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Quantum frames of reference

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Frames of reference attached to quantum-mechanical objects of finite mass are considered. A consistent description of such frames is obtained which resolves a variety of apparent paradoxes associated with such a description. The main result of the present work is a formalism wherein the principle of equivalence is extended to reference frames described by quantum states.

INTRODUCTION

In classical physics, the measuring device by means of which information about a physical system is obtained is always ignored. This is so because the interaction with the system may always, in principle at least, be made as small as desired. This approach has been carried over to quantum theory. Here, certainly, it has been recognized that due to the uncertainty principle, a finite quantum of action, which is uncontrollable and unpredictable, has to be exchanged during any interaction between object and measuring device. But it has been argued that the measuring device is sufficiently heavy so that the disturbance it suffers as a result of these quantum exchanges could be neglected, and the device be described as a classical system. Thus, the separation between the observed system and the measuring device seems, in this sense, justified in quantum theory too. Still it is clear that in the quantum-mechanical case there is a difficulty that does not arise classically. We call it the paradox of the measuring device. To illustrate the idea involved, let us consider the measurement of position.

Supposing a separation of the order $\sim x_0$ is measured, and an accuracy

$$\Delta x \ll x_0$$

is desired. From the uncertainty principle it follows that a finite amount of momentum

$$\Delta p > \frac{\hbar}{\Delta x} \gg \frac{\hbar}{x_0}$$

must be exchanged between object and measuring device. If the mass of the measuring device is finite, it will be accelerated during measurement by a finite, but uncertain, amount, and after a finite period of time, the device's position will become uncertain by

$$\Delta X(T) = \Delta X(0) + \frac{\Delta p}{M} T ,$$

relative to a given external reference frame. This, in turn, introduces a corresponding uncertainty into the separation, and hence into the result of the measurement. But then, the use of such a measuring device for repeated measurements becomes impracticable. In other words, the measuring device no longer fulfills those very ends which are the reason for its existence.

By contrast, if the mass of the measuring device is infinite, the disturbances caused to it will tend to zero in all cases. Hence, the above difficulty will not arise and it will be described by classical physics.

The paradox of the finite-mass measuring device was noticed early in the history of quantum theory, but has remained an open question since.[1] As said, the measuring device has always been considered a macroscopic system, i.e., one whose mass is infinite, in which case this difficulty does not arise.

Against this, two counterarguments can be advanced. First, out of necessity any measuring device's mass cannot be but finite. Approximating it by an infinite mass constitutes in truth a relative statement involving both the mass and the energy exchanged between the measuring device and the object of measurement. Hence, it is not hard to envisage a measurement where the energy that has to be exchanged is so large that the above approximation can no longer be justly used.

Second, it appears as though there is an unavoidable difficulty here, which is an outcome of the very nature of quantum theory.

The real question that is being raised here concerns the consistency of the quantum description. It seems as if the theory is, in principle, unable to encompass the whole universe in its description; that it cannot even be made to consistently support the notion of a finite-mass observer and that consequently it needs a classical-type theory to augment it.

A measuring device may also be taken to define a space-time reference frame.[2] This presents a somewhat different viewpoint of its function, and one which touches the issue at hand just the same.

In this work, we shall solve the problem of the consistency of the quantum description relative to a finite-mass measuring device, within the framework of nonrela-

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tivistic quantum theory. The central principle involved, and which provides the key to the solution of the consistency problem, is the principle of equivalence.

In this solution, a canonical description of a general N-particle quantum system relative to a finite-mass reference frame is given in closed form, by means of a covariant Hamiltonian, together with the transformation law between different reference frames.

Besides solving the consistency problem, the solution also constitutes an extension of the equivalence principle to quantum theory, since a covariant description relative to a *quantum* reference frame whose motion may be of the most general kind is given.

In the following sections, two thought experiments dealing in turn with the spatial and temporal aspects of the paradox of the measuring device will be discussed (Secs. I and VI). Following these (Secs. II—V), the full mathematical solution will be given.

I. SPATIAL ASPECT OF THE PARADOX OF THE MEASURING DEVICE

In the present section a paradox involving quantum reference frames ("observers") will be discussed. Besides bringing into focus the difficulties inherent in the program of formulating a finite-mass reference frame in quantum theory, it will also shed light on the fundamental ideas involved in the solution of this problem.

Our example involves two reference frames, an "internal" frame O_1, an "external" frame O_2, and a single particle Q. The reference frames both have finite mass. They may each be thought of as laboratories containing rulers, clocks, etc., all of which are rigidly attached to the walls of the laboratory. Thus, a reference frame will here be represented by a single degree of freedom, whose mass is finite, and whose center of mass defines the origin. Consequently, these reference frames may be put into a well-defined quantum state relative to one another, which would not be possible if the mass were infinite.

For the sake of simplicity, we shall restrict the discussion here to a one-dimensional problem. We shall denote observables by unprimed variables when referring to O_1, and by primed variables when referring to O_2.

Let us now return to the setup described above, including two observers and a single particle, and consider the set of all possible measurements therein. With regard to these, it is necessary to make the following statement: It should not be possible, by means of an experiment performed within one given reference frame, to discover its state of motion. This requirement is familiar from the special and the general theories of relativity, but it acquires here richer meaning, for in the present case the states in question are quantum states. Thus, O_1 may be in an eigenstate of position or of momentum relative to the external frame. The above requirement then means that it should be impossible, for an experimenter active inside O_1, to distinguish between these states. Since O_1 may either locate Q, or measure its velocity, while O_2 may either measure the velocity of the center of mass of O_1 plus Q, we observe that,

$$\Delta v_1 = 0 \text{ and } \Delta x'_{\text{c.m.}_1} = 0 , \tag{1a}$$

where subscript 1 stands for Q, and subscript c.m.$_1$ stands for the center of mass of O_1 plus Q; or,

$$\Delta x_1 = 0 \text{ and } \Delta v'_{\text{c.m.}_1} = 0 . \tag{2a}$$

That is, the particle's velocity and the position of the center of mass of O_1 plus Q, or, alternatively, the particle's position and the velocity of the center of mass of O_1 plus Q, may be simultaneously sharp. It is entirely equivalent to the above to observe that

$$[x'_{\text{c.m.}_1}, v_1] = 0 , \tag{1b}$$

$$[v'_{\text{c.m.}_1}, x_1] = 0 . \tag{2b}$$

Let us now calculate the left-hand side in (1b). With $v_1 = p_1/m_1$ and using the notation given above, and

$$x'_{\text{c.m.}_1} = \frac{m_1 x'_1 + M_1 x'_{O_1}}{M_1 + m_1} , \tag{3}$$

where m_1 and M_1 are the masses of the particle and of O_1, respectively,

$$x'_{O_1} = -x_{O_2} , \tag{4}$$

$$x'_1 = x_1 + x'_{O_1} = x_1 - x_{O_2} , \tag{5}$$

the left-hand side in (1b) can be rewritten

$$\left[\frac{-M_1 x_{O_2} + m_1 (x_1 - x_{O_2})}{M + m_1}, \frac{p_1}{m_1} \right] = \frac{i}{M_1 + m_1} \neq 0 \tag{6}$$

unless $M_1 = \infty$. A similar result can be shown to follow in the case of (2b). This is the paradox of the quantum reference frame. It seems to preclude the possibility of a consistent formulation of a finite-mass reference frame. We shall now show that this is not so and that the paradox can be resolved.

The clue to its solution comes from the canonical description of a particle under the influence of a force, which includes, in the most general case, both vector and scalar potentials. While in classical physics only the force is regarded as a basic physical quantity, and the potentials as auxiliary quantities, in quantum theory they are necessary even in the absence of a force. Let us then put

$$m_1 v_1 = p_1 + m_1 A \tag{7}$$

with the objective of finding a vector potential A that will restore the commutation relations (1b) and (2b). However, and this constitutes the central idea in the solution, because there should not be any one preferred reference frame, we shall put the two frames, the internal and the external, on equal footing, i.e., we shall demand

$$m_1 v'_1 = p'_1 + m_1 A' , \tag{8}$$

and where, in particular, A' must be the same function of variables in O_2 as is A of the corresponding variables in O_1.

We proceed to calculate A by substituting (7) into (1b),

$$\left[x'_{\text{c.m.}_1}, \frac{p_1}{m_1} + A \right] = 0 . \tag{9}$$

Using (3), (4), and (5) above it follows that the most general solution for A must be of the form

$$A = \alpha p_1 + \beta p_{O_2} + f(x_1, x_{O_2}) + C \ .$$

We further may choose $f=0$. We find that (see Appendix A)

$$\alpha = \frac{f_1(m_1,M_1)}{M_1}$$

and

$$\beta = \frac{f_2(m_1,M_1)}{M_1} \ ,$$

or that

$$v_1 = \frac{p_1}{m_1} + \frac{f_1(m_1,M_1)p_1 + f_2(m_1,M_1)p_{O_2}}{M_1} + C \ . \tag{10a}$$

To find f_1 and f_2, in the spirit of the foregoing discussion, one has to turn the particle Q into an observer as well, defining a third reference frame, the "double primed" frame. We shall thus write

$$v_1' = \frac{p_1'}{m_1} + \frac{f_1(m_1,M_2)p_1' + f_2(m_1,M_2)p_{O_1}'}{M_2} \ , \tag{10b}$$

$$v_{O_1}' = \frac{p_{O_1}'}{M_1} + \frac{f_1(M_1,M_2)p_{O_1}' + f_2(M_1,M_2)p_1'}{M_2} \ , \tag{10c}$$

and also

$$[x'_{\text{c.m.}_1}, v_1] = 0 \ , \tag{1b}$$

$$[x_{\text{c.m.}_2}, v_1'] = 0 \ , \tag{11}$$

$$[x''_{\text{c.m.}O_1O_2}, v_1] = 0 \ . \tag{12}$$

With these and using (2b) the coefficients $f_1(m_i,m_j)$, $f_2(m_i,m_j)$ in (10) may be calculated (see Appendix A). The result is

$$f_1(m_i,m_j) = f_2(m_i,m_j) \equiv 1 \ ,$$

or

$$v_1 = \frac{p_1}{m_1} + \frac{p_1 + p_{O_2}}{M_1} + C \ , \tag{13a}$$

$$v_1' = \frac{p_1'}{m_1} + \frac{p_1' + p_{O_1}'}{M_2} + C \ . \tag{13b}$$

More generally, if N particles are given,

$$v_k = \frac{p_k}{m_k} + \frac{1}{M_1} \sum_{n \neq 1}^{N} p_n + C \ , \tag{14a}$$

$$v_k' = \frac{p_k'}{m_k} + \frac{1}{M_2} \sum_{n \neq 2}^{N} p_n' + C \ . \tag{14b}$$

This is the solution of the paradox. Its main significance lies in that it demonstrates that a consistent formulation of a finite-mass reference frame is possible. Further, two points should be noted about this solution. The first is that all coordinates involved in it have been defined as relative variables. In order to see the second point, let us tentatively assume that the momentum of only one particle, the lth say, has changed. Then (14) shows that this entails an acceleration of all particles in the universe. The explanation for this follows readily, for if only p_l has changed without a corresponding change in the momentum of any other particle, then it must be that the momentum of the reference frame itself has changed (by $-\Delta p_l$), affecting, in turn, a relative acceleration (of $\Delta p_l / M$) in all other particles. In short, the vector potential represents the "kickback" of the finite-mass reference frame, but it cannot be ignored even if no forces are acting, since via its quantum spread it represents the quantum state of the reference frame.

II. ONE-DIMENSIONAL CASE

We now proceed with the full covariant solution for quantum reference frames. In the present section, the one-dimensional case will be taken up.

