our purposes Eq. (3.7) is of a more useful form than Eq. (3.8), [Note that we return to Eq. (3.7) from Eq. (3.8) by integrating over P^0.] In Eq. (3.7) we have a factor $\exp[-i\omega_P(Y^0-X^0)]$. Thus we see that if we analytically continue from Minkowski to Euclidean times $(x_j^0\to -ix_{j4}, y_j^0\to -iy_{j4}$ etc.), and examine the behavior of the Euclidean form of Eq. (3.7) as we approach the limits $Y_4\to\infty$, $X_4\to -\infty$, then the $\omega_P=M_0$ (i.e., $\vec{P}=0, M=M_0$) contribution will completely dominate.[11] The relative Euclidean times $(x'_{j4}, y'_{j4}$, etc.) are to remain constant when taking these limits. More precisely for increasing (Y_4-X_4) the Euclidean form of the RHS of Eq. (3.7) approaches

$$R(M_0,Y_4,X_4)\left[-\left[\frac{1}{2\pi}\right]^3\frac{1}{2M_0}\left\{\sum_r \chi_{Pr}(y'_i,y'_j,x'_k)\bar{\chi}_{Pr}(x'_i,x'_j,y'_k)\right\}_E\Bigg|_{\substack{\vec{P}=0\\ M=M_0}}\right], \tag{3.10}$$

where $\{\cdots\}_E$ is used to denote analytic continuation to Euclidean space and where we define

$$R(M_0,Y_4,X_4)=\int d^3P\exp[-(\vec{P}^2+M_0{}^2)^{1/2}(Y_4-X_4)]\ . \tag{3.11}$$

This argument makes the reasonable assumptions that χ_{Pr} and $\bar{\chi}_{Pr}$ of Eq. (3.4) can be analytically continued in time and that they are continuous functions of $\vec{P}$ (at least at $\vec{P}=0$). We note that M_0 can in principle be deduced from the dependence of the Euclidean Green's function on (Y_4-X_4) for sufficiently large values of (Y_4-X_4).

From Eqs. (3.7) and (3.10) we see that

$$-\left[\frac{1}{2\pi}\right]^3\frac{1}{2M_0}\left[\sum_r \chi_{Pr}(y'_i,y'_j,x'_k)\bar{\chi}_{Pr}(x'_i,x'_j,y'_k)\right]\Bigg|_{\substack{\vec{P}=0\\ M=M_0}} = \left\{\lim_{\substack{Y_4\to\infty\\ X_4\to-\infty}} R(M_0,Y_4,X_4)^{-1}\{\langle 0|T(\psi^i(y_i)\psi^j(y_j)\psi^k(y_k)\bar{\psi}^k(x_k)\bar{\psi}^j(x_j)\bar{\psi}^i(x_i))|0\rangle\}_E\right\}_M \tag{3.12}$$

where $\{\cdots\}_M$ denotes the analytic continuation back from Euclidean to Minkowski times. Both sides of Eq. (3.12) are independent of the center-of-mass coordinates Y and X. Let us consider the sector ground-state contribution to the multifermion Green's function in Eq. (3.7),

$$-\left[\frac{1}{2\pi}\right]^3\frac{1}{2M_0}\left[\sum_r \chi_{Pr}(y'_i,y'_j,x'_k)\bar{\chi}_{Pr}(x'_i,x'_j,y'_k)\right]\Bigg|_{\substack{\vec{P}=0\\ M=M_0}} e^{-iM_0(Y^0-X^0)}\ . \tag{3.13}$$

For brevity we omit the step function and simply assume that

$$(Y^0-X^0)>(\min[y_i'^0,y_j'^0,x_k'^0]-\max[x_i'^0,x_j'^0,y_k'^0])\ .$$

Now using Eq. (3.12) we see that this sector ground-state contribution (3.13) can be written

$$\left\{\lim_{\substack{Y_4\to\infty\\ X_4\to-\infty}} R(M_0,Y_4,X_4)^{-1}\{\langle 0|T(\psi^i(y_i)\psi^j(y_j)\psi^k(y_k)\bar{\psi}^k(x_k)\bar{\psi}^j(x_j)\bar{\psi}^i(x_i))|0\rangle\}_E\right\}_M e^{-iM_0(Y^0-X^0)}\ . \tag{3.14}$$

Thus we have seen that from the fermion Green's function we can obtain the mass (M_0) of the sector ground state, and subsequently isolate the contribution to this Green's function from this sector ground state using (3.14). Again we point out that this sector ground state may or may not be a bound state of the fermions.

