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Symmetry operators for neutrino and Dirac fields on curved spacetime

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We characterize tensorially all first-order differential operators whose commutator with the massless Dirac operator is proportional to it on a general curved background in terms of skew-symmetric tensors satisfying conformally invariant generalizations of the Killing-Yano equations. The same problem is also solved for the massive charged Dirac equation on a general curved background.

I. INTRODUCTION

Symmetries play a crucial role in the analytic solution of the equations satisfied by the variables describing a physical system. A well-known example of this is given by Pauli's[1] solution of the hydrogen-atom problem via the O(4,R) symmetry group. This symmetry group appears as the product of a group of geometrical symmetries of the system, which turns out to be a subgroup of the isometry group of the ambient Euclidean space with a group of dynamical symmetries whose existence reflects the hidden symmetry associated with the Runge-Lenz vector. In particular, when the analytic solutions are obtained by seeking separable or R-separable[2] solutions, the symmetries appear in the form of *symmetry operators,*[3] that is (matrix) differential operators whose commutators with the differential operator appearing in the field equations are proportional to it. Symmetry operators thus map the space of solutions into itself. Commuting operators are the most familiar examples of symmetry operators and are interpreted as *constants of the motion,* their eigenvalues yielding quantum numbers for the system. They have been studied by Carter[4] in the general context where the equations satisfied by the variables describing the physical system are rth-order self-adjoint linear homogeneous partial differential equations and they were characterized by him as those operators leaving the Lagrangian density admitted by such equations invariant up to a divergence.

In the case of the equation of a massive charged test scalar field (or the motion of a massive charged test particle) interacting with background gravitational and electromagnetic fields, the commuting symmetry operators may be characterized tensorially in terms of valence-two symmetric *Killing tensors.* The existence of irreducible Killing tensors reflects the presence of dynamical symmetries which are not isometries. A remarkable example of such a situation was given by Carter[5] when he discovered the fourth constant of the motion for the massive-particle orbits in the Kerr solution. The existence of such tensors is also required in the theorems[6] ensuring the existence of separable coordinates for the Hamilton-Jacobi equation for the massive-particle orbits, or the Klein-Gordon equation for massive scalar fields, on a curved background.

Operators commuting with the Dirac operator have also been considered, an important example of which is given in flat spacetime by the total angular momentum[7] $\vec{J}$ which is associated with the separability in spherical coordinates of the Dirac equation for a central potential, and which admits the separated solutions as eigenfunctions with the separation constants arising therefrom as eigenvalues. First-order operators commuting with the Dirac operator on a curved background were obtained by Carter and McLenaghan[8,9] through their analysis of the separation-of-variables procedure devised for the Dirac equation in the Kerr spacetime by Chandrasekhar.[10] Carter and McLenaghan showed that these operators admit the separated solutions as eigenfunctions with the separation constants as eigenvalues, and characterized one of them tensorially in terms of a skew-symmetric valence-two Penrose-Floyd tensor[11] $D_{\alpha\beta}$, whose spinor equivalent K_{AB} is a *Killing spinor*[12] satisfying a further skew-Hermiticity condition.[13] Spindel and McLenaghan[14] subsequently gave a tensorial expression for the most general first-order operator commuting with the charged Dirac operator on a general curved background in terms of a Killing vector, a Penrose-Floyd tensor, and a Killing-Yano tensor[15] of valence three.

In contrast with the fact that one obtains commuting symmetry operators when separating the massive (charged) Klein-Gordon or Dirac equations, when dealing with the conformally invariant Klein-Gordon or Dirac equations for *zero-rest-mass* particles, one has to consider symmetry operators which are *not necessarily* commuting operators. Nontrivial examples of such operators have long been known[16] for the Laplace equation in Euclidean space, where they are associated with the R-separable coordinate systems. On the other hand, examples of such operators for the neutrino equation for massless spin-$\frac{1}{2}$ particles in curved space have only recently been given.[17] They arise in the class $\mathscr{D}$ of solutions[18] of Einstein's vacuum and electrovac field equations with cosmological constant for a nonsingular aligned Maxwell field through the existence for the entire class of a separable coordinate system and spin frame, the separated solutions being eigenfunctions of these operators with the separation constants as eigenvalues.