Let us then consider $N+1$ particles in one dimension. Their masses are

$$m_0, \ldots, m_N \ .$$

We regard the physical variables pertaining to these particles as defined over an absolute coordinate system which here will serve the purpose of an auxiliary frame, to be completely abandoned in the future. In this frame, the particles' coordinates are

$$x_0, \ldots, x_N$$

and their momenta are

$$p_0, \ldots, p_N \ .$$

Also, in this reference frame, the usual commutation relations hold, i.e.,

$$[x_i, p_j] = i\delta_{ij}, \quad i,j = 0, \ldots, N$$

and the Hamiltonian is given by

$$H = \sum_{n=0}^{N} \frac{p_n^{\,2}}{2m_n} \ . \tag{15}$$

Because this commutes with the total momentum, we can, without loss of generality, assume that the system is in an eigenstate of $P_T = 0$ (P_T denoting the total momentum).

Let us now choose particle 0 to define the origin of our relative reference frame. In this capacity, this particle resembles one of the quantum observers O_1 and O_2 of the previous section. We further define a new set of canonical coordinates and momenta thus:

$$q_0 = x_0, \quad \pi_0 = P_T \ , \tag{16a}$$

$$q_n = x_n - x_0, \quad \pi_n = p_n, \quad n = 1, N \ . \tag{16b}$$

Clearly,

$$[q_i, \pi_j] = i\delta_{ij}, \quad i,j = 0, \ldots, N \ .$$

The characteristic feature about this choice is that the coordinates have been defined as relative variables, in particular, as distances from particle 0.

The unitary transformation that connects the relative frame with the absolute one is given by

$$U=\exp\left[-i\sum_{n=1}^{N}p_nx_0\right]. \tag{17}$$

Thus,

$$q_n=Ux_nU^\dagger,\ \pi_n=Up_nU^\dagger \text{ for } n=0,\ldots,N .$$

Using U and the "absolute" Hamiltonian, the Hamiltonian in the relative frame may be calculated (see Appendix B):

$$H'(q_n,\pi_n)=U\bar{H}(x_n,p_n)U^\dagger , \tag{18}$$

$$\bar{H}(x_n,p_n)=U^\dagger H(x_n,p_n)U . \tag{19}$$

With U and $H(p_n)$ as given by (17) and (15), respectively, we have

$$\bar{H}=\frac{\left[p_0-\sum_1^N p_n\right]^2}{2m_0}+\sum_1^N\frac{p_n^2}{2m_n}$$

and hence, from (18),

$$H'=\frac{\left[\pi_0-\sum_1^N \pi_n\right]^2}{2m_0}+\sum_1^N\frac{\pi_n^2}{2m_n} .$$

With the system in an eigenstate of $\pi_0=P_T=0$, we obtain

$$H'=\sum_1^N\frac{\pi_n^2}{2m_n}+\frac{\Pi^2}{2m_0}$$

where

$$\Pi\equiv\sum_1^N\pi_n .$$

This can be rewritten in the form

$$H'=\sum_1^N\frac{(\pi_n+m_n\Pi/m_0)^2}{2m_n}-\frac{M\Pi^2}{2m_0^2} , \tag{20}$$

where M is the total mass. It may be observed that particle 0 has been eliminated from the Hamiltonian. It is, however, compensated for by the appearance of a vector potential. This last result may have been anticipated on the grounds of the example of the previous section, viz., Eqs. (14a) and (14b) and the discussion that follows.

Combining (20) with (16b), we summarize by saying that we have here a description of the physics of N particles where observables are measured relative to a physical reference frame. Such a reference frame evolves under the influence of a vector potential. Consequently, a free particle no longer moves with a velocity that equals the momentum divided by the mass, but, rather,

$$v_l=\frac{\partial H'}{\partial\pi_l}=\frac{\pi_l}{m_l}+\frac{\Pi}{m_0} . \tag{21}$$

Considering another reference frame, where positions relative to particle 1, say, are measured, the coordinates and momenta will be

$$q_0'=x_1,\ \ \pi_0'=P_T , \tag{22a}$$

$$q_1'=x_0-x_1,\ \ \pi_1'=p_0 , \tag{22b}$$

$$q_n'=x_n-x_1,\ \ \pi_n'=p_n,\ \ n=2,\ldots,N , \tag{22c}$$

or

$$q_0'=q_0+q_1,\ \ \pi_0'=\pi_0 , \tag{23a}$$

$$q_1'=-q_1,\ \ \pi_1'=\pi_0-\sum_1^N\pi_n , \tag{23b}$$

$$q_n'=q_n-q_1,\ \ \pi_n'=\pi_n,\ \ n=2,\ldots,N . \tag{23c}$$

The unitary transformation that connects the quantum frames is

$$U=P_1\exp\left[i\sum_2^N\pi_nq_1\right]e^{-i\pi_0q_1} , \tag{24}$$

where P_1 is the parity operator for particle 1. Thus, $q_n'=Uq_nU^\dagger$, $\pi_n'=U\pi_nU^\dagger$ for $n=0,\ldots,N$, as can easily be verified.

III. TWO-DIMENSIONAL CASE

Let us now consider two-dimensional reference frames. With $N+1$ particles, as before, let $\vec{r}_n=(x_n,y_n)$, $\vec{p}_n=(p_{xn},p_{yn})$, $L_{zn}=\vec{r}_n\times\vec{p}_n$ $(n=0,\ldots,N)$ denote, respectively, the positions, momenta, and angular momenta relative to an "absolute" coordinate system. The usual commutation relations are assumed, the Hamiltonian is

$$H=\sum_0^N\frac{p_n^2}{2m_n} , \tag{25}$$

and again we assume the system to be in an eigenstate of $\vec{P}_T=0$. We could choose $L_{zT}=0$ as well (L_{zT} being the total angular momentum in the z direction), but since L_{zT} may be found by a measurement performed inside the system, it is important to keep it unspecified, and so consider the general case.

We now replace the variables of particles 0 and 1 by the corresponding "center-of-mass" and "relative" variables.[3] To this end, let us define

$$M=m_0+m_1 , \tag{26a}$$

$$\mu=\frac{m_0m_1}{m_0+m_1} , \tag{26b}$$

$$\vec{R}=\frac{m_0\vec{r}_0+m_1\vec{r}_1}{m_0+m_1} , \tag{26c}$$

$$\vec{r}=\vec{r}_1-\vec{r}_0 , \tag{26d}$$

$$\vec{P}=\vec{p}_0+\vec{p}_1 , \tag{26e}$$

$$\vec{p}=\frac{m_0m_1}{m_0+m_1}\left[\frac{\vec{p}_1}{m_1}-\frac{\vec{p}_0}{m_0}\right] , \tag{26f}$$

$$L_z^{\text{c.m.}}=\vec{R}\times\vec{P} , \tag{26g}$$

$$l_z=\vec{r}\times\vec{p} . \tag{26h}$$

The Cartesian components of $\vec{R}$ and $\vec{r}$ are

$$\vec{R}=(X,Y)\ , \tag{27a}$$

$$\vec{r}=(x,y)\ , \tag{27b}$$

and their plane-polar coordinates

$$\vec{R}=(R,\Theta)\ , \tag{28a}$$

$$\vec{r}=(r,\theta)\ . \tag{28b}$$

As is well known, the above define two independent canonical degrees of freedom. With this modification in our set of variables, the Hamiltonian becomes

$$H=\frac{P^2}{2M}+\frac{p^2}{2\mu}+\sum_2^N\frac{p_n{}^2}{2m_n}\ . \tag{29}$$

We now introduce a two-dimensional quantum reference frame. It will be characterized by the following coordinates and momenta:

$$Q_x=X\cos\theta+Y\sin\theta\ , \tag{30a}$$

$$Q_y=-X\sin\theta+Y\cos\theta\ , \tag{30b}$$

$$q_x=x\ , \tag{30c}$$

$$q_y=y\ , \tag{30d}$$

$$q_{nx}=(x_n-X)\cos\theta+(y_n-Y)\sin\theta\ , \tag{30e}$$

$$q_{ny}=-(x_n-X)\sin\theta+(y_n-Y)\cos\theta\ , \tag{30f}$$

where $n=2,\ldots,N$ and

$$\Pi_x=P_{Tx}\cos\theta+P_{Ty}\sin\theta\ , \tag{31a}$$

$$\Pi_y=-P_{Tx}\sin\theta+P_{Ty}\cos\theta\ , \tag{31b}$$

$$\pi_x=p_x-\frac{\sin\theta}{r}L_z\ , \tag{31c}$$

$$\pi_y=p_y+\frac{\cos\theta}{r}L_z\ , \tag{31d}$$

where $L_z=\sum_{n\neq\text{rel}}L_{zn}$. (This summation includes the center-of-mass degree of freedom.)

Continuing,

$$\pi_{nx}=p_{nx}\cos\theta+p_{ny}\sin\theta\ , \tag{31e}$$

$$\pi_{ny}=-p_{nx}\sin\theta+p_{ny}\cos\theta \tag{31f}$$

$$(n=2,\ldots,N)\ .$$

The point to notice about these definitions is that in the relative frame, positions are measured in terms of distances from the "center-of-mass particle," which, in this sense, defines the origin, while directions in the plane are measured relative to the "relative particle," which, accordingly, defines the x axis.

The transformation from the absolute to the quantum frame is generated by (see Appendix C 1)

$$U=\exp\left[-i\sum_2^N\vec{p}_n\cdot\vec{R}\right]\exp\left[-i\sum_{n\neq\text{rel}}L_{zn}\theta\right]\ . \tag{32}$$

With the help of U, the Hamiltonian in the quantum frame may be calculated from the Hamiltonian in the absolute frame (29). Using the same definitions for $H,\bar{H},H'$ as in the one-dimensional case, we obtain (see Appendix C 2)

$$\begin{aligned}\bar{H}(\vec{p}_n)&=U^\dagger H(\vec{p}_n)U\\&=\sum_2^N\frac{p_n{}^2}{2m_n}+\frac{K^2}{2M}+\frac{p_r{}^2}{2\mu}+\frac{(p_\theta-L_z)^2}{2\mu r^2}\\&=\sum_2^N\frac{p_n{}^2}{2m_n}+\frac{K^2}{2M}+\frac{1}{2\mu}\left[p_x+\frac{yL_z}{x^2+y^2}\right]^2\\&\quad+\frac{1}{2\mu}\left[p_y-\frac{xL_z}{x^2+y^2}\right]^2\ ,\end{aligned} \tag{33}$$

where

$$\vec{K}=\sum_2^N\vec{p}_n\ , \tag{34a}$$

$$L_z=\sum_{n\neq\text{rel}}L_{zn}\ , \tag{34b}$$

and

$$p_\theta=\frac{1}{i}\frac{\partial}{\partial\theta}\ ,$$

also denoted sometimes l_z [as in (26h)]. It is interesting to find that the vector potential appearing in the kinetic energy of the reference particle corresponds to a singular line of quantized flux. Because of the quantization of angular momentum, this flux introduces no observable effect. In the next section, the origin of this singularity will be clarified in a simple way.