It is now a simple matter to generalize these arguments. Had the roles of the x_i, x_j, x_k and y_i, y_j, y_k been exchanged in the definitions of X and Y, we would have isolated the corresponding antiparticle sector ground-state contribution, i.e., the contribution from the $N^i = N^j = -1$, $N^k = 1$ sector. If $i = j$, then we would have been in the $N^i = 2$, $N^k = -1$ sector. Alternatively, if $j = k$, then we would have been in the $N^i = 1$ sector. We access other fermion-number sectors by considering different definitions of Y and X and/or different fermion Green's functions.

IV. NONTOPOLOGICAL SOLITONS

In isolating the sector ground-state contribution (3.14) to the Green's function of Eq. (3.7), we made use of the Euclidean form of the Green's function, which we denoted by $\{\langle 0 | T(\cdots) | 0 \rangle\}_E$, i.e., the analytic continuation of the Minkowski Green's function into Euclidean space. The Euclidean Green's functions can be obtained from the Euclidean generating functional by successive differentiations with respect to the sources, just as for the Minkowski case. The Euclidean generating functional is given by

$$W_E[j,\bar{\eta}^k,\eta^k] = \frac{\int \mathscr{D}\phi \int \prod_{k=1}^{n} \mathscr{D}\bar{\psi}^k \mathscr{D}\psi^k \exp(-S_E[\phi,\bar{\psi}^k,\psi^k,j,\bar{\eta}^k,\eta^k])}{\int \mathscr{D}\phi \int \prod_{k=1}^{n} \mathscr{D}\bar{\psi}^k \mathscr{D}\psi^k \exp(-S_E[\phi,\bar{\psi}^k,\psi^k])} , \tag{4.1}$$

where

$$S_E[\phi,\bar{\psi}^k,\psi^k,j,\bar{\eta}^k,\eta^k] = \int d^4x \left[\tfrac{1}{2}\partial_\mu\phi\partial_\mu\phi + U(\phi) - j\phi + \sum_{k=1}^{n} [\bar{\psi}^k(\gamma_\mu\partial_\mu + m + g\phi)\psi^k - \bar{\eta}^k\psi^k - \bar{\psi}^k\eta^k] + \text{counterterms} \right] . \tag{4.2}$$

We use the definitions $\gamma_4 = \gamma^0$ and $\gamma_j = -i\gamma^j$ for $j = 1,2,3$. Setting the scalar source to zero and carrying out the fermion functional integrations gives

$$W_E[\bar{\eta}^k,\eta^k] = \frac{\int \mathscr{D}\phi \exp\left[- \left[S_E[\phi] - n\,\mathrm{Tr\,Ln}[(G_E\gamma_4)^{-1}] - \sum_{k=1}^{n} \int d^4x \int d^4y\, \bar{\eta}^k G_E \eta^k \right]\right]}{\int \mathscr{D}\phi \exp(-\{S_E[\phi] - n\,\mathrm{Tr\,Ln}[(G_E\gamma_4)^{-1}]\})} , \tag{4.3}$$

where

$$S_E[\phi] = \int d^4x [\tfrac{1}{2}\partial_\mu\phi\partial_\mu\phi + U(\phi)] \tag{4.4}$$

and

$$G_E[\phi](y,x) = [(\gamma_\mu\partial_\mu + m + g\phi)\delta^4(u-v)]^{-1}(y,x) . \tag{4.5}$$