As has been previously noted, the existence of an operator which commutes with the Dirac operator on the Kerr background gives rise to a valence-two Killing spinor which satisfies an additional skew-Hermiticity condition. In view of the existence in the *whole class* $\mathscr{D}$ of a valence-two Killing spinor not necessarily satisfying this additional condition, the following question arises naturally: upon which geometric objects is the most-general first-order symmetry operator for the neutrino operator constructed on a general curved background? This question is answered in theorem I of our paper (which was given without proof in a previous note[19]), where this symmetry operator is characterized in terms of a conformal Killing vector, a conformal generalization of the Penrose-Floyd tensor,[20] and a conformal generalization of the Yano-Killing tensor of valence three. As expected, the two-index spinor associated with the conformal Penrose-Floyd tensor satisfies the Killing spinor equation, but not necessarily the skew-Hermiticity condition. We are thus able to *derive* the existence of the Killing spinor in the whole class $\mathscr{D}$ via the separability of the neutrino equation therein. In theorem II, we show that modulo a "trivial" symmetry operator, every symmetry operator for the massive Dirac equation without electromagnetic interaction is a commuting operator, thus showing that McLenaghan and Spindel had in fact determined essentially all the symmetry operators for this operator. In theorem III, we generalize the McLenaghan-Spindel result by determining the most general first-order symmetry operator for the massive charged Dirac operator. In the last section of our paper, we give the integrability conditions for the equations of the conformal Penrose-Floyd tensor and draw some general conclusions about the spacetimes admitting such a tensor. We also discuss further the connection between the separation of variables for the neutrino equation in $\mathscr{D}$ and the Killing spinor possessed by the class-$\mathscr{D}$ solutions.

II. CONSTRUCTION OF THE MOST-GENERAL FIRST-ORDER SYMMETRY OPERATORS FOR THE WEYL NEUTRINO AND DIRAC OPERATORS ON CURVED SPACETIME

We consider the neutrino and Dirac equations

$$H\psi \equiv i\gamma^\alpha \nabla_\alpha \psi = 0 \ , \tag{2.1}$$

$$(H + e\gamma^\alpha A_\alpha - mI)\psi = 0 \ , \tag{2.2}$$

where $\{\gamma^\alpha\}$ is a set of Dirac matrices associated with a four-dimensional Lorentzian metric $g_{\alpha\beta}$ by the anticommutation relations

$$\{\gamma^\alpha,\gamma^\beta\} = 2g^{\alpha\beta} I \ , \tag{2.3}$$

where ∇_α denotes the covariant differentiation operator[21] on four-spinors associated with the choice of γ^α's and the Levi-Civita connection of $g_{\alpha\beta}$, where A_α is a covector field on spacetime, e is the charge, m is the mass, and ψ is the four-component Dirac spinor.

We seek the most general first-order differential operators $\widetilde{K}$, K, and L which satisfy the commutation relations

$$[\widetilde{K},H] = \widetilde{\mathscr{R}} H \ , \tag{2.4}$$

$$[K, H - mI] = \mathscr{R}(H - mI) \ , \tag{2.5}$$

$$[L, H + e\gamma^\alpha A_\alpha - mI] = \mathscr{S}(H + e\gamma^\alpha A_\alpha - mI) \ , \tag{2.6}$$

where $\mathscr{R}_0$, $\mathscr{R}$, and $\mathscr{S}$ are 4×4 matrices to be determined. It should be noted at this point that K_0, K, and L map, respectively, the spaces of solutions of the neutrino equation, the Dirac equation in the absence of electromagnetic interactions, and the Dirac equation with electromagnetic interactions into themselves, and that we recover as a special case of Eqs. (2.4), (2.5), and (2.6), when $\mathscr{R}_0$, $\mathscr{R}$, and $\mathscr{S}$ vanish identically, the relation defining a constant of the motion.[22]

We shall treat Eqs. (2.4), (2.5), and (2.6), as far as possible, simultaneously by writing

$$L = F^\alpha \nabla_\alpha + G \ , \tag{2.7}$$

where F^α and G are 4×4 matrices of complex-valued functions, and then expanding the commutator in Eq. (2.6), distinguishing between the cases $m=0$ and $m\neq0$ only when necessary.