To understand the meaning of the vector potential, let us consider a case where in the original Hamiltonian[4] there is included an interaction that depends on θ_1, i.e.,

$$H=\frac{p_1{}^2}{2m_1}+\sum_{n\neq1}\frac{p_n{}^2}{2m_n}+\sum_{n\neq1}V_n(\theta_n-\theta_1)\ .$$

Then

$$\begin{aligned}\dot{\vec{p}}_1&=\sum_{n\neq1}-\frac{\partial V_n}{\partial\theta_1}\frac{\partial\theta_1}{\partial\vec{x}_1}\\&=\sum_{n\neq1}\frac{\partial V_n}{\partial\theta_n}\frac{\partial\theta_1}{\partial\vec{x}_1}\\&=\left[\sum_{n\neq1}\dot{L}_{zn}\right]\frac{\partial\theta_1}{\partial\vec{x}_1}=\dot{L}_z\frac{\partial\theta_1}{\partial\vec{x}_1}\ .\end{aligned}$$

The transformed Hamiltonian is

$$\bar{H}=\frac{(\vec{p}_1+\vec{A})^2}{2m_1}+\sum_{n\neq1}\frac{\vec{p}_n{}^2}{2m_n}+\sum_{n\neq1}V_n(\theta_n')\ ,$$

where

$$\vec{A}=-L_z\frac{\partial\theta_1}{\partial\vec{x}_1}\ .$$

Thus

$$\vec{F}_1=-\frac{\partial\vec{A}}{\partial t}=\dot{L}_z\frac{\partial\theta_1}{\partial\vec{x}_1}\ ,$$

which agrees with the original force. Since in our transformation

$$\dot{\vec{x}}'_1=\dot{\vec{x}}_1\ ,$$

therefore

$$\ddot{\vec{x}}'_1=\ddot{\vec{x}}_1$$

and

$$\vec{F}'_1=\vec{F}_1\ .$$

Finally, we want to show how to transform between different two-dimensional quantum reference frames. Let us first give the space of "absolute" variables over which the relative frames are to be defined. These are

$$\vec{R}^{\,\rm c.m.}_{10}=\frac{m_0\vec{r}_0+m_1\vec{r}_1}{m_0+m_1}\ , \tag{35a}$$

$$\vec{r}^{\,\rm rel}_{10}=\vec{r}_1-\vec{r}_0\ , \tag{35b}$$

$$\vec{R}^{\,\rm c.m.}_{23}=\frac{m_2\vec{r}_2+m_3\vec{r}_3}{m_2+m_3}\ , \tag{35c}$$

$$\vec{r}^{\,\rm rel}_{23}=\vec{r}_3-\vec{r}_2\ , \tag{35d}$$

$$\vec{r}_n,\quad n=4,\ldots,N \tag{35e}$$

and

$$\vec{P}^{\,\rm c.m.}_{10}=\vec{p}_0+\vec{p}_1\ , \tag{35f}$$

$$\vec{p}^{\,\rm rel}_{10}=\frac{m_0m_1}{m_0+m_1}\left[\frac{\vec{p}_1}{m_1}-\frac{\vec{p}_0}{m_0}\right]\ , \tag{35g}$$

$$\vec{P}^{\,\rm c.m.}_{23}=\vec{p}_2+\vec{p}_3\ , \tag{35h}$$

$$\vec{p}^{\,\rm rel}_{23}=\frac{m_2m_3}{m_2+m_3}\left[\frac{\vec{p}_3}{m_3}-\frac{\vec{p}_2}{m_2}\right]\ , \tag{35i}$$

$$\vec{p}_n,\quad n=4,\ldots,N\ . \tag{35j}$$

With this choice of degrees of freedom, let us define the following two relative frames: The unprimed frame, where indices c.m. and 10 define the origin, and rel and 10 define the x axis. We here have

$$q_{nx}=(x_n-X_{10})\cos\theta_{10}+(y_n-Y_{10})\sin\theta_{10}\ ,$$

$$q_{ny}=-(x_n-X_{10})\sin\theta_{10}+(y_n-Y_{10})\cos\theta_{10}$$

for $n=4,\ldots,N$, where

$$X_{10}=\frac{m_1x_1+m_0x_0}{m_1+m_0},\quad Y_{10}=\frac{m_1y_1+m_0y_0}{m_1+m_0}$$

and

$$\theta_{10}=\arctan\frac{y_1-y_0}{x_1-x_0}\ .$$

The double-primed frame, where indices c.m. and 23 define the origin, and rel and 23 the x axis. Or, for instance,

$$q''_{nx}=(x_n-X_{23})\cos\theta_{23}+(y_n-Y_{23})\sin\theta_{23}\ ,$$

$$q''_{ny}=-(x_n-X_{23})\sin\theta_{23}+(y_n-Y_{23})\cos\theta_{23}\ ,$$

for $n=4,\ldots,N$,

$$X_{23}=\frac{m_2x_2+m_3x_3}{m_2+m_3},\quad \theta_{23}=\arctan\frac{y_3-y_2}{x_3-x_2}\ .$$

Let us introduce some further notations. In the relative frames, we denote the coordinates and momenta of the reference particles by the subscript a. Thus $\vec{Q}_a$, $\vec{\Pi}_a$, and $\vec{q}_a,\vec{\pi}_a$ for the degrees of freedom defining the origin and the direction of the x axis, respectively. With the above choice these refer to indices c.m. and 12 and rel and 12, respectively. In the double-primed frame, we accordingly have $\vec{Q}''_a$, $\vec{\Pi}''_a$, $\vec{q}''_a$, and $\vec{\pi}''_a$. Comparing with (30) these are

$$Q_{ax}=X_{10}\cos\theta_{10}+Y_{10}\sin\theta_{10}\ ,$$

$$Q_{ay}=-X_{10}\sin\theta_{10}+Y_{10}\cos\theta_{10}\ ,$$

$$q_{ax}=x_{10}=x_1-x_0\ ,$$

$$q_{ay}=y_{10}=y_1-y_0$$

and

$$Q''_{ax}=X_{23}\cos\theta_{23}+Y_{23}\sin\theta_{23}\ ,$$

$$Q''_{ay}=-X_{23}\sin\theta_{23}+X_{23}\cos\theta_{23}\ ,$$

$$q''_{ax}=x_3-x_2=x_{23}\ ,$$

$$q''_{ay}=y_3-y_2=y_{23}\ .$$

Also, we shall use the subscript b for the second pair of c.m. and rel degrees of freedom [see Eq. (35)]. Thus,

$$Q_{bx}=(X_{23}-X_{10})\cos\theta_{10}+(Y_{23}-Y_{10})\sin\theta_{10}\ ,$$

$$Q_{by}=-(X_{23}-X_{10})\sin\theta_{10}+(Y_{23}-Y_{10})\cos\theta_{10}\ ,$$

$$q_{bx}=x_{23}\cos\theta_{10}+y_{23}\sin\theta_{10}\ ,$$

$$q_{by}=-x_{23}\sin\theta_{10}+y_{23}\cos\theta_{10}\ ,$$

and

$$Q''_{bx}=(X_{10}-X_{23})\cos\theta_{23}+(Y_{10}-Y_{23})\sin\theta_{23}\ ,$$

$$Q''_{by}=-(X_{10}-X_{23})\sin\theta_{23}+(Y_{10}-Y_{23})\cos\theta_{23}\ ,$$

$$q''_{bx}=x_{10}\cos\theta_{23}+y_{10}\sin\theta_{23}\ ,$$

$$q''_{by}=-x_{10}\sin\theta_{23}+y_{10}\cos\theta_{23}\ .$$

The transformation from the unprimed to the double-primed frame is

$$U=U_1U_2\ , \tag{36a}$$

where

$$U_1=P_{{\rm c.m.},b}\exp\left[i\sum_4^N\vec{\pi}_n\cdot\vec{Q}_b\right]\exp\left[-i\vec{\Pi}_a\cdot\vec{Q}_b\right]\ , \tag{36b}$$

$$U_2=P_{{\rm rel},b}\exp\left[i\sum_{n\neq{\rm rel}}\pi_{\theta_n}\theta_b\right]\exp\left[-i\pi^{\rm rel}_{\theta_a}\theta_b\right]\ , \tag{36c}$$

$P_{c.m.,b}$,$P_{rel,b}$ are the parity operators associated with the degrees of freedom c.m.,b and rel,b, respectively. In Appendix C 3, this transformation will be discussed further. In particular, we show there how (36) has been reached, thus also elucidating its precise meaning.

IV. THREE-DIMENSIONAL CASE—PARTIAL SOLUTION

The extension to three-dimensional reference frames is somewhat more difficult. In the present section, we shall give only a partial solution to the problem. Even so, this solution brings out some of the more characteristic features of the three-dimensional case. The general case will be treated in the next section.

In principle, a three-dimensional reference frame needs three particles to define, say, the z axis and the x-z plane, in relation to which all directions in space may be determined without ambiguity. Hence, in the spirit of the foregoing work one has to choose three particles, perform a displacement followed by three rotations, leaving 0 at the origin, 1 on the z axis, and 2 on the x-z plane. But, rather than repeating what has already been done before, we shall treat only the rotational part of the problem. As mentioned earlier, we shall simplify our discussion in this section by assuming that the z axis lies in the plane defined by particles 0, 1, and 2. In this case, the rotation of (37) below connects the external and the internal frames.

We have to rotate particle 1 to the z axis. This involves two rotations, the first by the angle φ_1 about the z axis, the second by the angle θ_1' about the new y' axis. Thus,

$$U=e^{-iL_y'\theta_1'}e^{-iL_z\varphi_1}$$
$$=e^{-iL_z\varphi_1}e^{-iL_y\theta_1}, \quad (37)$$

where

$$L_y=\sum_n{}' L_{yn},\quad L_z=\sum_n{}' L_{zn},$$

and the primed summation excludes $n=1$. With this, the Hamiltonian will be (we calculate only $\overline{H}$)

$$\overline{H}=U^\dagger HU=U^\dagger\sum_0^N \frac{p_n{}^2}{2m_n}U$$
$$=\sum_0^N{}' \frac{p_n{}^2}{2m_n}+U^\dagger\frac{p_1{}^2}{2m_1}U$$

wherein we focus on $U^\dagger\vec{p}_1U$. One finds that

$$e^{iL_y\theta_1}e^{iL_z\phi_1}\vec{p}_1e^{-iL_z\phi_1}e^{-iL_y\theta_1}=\vec{p}_1+\vec{A},$$

where (see Appendix D 1)

$$A_x=\frac{1}{r}\left[-x\left[\frac{zL_y}{r(x^2+y^2)^{1/2}}\right]+y\left[\frac{-L_x}{(x^2+y^2)^{1/2}}+\frac{zL_z}{x^2+y^2}\right]\right], \quad (38a)$$

$$A_y=\frac{-1}{r}\left[y\left[\frac{zL_y}{r(x^2+y^2)^{1/2}}\right]+x\left[\frac{-L_x}{(x^2+y^2)^{1/2}}+\frac{zL_z}{x^2+y^2}\right]\right], \quad (38b)$$

$$A_z=\frac{(x^2+y^2)^{1/2}}{r^2}L_y, \quad (38c)$$

and the Hamiltonian is

$$\overline{H}=\sum_{n\neq1}\frac{p_n{}^2}{2m_n}+\frac{(p_{1x}+A_x)^2}{2m_1}+\frac{(p_{1y}+A_y)^2}{2m_1}+\frac{(p_{1z}+A_z)^2}{2m_1}.$$

We observe that the Coriolis-type fields (see Appendix D 2) vanish, i.e.,

$$B_k=F_{ij}\equiv\frac{\partial A_i}{\partial x_j}-\frac{\partial A_i}{\partial x_i}+i[A_i,A_j]=0, \quad (39)$$

for, starting with a free Hamiltonian, $U^\dagger$ could not possibly produce any fields within the reference frame. However, if one calculates the Abelian parts of the fields, i.e.,

$$\frac{\partial A_i}{\partial x_j}-\frac{\partial A_j}{\partial x_i},$$

one finds that the contribution which is proportional to L_z behaves like a Coulomb field, analogous to the field of a magnetic monopole. We denote this Abelian part by

$$\overline{B}_k=\frac{\partial A_i}{\partial x_j}-\frac{\partial A_j}{\partial x_i}.$$

Then, using (38) and (39), one immediately finds (see Appendix D 3)

$$\overline{B}_x(L_z)=\frac{x}{r^3}L_z,$$
$$\overline{B}_y(L_z)=\frac{y}{r^3}L_z, \quad (40)$$
$$\overline{B}_z(L_z)=\frac{z}{r^3}L_z.$$

We next calculate the line integral of $\vec{A}$ around a small circle, surrounding the z axis, and whose plane is parallel to the x-y plane. Let $r_{xy}=(x^2+y^2)^{1/2}=$const be the radius of the circle of the integration. We obtain (see Apendix D 4)

$$\lim_{(r_{xy}\to0)}\oint\vec{A}\cdot d\vec{l}=\begin{cases}-2\pi L_z, & z>0\\ 2\pi L_z, & z<0.\end{cases} \quad (41)$$

From (40) and (41) we see that the vector potential has a singularity along the z axis. It corresponds to two monopoles[5] of strength $2\pi L_z$ each, whose strings lie along the positive, negative, z axis, respectively.