For the single-fermion-field case we have, in analogy to Eqs. (2.6) and (2.7),

$$\langle 0 | \hat{T}(\psi(y_1)\cdots\psi(y_j)\bar{\psi}(x_1)\cdots\bar{\psi}(x_k)) | 0 \rangle_E = (-1)^k \left[\frac{\delta^{j+k}}{\delta\bar{\eta}(y_1)\cdots\delta\bar{\eta}(y_j)\delta\eta(x_1)\cdots\delta\eta(x_k)} W_E[\bar{\eta},\eta] \right] \Bigg|_{\bar{\eta}=\eta=0} , \tag{4.6}$$

and hence

$$\langle 0 | \hat{T}(\psi(y_1)\cdots\psi(y_m)\bar{\psi}(x_m)\cdots\bar{\psi}(x_1)) | 0 \rangle_E = \frac{\int \mathscr{D}\phi (G_E)^m(y_1,\ldots,y_m,x_1,\ldots,x_m) \exp(-\{S_E[\phi] - \mathrm{Tr\,Ln}[(G_E\gamma_4)^{-1}]\})}{\int \mathscr{D}\phi \exp(-\{S_E[\phi] - \mathrm{Tr\,Ln}[(G_E\gamma_4)^{-1}]\})} , \tag{4.7}$$

where $\hat{T}$ implies Euclidean time ordering, where the subscript E on the VEV is a reminder that times are Euclidean times, and where $(G_E)^m(\cdots)$ is defined by the analogous form of Eq. (2.8).

Thus for n fermion fields we have, in place of (2.10a) and (2.11),

$$\bar{S}_{E\,\text{eff}}[\phi] = S_E[\phi] - n\,\mathrm{Tr\,Ln}[(G_E\gamma_4)^{-1}] - \ln\left[\prod_{k=1}^{n} (G_E)^{m_k} \right] \tag{4.8}$$

and (neglecting fermion-loop contributions) the stationary-configuration equation

$$-\partial_\mu\partial_\mu\phi(z)+U'(\phi(z))-\sum_{k=1}^{n}[(G_E)^{m_k}(\cdots)]^{-1}\times\left[\frac{\delta}{\delta\phi(z)}(G_E)^{m_k}(\cdots)\right]=0. \quad (4.9)$$

As in Sec. II we will need to restrict our attention to time-independent nontopological stationary configurations only, since then using eigenvalue equation (2.13), we can write, similarly to Eq. (2.14),

$$G_E[\phi_0](y,x)=\sum_\alpha \text{sgn}(\epsilon_\alpha)\times\theta(\epsilon_\alpha(y_4-x_4))\psi_\alpha(\vec{y})\bar{\psi}_\alpha(\vec{x})e^{-\epsilon_\alpha(y_4-x_4)} \quad (4.10)$$

For the purposes of illustration we again consider the simple case of one fermion field ($n=1$) and $m=1$. Then, corresponding to Eq. (2.15), we have

$$-\nabla^2\phi_0(\vec{z})+U'(\phi_0(\vec{z}))=\left[-g\sum_\alpha\sum_\beta \text{sgn}(\epsilon_\alpha)\text{sgn}(\epsilon_\beta)\theta(\epsilon_\alpha(y_4-z_4))\theta(\epsilon_\beta(z_4-x_4))\times e^{z_4(\epsilon_\alpha-\epsilon_\beta)}e^{-\epsilon_\alpha y_4+\epsilon_\beta x_4}(\bar{\psi}_\alpha(\vec{z})\psi_\beta(\vec{z}))\psi_\alpha(\vec{y})\bar{\psi}_\beta(\vec{x})\right]\Big/\left[\sum_\alpha \text{sgn}(\epsilon_\alpha)\theta(\epsilon_\alpha(y_4-x_4))e^{-\epsilon_\alpha(y_4-x_4)}\psi_\alpha(\vec{y})\bar{\psi}_\alpha(\vec{x})\right], \quad (4.11)$$

where this equation has no solutions ϕ_0. This is of course also true for all choices of n and m. Thus as before, there can be no time-independent nontopological solutions of Eq. (4.9), and in fact we expect no time-independent stationary configurations at all.