It is possible to see qualitatively how singular lines of flux may appear in the transformed Hamiltonian from the following considerations:[6] The transformed state is

$$\Psi'=e^{-iL_z\phi_1}e^{-iL_y\theta_1}\Psi(\vec{r}_1,\vec{r}_2,\ldots,\vec{r}_N). \quad (42)$$

Let Ψ be an eigenstate of L_z so that

$$L_z\Psi=m\Psi\ . \tag{43}$$

Suppose that particle 1 describes a small circle around the positive z axis, as in (41). Then this induces the following transformation on Ψ. We have $\theta_1\cong 0$ and $e^{-iL_y\theta_1}\cong 1$ so that

$$\Psi'\cong e^{-iL_z\phi_1}\Psi=e^{-im\phi_1}\Psi\ . \tag{44}$$

Thus the phase changes by $-2\pi m$ along this circle. This corresponds to a flux line of $-2\pi m$, along the positive z axis. Consider now a small circle around the negative z axis. In this case $e^{-iL_y\theta_1}\cong e^{-iL_y\pi}$. Therefore,

$$L_z e^{-iL_y\theta_1}\Psi=-m(e^{-iL_y\theta_1}\Psi)\ . \tag{45}$$

Thus the phase along this circle changes by $+2\pi m$, which corresponds to a flux line of $2\pi m$ along the negative z axis. We thus find the singular line in this case to correspond to a "magnetic" monopole of strength $2\pi 2m$ at the origin plus a singular flux line of $-m2\pi$ along the z axis.

V. MOST GENERAL THREE-DIMENSIONAL QUANTUM FRAMES

In the present section, the most general type of a three-dimensional quantum frame will be described. We set out from the initial variables $\vec{r}_0,\vec{r}_1,\ldots,\vec{r}_N$ and $\vec{p}_0,\vec{p}_1,\ldots,\vec{p}_N$ and the Hamiltonian

$$H=\sum_0^N\frac{{p_n}^2}{2m_n}\ .$$

As in the previous section, we assume about this set of variables that the origin 0 stands for the center-of-mass coordinate that all other variables are measured relative to it.

With this choice of variables, it follows that the passage to the most general quantum frame involves three rotations by the pertinent Eular angles,

$$U=e^{-iL_z\phi_1}e^{-iL_y\theta_1}e^{-iL_z\psi_{1,2}}\ .$$

The three Euler angles are chosen so that the rotated z axis lies in the direction of particle 1, and the rotated x-z plane coincides with the plane defined by particles 1 and 2, and the origin. Thus, ϕ_1,θ_1 are, respectively, the azimuthal and polar angles of particle 1, and[7]

$$\tan\psi_{1,2}=\frac{r_1(x_1y_2-x_2y_1)}{z_1\vec{r}_1\cdot\vec{r}_2-{r_1}^2z_2}\ ,$$

$$L_y=\sum_{n=3}^N L_{yn},\quad L_z=\sum_{n=3}^N L_{zn}\ .$$

According to the by-now-familiar procedure, we calculate the Hamiltonian in the quantum frame by acting on H with $U^\dagger$,

$$\bar{H}=U^\dagger HU=\sum_{n=3}^N\frac{{p_n}^2}{2m_n}+U^\dagger\frac{{p_1}^2}{2m_1}U+U^\dagger\frac{{p_2}^2}{2m_2}U\ .$$

Denoting

$$U^\dagger\vec{p}_1U=\vec{p}_1+\vec{A}_1$$

and

$$U^\dagger\vec{p}_2U=p_2+\vec{A}_2\ ,$$

we find for the potentials

$$\begin{aligned}A_{1x}=&-\left[-\cos\psi\sin\theta_1\frac{\partial\phi_1}{\partial x_1}+\sin\psi\frac{\partial\theta_1}{\partial x_1}\right]L_x\\&-\left[\sin\psi\sin\theta_1\frac{\partial\phi_1}{\partial x_1}+\cos\psi\frac{\partial\theta_1}{\partial x_1}\right]L_y\\&-\left[\cos\theta_1\frac{\partial\phi_1}{\partial x_1}+\psi\frac{\partial\psi}{\partial x_1}\right]L_z\ ,\\A_{1y}=&-\left[-\frac{\partial\phi_1}{\partial y_1}\sin\theta_1\cos\psi+\frac{\partial\theta_1}{\partial y_1}\sin\psi\right]L_x\\&-\left[\frac{\partial\phi_1}{\partial y_1}\sin\theta_1\sin\psi+\frac{\partial\theta_1}{\partial y_1}\cos\psi\right]L_y-\frac{\partial\phi_1}{\partial y_1}\cos\theta_1L_z\ ,\end{aligned}$$

$$A_{1z}=-\frac{\partial\theta_1}{\partial z_1}\sin\psi L_x+\frac{\partial\theta_1}{\partial z_1}\cos\psi L_y-\frac{\partial\psi}{\partial z_1}L_z$$

and

$$A_{2x}=-\frac{\partial\psi}{\partial x_2}L_z\ ,$$

$$A_{2y}=-\frac{\partial\psi}{\partial y_2}L_z\ ,$$

$$A_{2z}=-\frac{\partial\psi}{\partial z_2}L_z\ .$$

From considerations similar to those used in the previous section [viz. Eq. (42)–(45) and the discussion there], we find that in the present case the vector potential of particle 2 corresponds to a singular line of flux of strength $-L_z2\pi$, along the line connecting particle 1 with the origin (since a small circle around this line changes $\psi_{1,2}$ by 2π). The vector potential of particle 1 corresponds to the same singular flux as described in the previous section, plus another line of strength $-L_z2\pi$ along the line connecting particle 2 with the origin.

VI. TEMPORAL ASPECT OF THE PARADOX

To complete the picture of a quantum reference frame within Galilean relativity, we now examine the temporal aspect of the problem. We shall introduce an internal time axis into the reference frame by including a clock within the system. Consider the Hamiltonian (see Appendix E).

$$H_0^{\text{ext}}=\pi_1+\pi_2+H^{\text{remainder}}\ ,$$

where H_0^{ext} stands for the Hamiltonian in the absolute frame, "particles" 1 and 2 are clocks (see Appendix E), and $H^{\text{remainder}}$ represents the Hamiltonian of all other particles, excluding 1 and 2.

For the sake of simplicity, let us at this point consider only the two clocks by themselves. We then have

$$H_0^{\text{ext}}=\pi_1+\pi_2\ .$$

We further assume that the system is isolated, so that its energy does not change with time, and, in particular, that it is in a stationary state, and that the total energy equals zero. Thus,

$$H_0^{\text{ext}}=0 \; .$$

As mentioned earlier, we regard the two clocks as defining the internal times of two laboratories, or reference frames. The Schrödinger equation in reference frame (1), where clock (1) is used to measure the time, is

$$i\frac{\partial}{\partial t_1}\psi(q_2,t_1)=\pi_2\psi(q_2,t_1) \; .$$

A particular solution of this equation is

$$\Psi(q_2,t_1)=f(q_2-t_1)+f(q_2-t_1-T) \tag{46}$$

where f differs from zero in a region smaller than T. In reference frame (2), the Schrödinger equation is

$$i\frac{\partial}{\partial t_2}\psi(q_1,t_2)=\pi_1\psi(q_1,t_2)$$

and the corresponding solution is given by

$$\Psi(q_1,t_2)=f(q_1-t_2)+f(q_1-t_2+T) \; . \tag{47}$$

With the choice of initial conditions as in (46), the two reference frames (1) and (2) stand in time uncertainty relative to one another. For at a given time, $t_1=T_0$, say, in reference frame (1),

$$\Psi(t_1=T_0,q_2)=f(q_2-T_0)+f(q_2-T_0-T) \; ,$$

which describes clock (2) to be in a superposition of the times (Fig. 1) $t_2\equiv q_2=T_0$ and $t_2\equiv q_2=T_0+T$, while from the point of view of reference frame (2), it is reference frame (1) whose time is uncertain (Fig. 2),

$$\Psi(t_2=T_0,q_1)=f(q_1-T_0)+f(q_1-T_0+T) \; .$$

Let us now consider the Hamiltonian

$$H^{\text{ext}}=\pi_1+\pi_2+V(q_1) \; ,$$

where

$$V(q_1)=\begin{cases}V_0, & q_1\geq 0\\ 0 & \text{otherwise} \; .\end{cases}$$

The initial conditions will be given in accordance with (46) [or (47)]. In reference frame (2), the Schrödinger equation will be

$$i\frac{\partial}{\partial t_2}\psi=\frac{1}{i}\frac{\partial}{\partial y_1}\psi+V(q_1)\psi \; . \tag{48}$$

With the initial conditions

$$\Psi(t_2=0,q_1)=f(q_1)+f(q_1+T) \; ,$$

the solution at time t_2 will be

$$\Psi(t_2,q_1)=[f(q_1-t_2)+f(q_1-t_2+T)] \times\exp\left[-i\int_{q_0}^{q_1}V(q_1')dq_1'\right] \; , \tag{49}$$

as may easily be confirmed by substituting (49) back into (48) and solving. For $t_2\geq T$ we obtain

$$\Psi(t_2\geq T,q_1)=f(q_1-t_2)+e^{-iV_0T}f(q_1-t_2+T) \; . \tag{50}$$

We notice that the relative phase in the superposition has changed by the amount $-V_0T$. In reference frame (1), the Schrödinger equation will be written

$$i\frac{\partial}{\partial t_1}\psi=\frac{1}{i}\frac{\partial}{\partial q_2}\psi+V(t_1)\psi \; ,$$

and the initial conditions are

$$\Psi(t_1=0,q_2)=f(q_2)+f(q_2-T) \; .$$

The corresponding solution is

$$\Psi(t_1,q_2)=[f(q_2-t_1)+f(q_2-t_1-T)] \times\exp\left[-i\int_{t_0}^{t_1}V(t_1')dt_1'\right] \; ,$$

where $t_0\geq 0$. This may be rewritten as

$$\Psi(t_1,q_2)=[f(q_2-t_1)+f(q_2-t_1-T)] \times\exp[-iV_0\theta(t_1)t_1] \; ,$$

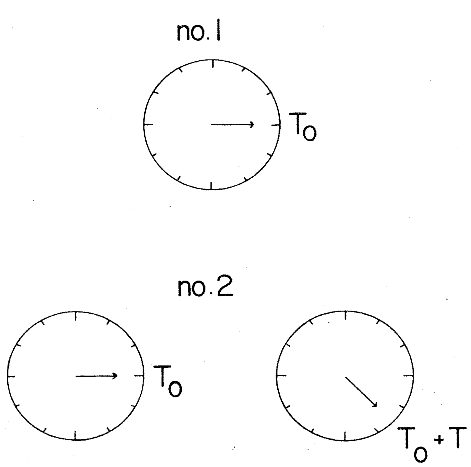

FIG. 1. Clock states as viewed in frame 1.

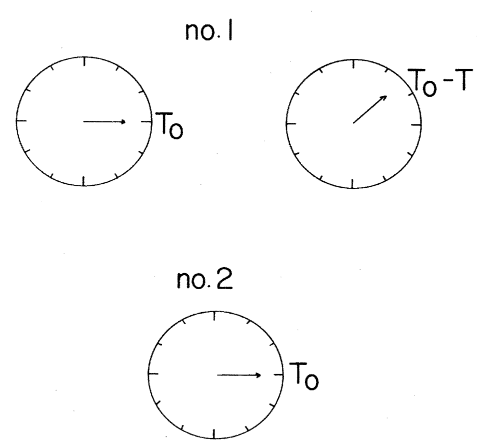

FIG. 2. Clock states as viewed in frame 2.

where $\theta(t_1)$ is the step function. We find that while in reference frame (1) a change in a physical quantity has occurred, in reference frame (2) no observable quantity has apparently changed. Thus, the descriptions in the two reference frames seem to yield conflicting results.

Let us next consider a new Hamiltonian,

$$H_2=\pi_1+V(q_1)-g(t_2)\frac{1+\sigma_z}{2}\pi_1 , \tag{51}$$

where

$$g(t_2)=\begin{cases}\frac{1}{2}, & T\le t_2\le 3T\\ 0 & \text{otherwise}\end{cases}$$

with the system prepared in the state

$$\Psi_s=[f(q_1-t_2)+e^{i\alpha}f(q_1-t_2-T)]\begin{bmatrix}1\\1\end{bmatrix} . \tag{52}$$

We recognize the expression appearing in the left parenthesis on the right-hand side of (52) to be the wave function of (50) above, with $-V_0T\equiv\alpha$. Using von Neumann's measurement theory,[8] the above describes an interaction that measures the phase, with the spin here representing the measuring device. Our aim is to follow up such a measurement, which, according to our previous results, is in principle possible in reference frame (2), but appears to be meaningless in reference frame (1).

In reference frame (2), then, we have

$$\dot{q}_1\equiv\frac{dq_1}{dt_2}=\frac{\partial H_2}{\partial\pi_1}=1-g(t_2)\frac{1+\sigma_z}{2}$$

$$=\begin{cases}\frac{1}{2}, & \sigma_z=+1\\ 1, & \sigma_z=-1\end{cases}\quad (T\le t_2\le 3T) .$$

It then follows that

$$\Psi_s(T\le t_2\le 3T)=[f(q_1-\tfrac{1}{2}(T_2-T)-T)$$
$$+e^{i\alpha}f(q_1-\tfrac{1}{2}(t_2-T))]\begin{bmatrix}1\\0\end{bmatrix}$$
$$+[f(q_1-t_2)+e^{i\alpha}f(q_1-t_2+T)]\begin{bmatrix}0\\1\end{bmatrix}$$

and

$$\Psi_s(t_2=3T)=[f(q_1-2T)+e^{i\alpha}f(q_1-T)]\begin{bmatrix}1\\0\end{bmatrix}$$
$$+[f(q_1-3T)+e^{i\alpha}f(q_1-2T)]\begin{bmatrix}0\\1\end{bmatrix}$$
$$=f(q_1-2T)\begin{bmatrix}1\\e^{i\alpha}\end{bmatrix}+\text{other terms} .$$

Thus, when the measurement is completed, the state of the measuring device

$$\begin{bmatrix}1\\e^{i\alpha}\end{bmatrix}$$

depends on α and distinguishes between the two orthogonal cases $\alpha=0, \pi$. Let us now examine this measurement, as seen from reference frame (1). We have

$$H_1=\pi_2+V(t_1)+g(q_2)\frac{1+\sigma_z}{2}H_1$$

or

$$i\left[1-g(q_2)\frac{1+\sigma_z}{2}\right]\frac{\partial}{\partial t_1}\Psi=i\frac{\partial}{\partial t_1}\left[1-g(q_2)\frac{1+\sigma_z}{2}\right]\Psi$$
$$=[\pi_2+V(t_1)]\Psi . \tag{53}$$

Defining

$$\Psi'=\left[1-g(q_2)\frac{1+\sigma_z}{2}\right]^{1/2}\Psi \tag{54}$$

and

$$H_1'=\frac{1}{\left[1-g(q_2)\frac{1+\sigma_z}{2}\right]^{1/2}}[\pi_2+V(t_1)]\times\frac{1}{\left[1-g(q_2)\frac{1+\sigma_z}{2}\right]^{1/2}} ,$$

(53) above can then be rewritten as

$$i\frac{\partial}{\partial t_1}\Psi'=H_1'\Psi' ,$$

which is the Schrödinger equation in reference frame (1). This equation may easily be solved for Ψ' as follows. Inspecting (53) above, one can solve for Ψ, noticing that the interaction involved in the measurement effectively introduces a magnetic field in the z direction that is proportional to $V(t_1)$. This interaction will rotate the spin state

$$\begin{bmatrix}1\\1\end{bmatrix} .$$

It follows that here, too, the phase may be found by reading the state of the measuring device. Finally, Ψ' may be calculated from Ψ according to (54). We thus see that the phase introduced by $V(t_1)$ can also be measured in reference frame (1).

CONCLUSIONS

This paper addresses itself to the problem of finite-mass quantum reference frames. We show that with the help of suitable geometrical potentials it is possible to construct a theory that takes account of the quantum disturbances of the reference frame in a consistent manner. Our formalism satisfies an equivalence principle that presents an extension of the classical equivalence principle to quantum theory. It also augments the usual continuous Galilean symmetry with additional discrete symmetries corresponding to the transformation from one physical relative frame to another. So far, nonrelativistic theory has been treated. We intent to extend the treatment to relativistic reference frames, in the hope that this approach will contribute towards the solution of problems associated with the quantization of general relativity.

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APPENDIX A

Substituting (7) into (1b)

$$\left[x'_{\text{c.m.}_1}, \frac{p_1}{m_1} + A\right] = 0 , \quad \text{(A1)}$$

using (3), (4), and (5),

$$\left[\frac{-M_1 x_{O_2} + m_1(x_1 - x_{O_2})}{M_1 + m_1}, \frac{p_1}{m_1} + A\right] = 0 , \quad \text{(A2)}$$

whence it follows that

$$A = \alpha p_1 + \beta p_{O_2} + f(x_1, x_{O_2}) + C . \quad \text{(A3)}$$

The term $f(x_1, x_{O_2})$ is related to the choice of gauge and may therefore be chosen as zero,

$$A = \alpha p_1 + \beta p_{O_2} + C . \quad \text{(A4)}$$

By inspection of (A2) above it follows that

$$\alpha = \frac{f_1(m_1, M_1)}{M_1}, \quad \beta = \frac{f_2(m_1, M_1)}{M_1}$$

or

$$v_1 = \frac{p_1}{m_1} + \frac{f_1(m_1, M_1) p_1 + f_2(m_1, M_1) p_{O_2}}{M_1} \quad \text{(A5a)}$$

and also

$$v'_1 = \frac{p'_1}{m_1} + \frac{f_1(m_1, M_2) p'_1 + f_2(m_1, M_2) p'_{O_1}}{M_2} , \quad \text{(A5b)}$$

$$v'_{O_1} = \frac{p'_{O_1}}{M_1} + \frac{f_1(M_1, M_2) p'_{O_2} + f_2(M_1, M_2) p'_1}{M_2} . \quad \text{(A5c)}$$

As explained in Sec. I, it is necessary that

$$[x'_{\text{c.m.}_1}, v_1] = 0 , \quad \text{(A6a)}$$

$$[x_{\text{c.m.}_2}, v'_1] = 0 , \quad \text{(A6b)}$$

$$[x''_{\text{c.m.}O_1,O_2}, v_1] = 0 . \quad \text{(A6c)}$$

Substituting (A5a) into (A6a), we obtain

$$0 = [x'_{\text{c.m.}_1}, v_1] = \left[\frac{-M_1 x_{O_2} + m_1(x_1 - x_{O_2})}{M_1 + m_1}, \frac{p_1}{m_1} + \frac{f_1(m_1, M_1) p_1 + f_2(M_1, M_1) p_{O_2}}{M_1}\right]$$

$$= \frac{1}{M_1 + m_1}\left[m_1 x_1 - (M_1 + m_1) x_{O_2}, \left[\frac{1}{m_1} + \frac{f_1(m_1, M_1)}{M_1}\right] p_1 + \frac{f_2(m_1, M_1)}{M_1} p_{O_2}\right] ,$$

whence, finally,

$$1 + \frac{f_1(m_1, M_1)}{M_1} m_1 - \frac{f_2(m_1, M_1)}{M_1}(m_1 + M_1) = 0 . \quad \text{(A7)}$$

Substituting (A5b) into (A6b) and using

$$x_{\text{c.m.}_2} = \frac{m_1 x_1 + M_2 x_{O_2}}{m_1 + M_2} ,$$

$$x_{O_2} = -x'_{O_1} ,$$

$$x_1 = x'_1 + x_{O_2} = x'_1 - x'_{O_1} ,$$

$$x_{\text{c.m.}_2} = \frac{m_1 x'_1 - (m_1 + M_2) x'_{O_1}}{m_1 + M_1} ,$$

we obtain, after a calculation similar to that leading to (A7),

$$1 + \frac{f_1(m_1, M_2)}{M_2} m_1 - \frac{f_2(m_1, M_2)}{M_2}(m_1 + M_2) = 0 . \quad \text{(A8)}$$

Substituting (A5c) into (A6c) and using

$$x''_{\text{c.m.}O_1O_2} = \frac{M_1 x''_{O_1} + M_2 x''_{O_2}}{M_1 + M_2} ,$$

$$x''_{O_1} = x'_{O_1} - x'_1 ,$$

$$x''_{O_2} = -x'_1 ,$$

$$x''_{\text{c.m.}O_1O_2} = \frac{M_1(x'_{O_1} - x'_1) + M_2(-x'_1)}{M_1 + M_2}$$

$$= \frac{-(M_1 + M_2) x'_1 + M_1 x'_{O_1}}{M_1 + M_2} ,$$

one obtains

$$1 + \frac{f_1(M_1, M_2)}{M_2} M_1 + \frac{f_2(M_1, M_2)}{M_2}(M_1 + M_2) = 0 . \quad \text{(A9)}$$

Substituting (A5b) and (A5c) into (2b) of Sec. I and using the definition

$$v'_{\text{c.m.}_1} = \frac{m_1 v'_1 + M_1 v'_{O_1}}{m_1 + M_1} ,$$

we obtain

$$0=[x_1,v'_{\mathrm{c.m.}_1}]$$

which implies

$$0=\Big[x'_1-x'_{O_1},p'_{O_1}+\frac{M_1}{M_2}[f_1(M_1,M_2)p'_{O_1}+f_2(M_1,M_2)p'_1]+p'_1+\frac{m_1}{M_1}[f_1(m_1,M_2)p'_1+f_2(m_1,M_2)p'_{O_1}]\Big] .$$

Since $x'_1-x'_{O_1}$ commutes with $p'_{O_1}+p'_1$, we have

$$0=\Big[x'_1-x'_{O_1},\frac{M_1}{M_2}[f_1(M_1,M_2)p'_{O_1}+f_2(M_1,M_2)p'_1]+\frac{m_1}{M_2}[f_1(m_1,M_2)p'_1+f_2(m_1,M_2)p'_{O_1}]\Big] .$$

This leads to the condition

$$M_1[f_1(M_1,M_2)-f_2(M_1,M_2)]=m_1[f_1(m_1,M_2)-f_2(m_1,M_2)] . \quad \text{(A10)}$$

Next, substracting (A8) from (A9) gives

$$M_1[f_1(M_1,M_2)-f_2(M_1,M_2)]-M_2f_2(M_1,M_2)=m_1[f_1(m_1,M_2)-f_2(m_1,M_2)]-M_2f_2(m_1,M_2) . \quad \text{(A11)}$$

Substracting (A10) from (A11) gives

$$f_2(M_1,M_2)=f_2(m_1,M_2) ,$$

which implies

$$f_2(m_i,M_2)\equiv f_2(M_2) .$$

With this, (A8) becomes

$$1+\frac{f_1(m_1,M_2)}{M_2}m_1-\frac{f_2(M_2)}{M_2}(m_1+M_2)=0 . \quad \text{(A12)}$$