As already stated, the Euclidean Green's functions are the analytic continuations of the corresponding Minkowski Green's functions, i.e.,

$$\langle 0|\hat{T}(\cdots)|0\rangle_E=\{\langle 0|T(\cdots)|0\rangle\}_E$$

in our notation. We now use this to inquire about the sector ground-state contributions to the fermion Green's functions, which are given by

$$\left\{\lim_{\substack{Y_4\to\infty\\ X_4\to-\infty}} R(M_0,Y_4,X_4)^{-1}\{\langle 0|T(\cdots)|0\rangle]\}_E\right\}_M\times e^{-iM_0(Y^0-X^0)} \quad (4.12)$$

for the m-fermion-to-m-fermion Green's function $\langle 0|T(\cdots)|0\rangle$. The fermion-number sector under consideration is determined by the definitions of Y and X. See, e.g., Eq. (3.14). Since we do not know the Green's functions we can hardly use them to isolate these sector ground-state contributions (4.12). However, we can make use of the FI representation to yield some useful information. The stationary-configuration equation for the FI representation of $R(M_0,Y_4,X_4)^{-1}\{\langle 0|T(\cdots)|0\rangle\}_E$ is just given by Eq. (4.9), since R has no functional dependence on ϕ. Then clearly the stationary configurations of Eq. (4.9) in the limits $Y_4\to\infty$, $X_4\to-\infty$ are expected to be important (in the sense of the semiclassical approximation) to the sector ground-state contributions of Eq. (4.12).

For reasons already discussed our considerations are restricted to time-independent nontopological stationary configurations. However, these are just the configurations that we expect on physical grounds to be important to unexcited fermion bound states at rest, and so this restriction is no real inconvenience. We return to the simple $n=1$, $m=1$ case of Eq. (4.11). This equation arises when considering the Green's function

$$\langle 0|T(\psi(y)\bar{\psi}(x))|0\rangle .$$

There are only two possible choices of Y and X, i.e., $Y=y$, $X=x$, or $Y=x$, $X=y$. The first of these corresponds to a choice of the $y^0>x^0$ region for the fermion Green's function and the fermion-number sector $N=1$, whereas the latter corresponds to the $x^0>y^0$ region and the $N=-1$ sector. Taking the limits $Y_4\to\infty$, $X_4\to-\infty$ for Eq. (4.11) yields for the first choice of Y and X,

$$-\nabla^2\phi_0(\vec{z})+U'(\phi_0(\vec{z}))=-g\psi_1^\dagger(\vec{z})\beta\psi_1(\vec{z}) , \quad (4.13)$$

where $\bar{\psi}_\alpha=\psi_\alpha^\dagger\gamma^0=\psi_\alpha^\dagger\beta$. The second choice yields on the right-hand side of Eq. (4.13)

$$-g(-1)\psi_{-1}^\dagger(\vec{z})\beta\psi_{-1}(\vec{z}) .$$

For the example considered in Sec. III, i.e., the $N^i=N^j=1$ and $N^k=-1(i\neq j\neq k)$ case, we again obtain Eq. (4.13) but where now we have on the RHS

$$-g\sum_{k=1}^{n}\sum_\alpha n_\alpha^k\psi_\alpha^\dagger(\vec{z})\beta\psi_\alpha(\vec{z})$$

with

$$n_1^i=n_1^j=1, n_{-1}^k=-1$$

and all others zero. This follows from the time-independent form of Eq. (4.9) and these limits. One first obtains the appropriate generalization of Eq. (4.11) and then takes the limits $Y_4\to\infty$, $X_4\to-\infty$. Recall that these limits are taken such that the relative Euclidean times (x'_{j4}, y'_{j4}, etc.) remain finite. This means that the actual Euclidean times satisfy y_{i4}, y_{j4}, $x_{k4} \to \infty$ and x_{i4}, x_{j4}, $y_{k4} \to -\infty$ but with $(y_{i4}-y_{j4})$, $(y_{i4}-x_{k4})$, $(y_{j4}-x_{k4})$ and $(x_{i4}-x_{j4})$, $(x_{i4}-y_{k4})$, $(x_{j4}-y_{k4})$ all remaining finite.

If for this example $i=j$ or $j=k$, then we have the $N^i=2$, $N^k=-1$ case and the $N^i=1$ case, respectively. For these situations the limit procedure is not entirely trivial, since for $m_k \neq 1$ the antisymmetric nature of $(G_E)^{m_k}$ becomes apparent with a corresponding increase in the number of terms. With a little care we obtain the same equation as before but now with $n_1^i = n_2^i = 1$ and $n_{-1}^k = -1$ if $i=j$ and with $n_1^i = 1$ if $j=k$, which is as we might expect.