Now the last term on the left-hand side above is linear in m_1, and hence

$$\frac{f_1(m_1,M_2)}{M_2}m_1$$

must also be linear in m_1. But then $f_1(m_1,M_2)=f_1(M_2)$, i.e., independent of m_1. Noting this, we write (A12) as

$$M_2[1-f_2(M_2)]+m_1[f_1(M_2)-f_2(M_2)]=0$$

In this equation, the coefficients of M_2 and m_1 must vanish independently. This yields

$$f_2(M_2)=1 ,$$

$$f_1(M_2)=f_2(M_2) ,$$

from which we finally obtain

$$f_1(m_i,m_j)=f_2(m_i,m_j)\equiv 1 .$$

APPENDIX B

$$q_n=Ux_nU^\dagger ,$$

$$\pi_n=Up_nU^\dagger .$$

Similarly, for a general operator $A(x_n,p_n)$, we have

$$A'=UA(x_n,p_n)U^\dagger=A(q_n,\pi_n) ,$$

$$\begin{aligned} i\dot{A}' &= i(\dot{U}AU^\dagger+U\dot{A}U^\dagger+UA\dot{U}^\dagger) \\ &= (UH-HU)AU^\dagger+U(AH-HA)U^\dagger+UA(U^\dagger H-HU^\dagger) \\ &= U(HA-U^\dagger HUA+AH-HA+AU^\dagger HU-AH)U^\dagger \\ &= U[A,U^\dagger HU]U^\dagger , \end{aligned}$$

$$\overline{H}(x_n,p_n)\equiv U^\dagger H(x_n,p_n)U ,$$

which implies

$$i\dot{A}(q_n,\pi_n)=[A(q_n,\pi_n),\overline{H}(q_n,\pi_n)]$$

Hence, in the quantum frame, the Hamiltonian is

$$H'=\overline{H}(q_n,\pi_n) .$$

APPENDIX C

1. Transformation from the absolute to the quantum frame in two dimensions

The transformation from the absolute to the quantum frame may be viewed as consisting of two steps, defined by

$$U=U'_2U_1 , \quad \text{(C1a)}$$

where

$$U_1=\exp\left[-\sum_2^N\vec{p}_n\cdot\vec{R}\right] , \quad \text{(C1b)}$$

$$U'_2=\exp\left[-i\sum_{n\neq\mathrm{rel}}\overline{\pi}_{\theta_n}\overline{\theta}\right] , \quad \text{(C1c)}$$

and

$$\overline{\theta}=\theta_{\mathrm{rel}}=\arctan\frac{y}{x} ,$$

$$\overline{\pi}_{\theta_n}=(\vec{r}_n-\vec{R})\times\vec{p}_n$$

are intermediate variables, obtained from the absolute by acting on them with U_1. Thus $\overline{\theta}=U_1\theta_{\mathrm{rel}}U_1^\dagger$ and $\overline{\pi}_{\theta_n}=U_1L_{zn}U_1^\dagger$. We have $U'_2U_1=U_1U_2U_1^\dagger U_1=U_1U_2$, hence (C1) may be rewritten

$$U=\exp\left[-i\sum_2^N\vec{p}_n\cdot\vec{R}\right]\exp\left[-i\sum_{n\neq\mathrm{rel}}L_{zn}\theta_{\mathrm{rel}}\right] , \quad \text{(C2)}$$

i.e., in terms of absolute variables.

Let us return now to (C1). The first step in the transformation consists of attaching the origin to the "c.m. particle." It is given by (C1b) and resembles (17) of the one-dimensional case. It should be noticed that U_1 leaves the relative degree of freedom invariant.

The set of intermediate variables that result from the action of U_1 on the absolute space will be denoted with bars [as in (C1c)] and are as follows:

$$\vec{\bar{q}}_n = U_1 \vec{r}_n U_1^\dagger = \vec{r}_n - \vec{R} , \tag{C3a}$$

$$\vec{\bar{\pi}}_n = U_1 \vec{p}_n U_1^\dagger = \vec{p}_n . \tag{C3b}$$

For the angular variables we have

$$\bar{\theta}_n = \arctan\frac{\bar{y}_n}{\bar{x}_n} = \arctan\frac{y_n - Y}{x_n - X} , \tag{C3c}$$

$$\vec{\bar{\pi}}_{\theta_n} = (\vec{r}_n - \vec{R}) \times \vec{p}_n . \tag{C3d}$$

The second step in the transformation is given by U_2' of (C1c). This involves a rotation of all particles except the "relative" by the angle $\bar{\theta}$. Thus,

$$\theta_n = \bar{\theta}_n - \bar{\theta} = \bar{\theta}_n - \theta_{\text{rel}} , \tag{C4a}$$

$$\vec{\pi}_{\theta_n} = \vec{\bar{\pi}}_{\theta_n} . \tag{C4b}$$

Alternatively, U as given by (C2) may be directly applied to the absolute variables. The resulting variables are as given in (30) and (31) of Sec. III, thus confirming the correctness of (C2) [and of (32)].

2. Calculation of the Hamiltonian

According to (32) and (33),

$$\bar{H}(\vec{p}_n) = U_2^\dagger U_1^\dagger H(\vec{p}_n) U_1 U_2$$

where

$$U_1 = \exp\left[-i \sum_2^N \vec{p}_n \cdot \vec{R}\right] ,$$

$$U_2 = \exp\left[-i \sum_{n \neq \text{rel}} L_{zn} \theta\right] ,$$

$$U_1^\dagger H(\vec{p}_n) U_1 = \exp\left[i \sum_2^N \vec{p}_n \cdot \vec{R}\right] \left[\frac{P^2}{2M} + \frac{p^2}{2\mu} + \sum_2^N \frac{p_n{}^2}{2m_n}\right] \times \exp\left[-i \sum_2^N \vec{p}_n \cdot \vec{R}\right]$$

$$= \frac{\left[\vec{P} - \sum_2^N \vec{p}_n\right]^2}{2M} + \frac{p^2}{2\mu} + \sum_2^N \frac{p_n{}^2}{2m_n} .$$

In the final step, when $\bar{H}(\vec{p}_n) \to H'(\vec{\pi}_n) = U\bar{H}(\vec{p}_n)U^\dagger$, $\vec{P}$ transforms into $\vec{P}_T$, which we have assumed to be equal to zero. We therefore drop $\vec{P}$. With

$$\vec{K} \equiv \sum_2^N \vec{p}_n ,$$

we obtain

$$U_1^\dagger H(\vec{p}_n) U_1 = \sum_2^N \frac{p_n{}^2}{2m_n} + \frac{p^2}{2\mu} + \frac{K^2}{2M} ,$$

$$\exp\left[i \sum_{n \neq \text{rel}} L_{zn} \theta\right] \left[\sum_2^N \frac{p_n{}^2}{2m_n} + \frac{K^2}{2M} + \frac{p^2}{2\mu}\right] \times \exp\left[-i \sum_{n \neq \text{rel}} L_{zn} \theta\right]$$

$$= \sum_2^N \frac{p_n{}^2}{2m_n} + \frac{K^2}{2M} + e^{iL_z\theta} \frac{p^2}{2\mu} e^{-iL_z\theta} , \tag{C5a}$$

where

$$L_z \equiv \sum_{n \neq \text{rel}} L_{zn} . \tag{C5b}$$

We now calculate the last term on the right-hand side. In two dimensions, using plane polar coordinates,

$$p_x = \frac{1}{i}\left[-\frac{\sin\theta}{r}\frac{\partial}{\partial\theta} + \cos\theta\frac{\partial}{\partial r}\right] ,$$

$$p_y = \frac{1}{i}\left[\frac{\cos\theta}{r}\frac{\partial}{\partial\theta} + \sin\theta\frac{\partial}{\partial r}\right] ,$$

$$e^{i\alpha\theta}\frac{1}{i}\frac{\partial}{\partial\theta}e^{-i\alpha\theta} = \frac{1}{i}\frac{\partial}{\partial\theta} - \alpha .$$

It then follows that

$$e^{iL_z\theta} p_x e^{-iL_z\theta} = p_x + \frac{\sin\theta}{r} L_z ,$$

$$e^{iL_z\theta} p_y e^{-iL_z\theta} = p_y - \frac{\cos\theta}{r} L_z .$$

Again, we have in the plane

$$\sin\theta = \frac{y}{r} = \frac{y}{(x^2+y^2)^{1/2}} ,$$

$$\cos\theta = \frac{x}{r} = \frac{x}{(x^2+y^2)^{1/2}} ,$$

and hence we obtain

$$e^{iL_z\theta} p^2 e^{-iL_z\theta} = \left[p_x + \frac{yL_z}{(x^2+y^2)^{1/2}}\right]^2 + \left[p_y - \frac{xL_z}{(x^2+y^2)^{1/2}}\right]^2 . \tag{C6}$$

We calculate now the corresponding expression in plane polar coordinates,

$$e^{iL_z\theta} p^2 e^{-iL_z\theta} = p_x{}^2 + p_y{}^2 + \frac{\sin^2\theta + \cos^2\theta}{r^2} L_z{}^2 + p_x \frac{\sin\theta}{r} L_z + \frac{\sin\theta}{r} p_x L_z - p_y \frac{\cos\theta}{r} L_z - \frac{\cos\theta}{r} p_y L_z$$

$$= p^2 + \frac{L_z{}^2}{r^2} + \left[p_x \frac{\sin\theta}{r} + \frac{\sin\theta}{r} p_x\right] L_z - \left[p_y \frac{\cos\theta}{r} + \frac{\cos\theta}{r} p_y\right] L_z , \tag{C7}$$

$$p_x \frac{\sin\theta}{r} - p_y \frac{\cos\theta}{r} = \frac{1}{i}\left[\left[-\frac{\sin\theta}{r}\frac{\partial}{\partial\theta}+\cos\theta\frac{\partial}{\partial r}\right]\frac{\sin\theta}{r} - \left[\frac{\cos\theta}{r}\frac{\partial}{\partial\theta}+\sin\theta\frac{\partial}{\partial r}\right]\frac{\cos\theta}{r}\right]$$
$$= \frac{1}{i}\left[-\frac{\sin\theta}{r}\left[\frac{\cos\theta}{r}+\frac{\sin\theta}{r}\frac{\partial}{\partial\theta}\right]+\cos\theta\sin\theta\frac{\partial}{\partial r}\frac{1}{r} - \frac{\cos\theta}{r}\left[-\frac{\sin\theta}{r}+\frac{\cos\theta}{r}\frac{\partial}{\partial\theta}\right]-\sin\theta\cos\theta\frac{\partial}{\partial r}\frac{1}{r}\right]$$
$$= -\frac{1}{i}\frac{\sin^2\theta+\cos^2\theta}{r^2}\frac{\partial}{\partial\theta} = -\frac{1}{r^2}\frac{1}{i}\frac{\partial}{\partial\theta} . \quad \text{(C8)}$$

A similar calculation gives

$$\frac{\sin\theta}{r}p_x - \frac{\cos\theta}{r}p_y = \frac{1}{i}\left[\frac{\sin\theta}{r}\left[-\frac{\sin\theta}{r}\frac{\partial}{\partial\theta}+\cos\theta\frac{\partial}{\partial r}\right]-\frac{\cos\theta}{r}\left[\frac{\cos\theta}{r}\frac{\partial}{\partial\theta}+\sin\theta\frac{\partial}{\partial r}\right]\right]$$
$$= -\frac{1}{r^2}\frac{1}{i}\frac{\partial}{\partial\theta} . \quad \text{(C9)}$$

Substituting (C8) and (C9) into (C7), we obtain

$$e^{iL_z\theta}p^2e^{-iL_z\theta} = p^2 - 2\frac{1}{i}\frac{\partial}{\partial\theta}\frac{L_z}{r^2}+\frac{{L_z}^2}{r^2} ,$$

or, denoting

$$\frac{1}{i}\frac{\partial}{\partial\theta} = p_\theta$$

[we used $(1/i)\partial/\partial\theta = l_z$ previously, as in (26h)], we have

$$e^{iL_z\theta}p^2e^{-iL_z\theta} = {p_r}^2 + \frac{{p_\theta}^2}{r^2} - 2p_\theta\frac{L_z}{r^2}+\frac{{L_z}^2}{r^2}$$
$$= {p_r}^2 + \frac{(p_\theta - L_z)^2}{r^2} . \quad \text{(C10)}$$

Substituting (C6) or (C10) into (C5), we finally obtain

$$\overline{H}(\vec{p}_n) = \sum_2^N \frac{{p_n}^2}{2m_n} + \frac{K^2}{2M} + \frac{{p_r}^2}{2\mu} + \frac{(p_\theta - L_z)^2}{2\mu r^2}$$
$$= \sum_2^N \frac{{p_n}^2}{2m_n} + \frac{K^2}{2M} + \frac{1}{2\mu}\left[p_x + \frac{yL_z}{(x^2+y^2)^{1/2}}\right]^2$$
$$+ \frac{1}{2\mu}\left[{p_y}^2 - \frac{xL_z}{(x^2+y^2)^{1/2}}\right]^2 .$$

3. Transforming between quantum frames

Our objective is to find the transformation from the unprimed to the double-primed frames (as defined in Sec. III). To this end, we shall define an auxiliary frame (called the primed frame) in the following manner: The origin is defined by indices c.m. and 23, and the x axis by rel and 10. Our transformation will then consist of a sequence of two steps. From the unprimed to the primed frame, and from the primed to the double-primed frame.

In the first step, indices c.m. and 10 are replaced by indices c.m. and 23. This furnishes a displacement of the origin by $\vec{R}_{23}^{\text{c.m.}} - \vec{R}_{10}^{\text{c.m.}}$. In the unprimed frame we have the variables

$$q_{nx} = (x_n - X_{10})\cos\theta_{10} + (y_n - Y_{10})\sin\theta_{10} ,$$
$$q_{ny} = -(x_n - X_{10})\sin\theta_{10} + (y_n - Y_{10})\cos\theta_{10} . \quad \text{(C11)}$$

A listing of the corresponding canonically conjugate momenta can also be rewritten down, with the help of (31) of Sec. IV. In the primed frame, we have the variables

$$q'_{nx} = (x_n - X_{23})\cos\theta_{10} + (y_n - Y_{23})\sin\theta_{10} ,$$
$$q'_{ny} = -(x_n - X_{23})\sin\theta_{10} + (y_n - Y_{23})\cos\theta_{10} . \quad \text{(C12)}$$

The transformation from the unprimed to the primed frame will be generated by

$$U_1 = P_{\text{c.m.},b}\exp\left[i\sum_4^N \vec{\pi}_n\cdot\vec{Q}_b\right]e^{-i\vec{\Pi}_a\cdot\vec{Q}_b} , \quad \text{(C13)}$$

where $P_{\text{c.m.},b}$ is the parity operator associated with degree of freedom c.m., b, i.e., $\vec{Q}_b, \vec{\Pi}_b$. The important point to notice here is that the "relative" degrees of freedom (rel, a and rel, b) are not affected by this transformation. We have

$$U_1\vec{Q}_aU_1^\dagger = \vec{Q}_a + \vec{Q}_b = \vec{Q}'_a ,$$
$$U_1\vec{Q}_bU_1^\dagger = -\vec{Q}_b = \vec{Q}'_b ,$$
$$U_1\vec{q}_nU_1^\dagger = \vec{q}_n - \vec{Q}_b = \vec{q}\,'_n ,$$

whereas

$$U_1\vec{q}_aU_1^\dagger = \vec{q}_a = \vec{q}\,'_a, \quad U_1\vec{q}_bU_1^\dagger = \vec{q}_b = \vec{q}\,'_b ,$$

where the second equality sign in each case may be verified by calculation of the expression on the left-hand side (of the said equality sign) and comparison with (C12) above.

We next transform from the primed to the double-primed frame. What we have to do here is to exchange rel 10 with rel 23 as the "particle" defining the direction of the x axis. This induces a rotation by an angle $\theta_{10}^{\text{rel}} - \theta_{23}^{\text{rel}}$ on all particles (except "little a" and "little b").

In the double-primed frame we have

$$q''_{nx} = (x_n - X_{23})\cos\theta_{23} + (y_n - Y_{23})\sin\theta_{23} ,$$
$$q''_{ny} = -(x_n - X_{23})\sin\theta_{23} + (y_n - Y_{23})\cos\theta_{23} . \quad \text{(C14)}$$

As for the angular variables, inspecting (C12), we see that in the primed frame we have

$$\theta'_n=\bar{\theta}'_n-\theta_{10}^{\rm rel},\quad \bar{\theta}'_n=\arctan\frac{y_n-Y_{23}}{x_n-X_{23}}\quad (n=4,\ldots,N) \tag{C15}$$

and in the double-primed frame [from (C14)],

$$\theta''_n=\bar{\theta}'_n-\theta_{23}^{\rm rel}. \tag{C16}$$

Comparing (C16) with (C15), we find for the transformation from the primed to the double-primed frame,

$$U'_2=P_{{\rm rel},b}\exp\left[i\sum_{n\neq{\rm rel}}\pi'_{\theta_n}\theta'_b\right]e^{-i\Pi'^{\rm rel}_{\theta_a}\theta b'}. \tag{C17}$$

Thus we have for the overall transformation

$$U=U'_2U_1 .$$

Since

$$U'_2U_1=U_1U_2 ,$$

we finally obtain for U

$$U=U_1U_2 ,$$

where now

$$U_2=P_{{\rm rel},b}\exp\left[i\sum_{n\neq{\rm rel}}\pi_{\theta_n}\theta_b\right]e^{-i\Pi^{\rm rel}_{\theta_a}\theta_b}, \tag{C18}$$

i.e., in terms of variables of the unprimed frame.

APPENDIX D

1. Derivation of the geometrical vector potential for the case considered in Sec. III

We now calculate

$$e^{iL_z\varphi_1}p_{1x}e^{-iL_z\varphi_1}=e^{iL_z\varphi_1}\frac{1}{i}\frac{\partial}{\partial x_1}e^{-iL_z\varphi_1}. \tag{D1}$$

Using

$$e^{iL_z\varphi_1}\frac{1}{i}\frac{\partial}{\partial\varphi_1}e^{-iL_z\varphi_1}=\frac{1}{i}\frac{\partial}{\partial\varphi_1}-L_z ,$$

we obtain

$$e^{iL_z\varphi_1}p_{1x}e^{-iL_z\varphi_1}=p_{1x}+\frac{1}{r_1}\frac{\sin\varphi_1}{\sin\theta_1}L_z .$$

Next we calculate

$$e^{iL_y\theta_1}\left[p_{1x}+\frac{1}{r_1}\frac{\sin\varphi_1}{\sin\theta_1}L_z\right]e^{-iL_y\theta_1}$$
$$=e^{iL_y\theta_1}\left[\frac{1}{i}\frac{\partial}{\partial x_1}\right]e^{-iL_y\theta_1}+\frac{1}{r_1}\frac{\sin\varphi_1}{\sin\theta_1}e^{iL_y\theta_1}L_ze^{-iL_y\theta_1}, \tag{D2}$$

$$e^{iL_y\theta_1}\frac{1}{i}\frac{\partial}{\partial x_1}e^{-iL_y\theta_1}=\frac{1}{i}\frac{\partial}{\partial x_1}-\frac{1}{r_1}\cos\theta_1\cos\varphi_1L_y . \tag{D3}$$

To calculate the second term in (D2), let us define

$$e^{iL_y\theta_1}L_ze^{-iL_y\theta_1}\equiv L_z(\theta) .$$

We note that

$$L_y(\theta)=L_y(0) .$$

Then

$$\begin{aligned} i\frac{\partial}{\partial\theta}L_z(\theta)&=i[iL_yL_z(\theta)-iL_z(\theta)L_y]\\ &=L_z(\theta)L_y-L_yL_z(\theta)\\ &=L_z(\theta)L_y(\theta)-L_y(\theta)L_z(\theta)\\ &=-iL_x(\theta) ,\end{aligned}$$

or

$$\frac{\partial}{\partial\theta}L_z(\theta)=-L_x(\theta) .$$

Also,

$$\frac{\partial}{\partial\theta}L_x(\theta)=L_z(\theta) .$$

The solution is

$$L_z(\theta)=L_z\cos\theta-L_x\sin\theta , \tag{D4a}$$

$$L_x(\theta)=L_z\sin\theta+L_x\cos\theta . \tag{D4b}$$

Substituting (D3) and (D4a) into (D2), the result is

$$\begin{aligned} U^\dagger p_{1x}U&=p_{1x}-\frac{1}{r_1}\cos\varphi_1\cos\theta_1L_y\\ &\quad+\frac{1}{r_1}\frac{\sin\varphi_1}{\sin\theta_1}(L_z\cos\theta_1-L_x\sin\theta_1)\\ &=p_{1x}-\frac{1}{r_1}\sin\varphi_1L_x\\ &\quad-\frac{1}{r_1}\cos\varphi_1L_y+\frac{1}{r_1}\sin\varphi_1\frac{\cos\theta_1}{\sin\theta_1}L_z .\end{aligned}$$

A similar calculation yields

$$\begin{aligned} U^\dagger p_{1y}(U)&=p_{1y}+\frac{1}{r_1}\cos\varphi_1L_x\\ &\quad-\frac{1}{r_1}\sin\varphi_1\cos\theta_1L_y-\frac{1}{r_1}\cos\varphi_1\frac{\cos\theta_1}{\sin\theta_1}L_z\end{aligned}$$

and

$$U^\dagger p_{1z}U=p_{1z}+\frac{1}{r_1}\sin\theta_1L_y ,$$

whence the potential may be read

$$A_x=\frac{1}{r_1}\left[-\sin\varphi_1L_x-\cos\varphi_1\cos\theta_1L_y+\sin\varphi_1\frac{\cos\theta_1}{\sin\theta_1}L_z\right],$$

$$A_y=\frac{1}{r_1}\left[\cos\varphi_1L_x-\sin\varphi_1\cos\theta_1L_y-\cos\varphi_1\frac{\cos\theta_1}{\sin\theta_1}L_z\right],$$

$$A_z=\frac{1}{r_1}\sin\theta_1L_y .$$

Substituting the sines and cosines in terms of Cartesian variables, i.e.,

$$\cos\theta=\frac{z}{r},\quad \frac{\cos\theta}{\sin\theta}=\frac{z}{(x^2+y^2)^{1/2}}\ ,$$

$$\sin\varphi=\frac{y}{(x^2+y^2)^{1/2}},\quad \cos\varphi=\frac{x}{(x^2+y^2)^{1/2}}\ ,$$

one obtains

$$A_x=\frac{1}{r}\left[-x\left[\frac{zL_y}{r(x^2+y^2)^{1/2}}\right]+y\left[-\frac{L_x}{(x^2+y^2)^{1/2}}+\frac{zL_z}{x^2+y^2}\right]\right],$$

$$A_y=\frac{1}{r}\left[-y\left[\frac{zL_y}{r(x^2+y^2)^{1/2}}\right]+x\left[\frac{L_x}{(x^2+y^2)^{1/2}}-\frac{zL_z}{x^2+y^2}\right]\right],$$

$$A_z=\frac{(x^2+y^2)^{1/2}}{r^2}L_y$$

(we have dropped the subscript 1 everywhere).