These arguments can be extended to the general case. For a fermion Green's function in a region with the associated fermion numbers N^k for $k=1,\ldots,n$, we can isolate the corresponding sector ground-state contribution (4.12). Using the FI representation and ignoring fermion-loop effects we find that we have the following time-independent stationary-configuration equation for this contribution,

$$-\nabla^2\phi_0(\vec{z}) + U'(\phi_0(\vec{z})) = -g \sum_{k=1}^{n} \sum_{\alpha} n_\alpha^k \psi_\alpha^{k\dagger}(\vec{z}) \beta \psi_\alpha^k(\vec{z}) , \tag{4.14}$$

where

$$\{-i\vec{\alpha}\cdot\vec{\nabla} + \beta[m^k + g\phi_0(\vec{x})]\}\psi_\alpha^k(\vec{x}) = \epsilon_\alpha^k \psi_\alpha^k(\vec{x}) , \tag{4.15}$$

and where the index k on the eigenvectors ψ_α^k and eigenvalues ϵ_α^k is to allow for the possibility of different fermion masses m^k. If N^k is positive then $n_\alpha^k = 1$ for $\alpha = 1, \ldots, N^k$, and if N^k is negative then $n_\alpha^k = -1$ for $\alpha = -1, \ldots, -N^k$. But these are just the nontopological-soliton equations with the associated fermion numbers N^k, $k=1,\ldots,n$ and with the fermions sequentially filling the lowest "energy" states. Hence, Eqs. (4.14) and (4.15) will somewhat loosely be referred to as the ground-state nontopological-soliton equations.

Because Eq. (4.14) contains none of the explicit times appearing in the original Green's function and because it is (z_4)-independent the return to Minkowski space $\{\cdots\}_M$ in Eq. (4.12) has no effect. Thus, the ground-state nontopological-soliton equations play an important role in the corresponding sector ground-state contributions to the fermion Green's functions.

It is also apparent that Eq. (4.14) contains none of the original space coordinates, which has allowed the equation to become space translationally invariant. This seems natural in the sense that isolating the sector ground state means isolating a state of definite momentum ($\vec{P}=0$). With a state of definite momentum we associate complete uncertainty in position. We also note that Eq. (4.14) contains none of the spinor indices present in the original Green's function. These canceled from the equation at the same time as the space coordinates.

V. CONCLUSION

The sector ground-state contributions to the fermion Green's functions were isolated. We subsequently inquired about the time-independent nontopological stationary configurations arising from a FI representation for these contributions and found that (neglecting fermion-loop effects) these were the (ground-state) nontopological-soliton equations. If a bound state of the fermions exists in some fermion-number sector, then the sector ground state is such a bound state at rest. Thus, for the system studied here it is natural to associate nontopological solitons with the corresponding fermion bound states. This is of course as we might have expected, since a nontopological soliton (soliton bag) is a mean-field model of such a fermion bound state.

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[8]Strictly speaking we are assuming here that the order of the scalar FI and the operation of setting the sources to zero (after the differentiations) can be changed.

[9]These fermion-loop contributions have also been neglected in lattice gauge theory studies, where this approximation has been referred to as the "quenched" approximation. See, for example, F. Fucito, G. Martinelli, C. Omero, G. Parisi, R. Petronzio, and F. Rapuano, Nucl. Phys. B210 [FS6], 407 (1982).

[10]The momentum expansion carried out here is a generalization of arguments used in discussions of Bethe-Salpeter wave functions. See, for example, D. Lurié, A. J. Macfarlane, and Y. Takahashi, Phys. Rev. 140, B1091 (1965).

[11]While this statement seems obvious the proof is not entirely trivial and will not be given here. It is a matter of demonstrating that as $(Y_4-X_4)\to\infty$ a δ function, $\delta^3(\vec{P})$, arises in the Euclidean form of Eq. (3.7) due to the $\exp[-\omega_P(Y_4-X_4)]$ term.