2. Calculation of the force in the general case of a non-Abelian vector potential

Given the Hamiltonian

$$H=\frac{\left[\vec{\mathrm{p}}-\vec{\mathrm{A}}\right]^2}{2m}+\varphi(\vec{\mathrm{x}})\ ,$$

$$F_i=\dot{p}_i-\dot{A}_i=-\frac{\partial H}{\partial x_i}-\frac{\partial A_i}{\partial t}-\frac{1}{i}[A_i,H]\ .$$

We assume that $\vec{\mathrm{A}}$ has no explicit time dependence. Then,

$$F_i=-\frac{\partial H}{\partial x_i}-\frac{1}{i}[A_i,H]$$

$$=\frac{-\partial\varphi}{\partial x_i}+\left\{\left[\frac{\partial A_j}{\partial x_i}-\frac{1}{i}[A_i,p_j-A_j]\right],\frac{p_j-A_j}{2m}\right\}_+ ,$$

$$\frac{p_j-A_j}{m}=v_j,\ [A_i,p_j]=-\frac{1}{i}\frac{\partial A_i}{\partial x_j}\ .$$

So that

$$F_i=-\frac{\partial\varphi}{\partial x_i}+\left\{\left[\frac{\partial A_j}{\partial x_i}-\frac{\partial A_i}{\partial x_j}-i[A_i,A_j]\right],\frac{v_j}{2}\right\}_+$$

$$=-\frac{\partial\varphi}{\partial x_i}+\epsilon_{ijk}\left\{\left[\frac{\partial A_j}{\partial x_i}-\frac{\partial A_i}{\partial x_j}-i[A_i,A_j]\right],\frac{v_j}{2}\right\}_+ .$$

3. Contributions of the term proportional to L_z to the "magnetic" field

We have

$$B_k=\frac{\partial A_i}{\partial x_j}-\frac{\partial A_j}{\partial x_i}+i[A_i,A_j]\ ,$$

$$\bar{B}_k=\frac{\partial A_i}{\partial x_j}-\frac{\partial A_j}{\partial x_i}=-i[A_i,A_j]\ ,$$

$$i[A_y,A_z]=\frac{ix}{r^3}\left[[L_x,L_y]-\frac{z}{r_{xy}}[L_z,L_y]\right]$$

$$=\frac{ix}{r^3}\left[iL_z-\frac{z}{r_{xy}}(-iL_x)\right]$$

$$=-\frac{xL_z}{r^3}-\frac{xzL_x}{r^3r_{xy}}\ .$$

Therefore,

$$\bar{B}_x(L_z)=\frac{xL_z}{r^3}\ ,$$

$$i[A_x,A_z]=\frac{iy}{r^3}\left[-[L_x,L_y]+\frac{z}{r_{xy}}[L_z,L_y]\right]$$

$$=\frac{y}{r^3}L_z+\frac{yzL_x}{r^3r_{xy}}\ .$$

Thus,

$$\bar{B}_y(L_z)=\frac{yL_z}{r^3}\ ,$$

$$i[A_x,A_y]=\frac{-i}{r^2}[-x\hat{a}+y\hat{b},y\hat{a}+x\hat{b}]$$

$$=\frac{-i}{r^2}(-x^2[\hat{a},\hat{b}]+y^2[\hat{b},\hat{a}])$$

$$=i\frac{(x^2+y^2)}{r^2}[\hat{a},\hat{b}]=i\frac{{r_{xy}}^2}{r^2}[\hat{a},\hat{b}]\ ,$$

where

$$\hat{a}=\frac{zL_y}{rr_{xy}},\quad \hat{b}=-\frac{L_x}{r_{xy}}+\frac{zL_z}{{r_{xy}}^2}\ ,$$

$$[\hat{a},\hat{b}]=-\frac{z}{r{r_{xy}}^2}[L_y,L_x]+\frac{z^2}{r{r_{xy}}^3}[L_y,L_z]$$

$$=\frac{izL_z}{r{r_{xy}}^2}+\frac{iz^2L_x}{r{r_{xy}}^3}\ ,$$

$$i[A_x,A_y]=-\frac{{r_{xy}}^2}{r^2}\left[\frac{zL_z}{r{r_{xy}}^2}+\frac{z^2L_x}{r{r_{xy}}^3}\right]$$

$$=-\frac{zL_z}{r^3}-\frac{z^2L_x}{r^3{r_{xy}}^3}\ .$$

Therefore,

$$\bar{B}_z(L_z)=\frac{zL_z}{r^3}\ .$$

4. Calculation of the line integral around the flux singularity

In Appendix D1 we have shown that

$$A_{1x}=-\frac{1}{r_1}\left[\sin\varphi_1L_x+\cos\varphi_1\cos\theta_1L_y-\sin\varphi_1\frac{\cos\theta_1}{\sin\theta_1}L_z\right],$$

$$A_{1y}=\frac{1}{r_1}\left[\cos\varphi_1 L_x-\sin\varphi_1\cos\theta_1 L_y-\cos\varphi_1\frac{\cos\theta_1}{\sin\theta_1}L_z\right].$$

In what is to follow, we shall, for the sake of convenience, drop the subscript 1. We now calculate

$$\oint_{r_{xy}=\text{const}} \vec{A}\cdot d\vec{l}=\int A_x dx+A_y dy\ ,$$

$$dx=-r_{xy}\sin\varphi\, d\varphi\ ,$$

$$dy=r_{xy}\cos\varphi\, d\varphi\ ,$$

$$\oint\vec{A}\cdot d\vec{l}=\frac{r_{xy}}{r}\int_0^{2\pi}\sin\varphi\left[\sin\varphi L_x+\cos\varphi\cos\theta L_y-\sin\varphi\frac{\cos\theta}{\sin\theta}L_z\right]d\varphi$$
$$+\frac{r_{xy}}{r}\int_0^{2\pi}\cos\varphi\left[\cos\varphi L_x-\sin\varphi\cos\theta L_y-\cos\varphi\frac{\cos\theta}{\sin\theta}L_z\right]d\varphi$$
$$=\frac{r_{xy}}{r}\int_0^{2\pi}\left[\sin^2\varphi L_x+\sin\varphi\cos\varphi\cos\theta L_y-\sin^2\varphi\frac{\cos\theta}{\sin\theta}L_z\right]d\varphi$$
$$+\frac{r_{xy}}{r}\int_0^{2\pi}\left[\cos^2\varphi L_x-\cos\varphi\sin\varphi\cos\theta L_y-\cos^2\varphi\frac{\cos\theta}{\sin\theta}L_z\right]d\varphi$$
$$=\frac{r_{xy}}{r}\int_0^{2\pi}\left[L_x-\frac{\cos\theta}{\sin\theta}L_z\right]d\varphi=\frac{r_{xy}}{r}\int_0^{2\pi}\left[L_x-\frac{z}{r_{xy}}L_z\right]d\varphi=\frac{r_{xy}}{r}2\pi L_x-\frac{z}{r}2\pi L_z\ .$$

Taking the limit $r_{xy}\to 0$, it follows that

$$\frac{|z|}{r}\to 1\ ,$$

and we obtain

$$\lim_{r_{xy}\to 0}\oint\vec{A}\cdot d\vec{l}=\begin{cases}-2\pi L_z, & z>0\\ 2\pi L_z, & z<0.\end{cases}$$

APPENDIX E

Suppose that we have a sufficiently heavy object, of mass M_1, such that its velocity changes may be neglected. Let its Hamiltonian be

$$H=\frac{\pi_1^2}{2M_1}+V(q_1) \tag{E1}$$

and assume that the average velocity $\bar{v}_1=1$ and that the uncertainty $\Delta v_1\ll 1$. Define

$$\delta\pi_1=\pi_1-\bar{\pi}_1\ , \tag{E2}$$

where π_1 is the momentum of the particle and $\bar{\pi}_1$ is the initial value of the momentum average. We then have

$$\frac{\bar{\pi}_1}{M_1}=1 \tag{E3}$$

and

$$\frac{\delta\pi_1}{M_1}\ll 1\ . \tag{E4}$$

Substituting (E2) into (E1), we obtain

$$H=\frac{(\bar{\pi}_1+\delta\pi_1)^2}{2M_1}+V(q_1)$$
$$=\frac{(\delta\pi_1)^2}{2M_1}+\frac{\bar{\pi}_1\delta\pi_1}{M_1}+\frac{\bar{\pi}_1^2}{2M_1}+V(q_1)\ .$$

In this expression,

$$\frac{(\delta\pi_1)^2}{2M_1}$$

is negligible because of (E4). $\bar{\pi}_1^2/2M_1$ is a constant and may therefore be removed.

Using (E3), we obtain

$$H=\delta\pi_1+V(q_1)\ .$$

Since $\delta\pi_1$ has the same commutation relation with q_1 as does π_1, we may write

$$H=\pi_1+V(q_1)\ .$$

Then

$$\dot{\pi}_1=-\frac{\partial V}{\partial q_1}$$

and

$$\dot{q}_1=1\ .$$

Solving for $q_1(t)$, we have

$$q_1(t)=q_1(0)+t\ . \tag{E5}$$

Thus $q_1(t)$ differs from the "external" time t by the constant operator $q_1(0)$.

Consider now the Hamiltonian

$$H=\pi_1+\sum_{i\neq 1}\frac{p_i^2}{2m_i}\ . \tag{E6}$$

Suppose that (E6) describes an isolated system so that the total energy is constant. Assume further that the system is in a stationary state whose energy equals zero. Then,

$$H=0$$

or

$$i\frac{\partial}{\partial q_1}=\sum_{i\neq 1}\frac{p_i^2}{2m_i}\ . \tag{E7}$$

Recalling the result of (E5) above, and in view of (E7), we can now identify the coordinate q_1 as the internal time of the system, particle 1 thus serving as a clock. In terms of this internal time, Eq. (E7) is the time-dependent Schrödinger equation of the system[9]

$$i\frac{\partial}{\partial t_1}=\sum_{i\neq 1}\frac{p_i^2}{2m_i}\ .$$

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[1]A. S. Eddington, *The Nature of the Physical World* (MacMillan, London, 1929), pp. 220—229; A. Einstein and L. Infeld, *The Evolution of the Physical World* (Simon and Schuster, New York, 1938), pp. 220—260; R. P. Feynman, R. B. Leighton, and M. Sands, *The Feynman Lectures on Physics* (Addison-Wesley, Reading, Mass., 1964), Vol. II, pp. 42-1 through 42-14; A. S. Eddington, *Fundamental Theory* (Cambridge University Press, Cambridge, England, 1946); N. Bohr, in *Einstein, Philosopher, Scientist,* edited by P. A. Schilpp (The Library of Living Philosophers, Evanston, Illinois, 1949), Vol. I, pp. 224—228.

[2]Y. Aharonov and G. Carmi, Found. Phys. 3, 493 (1973); 4, 75 (1974); E. Lubkin, Ann. Phys. (N.Y.) 56, 69 (1970). See also C. A. Mead, Chem. Phys. (Netherlands) 49, 1 (1980).

[3]Particles 0 and 1 are chosen as reference particles. The corresponding center-of-mass and relative degrees of freedom will most often be referred to in the future. Therefore, and for the sake of convenience, the latter will sometimes be called "the center-of-mass particle" and "the relative particle."

[4]Here meaning the unrotated Hamiltonian. Because $\vec{A}$ arises due to the rotation, we concern ourselves here with this part of the transformation only.

[5]P. A. M. Dirac, Proc. R. Soc. London A133, 60 (1931); Phys. Rev. 74, 817 (1948); J. D. Jackson, *Classical Electrodynamics,* 2nd ed. (Wiley, New York, 1975), pp. 251—260. For a more recent discussion of this topic, see also C. N. Yang and R. L. Mills, Phys. Rev. 1, 191 (1954).

[6]See also G. 't Hooft, Nucl. Phys. B79, 276 (1976).

[7]Note that in this case we have not handled the particles defining the frame of reference in the same manner as in the simpler one- and two-dimensional cases. There we were able to separate the degrees of freedom of those particles into two groups, the first corresponding to variables that can be measured internally (i.e., the separation between the two particles in the two-dimensional case), the second corresponding to those that can be measured only relative to an absolute frame (i.e., the angle θ). This separation is more complicated in the three-dimensional case and will be handled separately in a future paper on this subject.

[8]J. von Neumann, *Mathematical Foundations of Quantum Mechanics* (Princeton University Press, Princeton, New Jersey, 1955); D. Bohm, *Quantum Theory* (Prentice-Hall, Englewood Cliffs, New Jersey, 1951), Chap. 22.

[9]For a complete discussion on this subject, see P. M. Morse and H. Feshbach, *Methods of Theoretical Physics* (McGraw-Hill, New York, 1953), Part I, Sec. 2.6, pp. 251—254.