By equating to zero the coefficients of the covariant derivatives of each order in Eq. (2.6) and using the Ricci identity[23] on four-spinors, we obtain the following three conditions which are necessary and sufficient for Eq. (2.6) to be satisfied:

$$F^{(\alpha}\gamma^{\beta)} - \gamma^{(\alpha}F^{\beta)} = 0 \ , \tag{2.8a}$$

$$G\gamma^\beta - \gamma^\beta G - \gamma^\alpha \nabla_\alpha F^\beta - ieA_\alpha(F^\beta\gamma^\alpha - \gamma^\alpha F^\beta) - \mathscr{S}\gamma^\beta = 0 \ , \tag{2.8b}$$

$$\begin{aligned}(F^\alpha\gamma^\beta - \gamma^\alpha F^\beta)R_{\alpha\beta\gamma\delta}\gamma^\gamma\gamma^\delta + 8\gamma^\beta\nabla_\beta G \\ + 8ie[F^\alpha\gamma^\beta A_{\beta;\alpha} + A_\beta(G\gamma^\beta - \gamma^\beta G)] \\ - 8ie\mathscr{S}\gamma^\alpha A_\alpha + 8im\mathscr{S} = 0 \ .\end{aligned} \tag{2.8c}$$

To analyze Eqs. (2.8), we introduce the standard tensorial basis for $M_4(\mathbb{C})$

$$\left\{ I, \gamma^5 \equiv \frac{1}{4!}\epsilon_{\alpha\beta\gamma\delta}\gamma^\alpha\gamma^\beta\gamma^\gamma\gamma^\delta, \gamma^\alpha, \gamma^5\gamma^\alpha, \gamma^{\alpha\beta} \equiv \gamma^{[\alpha}\gamma^{\beta]} \right\} , \tag{2.9}$$

where we denote by $\epsilon_{\alpha\beta\gamma\delta}$ the components of the volume form of spacetime.

Equation (2.8a) will be identically satisfied if and only if

$$F^\alpha \equiv B^\alpha I + C\gamma^\alpha + D^\alpha{}_\beta\gamma^5\gamma^\beta + E^\alpha{}_{\beta\gamma}\gamma^{\beta\gamma} \ , \tag{2.10a}$$

where the tensor fields $D_{\alpha\beta}$ and $E_{\alpha\beta\gamma}$ satisfy

$$D_{\alpha\beta} = D_{[\alpha\beta]} \ , \quad E_{\alpha\beta\gamma} = E_{[\alpha\beta\gamma]} \ . \tag{2.10b}$$

To study conditions (2.8b) and (2.7c), we expand G and $\mathscr{S}$ in the basis (2.9):

$$G \equiv \Phi I + S\gamma^5 + J_\alpha\gamma^\alpha + K_\alpha\gamma^5\gamma^\alpha + L_{\alpha\beta}\gamma^{\alpha\beta} \ , \tag{2.11a}$$

$$\mathscr{S} \equiv MI + N\gamma^5 + P_\alpha\gamma^\alpha + Q_\alpha\gamma^5\gamma^\alpha + U_{\alpha\beta}\gamma^{\alpha\beta} \ . \tag{2.11b}$$

It is then readily shown that Eq. (2.8b) will be satisfied if and only if

$$C_{,\alpha} = -P_\alpha \ , \tag{2.12a}$$

$$K^\alpha = \tfrac{1}{2}D^{\beta\alpha}{}_{;\beta} + \tfrac{1}{2}Q^\alpha + ieA_\beta D^{\alpha\beta} \ , \tag{2.12b}$$

$$L_{\alpha\beta}=\tfrac{1}{4}(-B_{\alpha;\beta}+2E_{\alpha\beta}{}^{\gamma}{}_{;\gamma}-2U_{\beta\alpha}-Mg_{\alpha\beta}+4ieA_{\gamma}E_{\alpha}{}^{\gamma}{}_{\beta})\ , \tag{2.12c}$$

$$(2S-N)g_{\mu\nu}=-E_{\mu\gamma\delta;\alpha}\epsilon^{\alpha\gamma\delta}{}_{\nu}-U_{\alpha\delta}\epsilon^{\alpha\delta}{}_{\mu\nu}\ , \tag{2.12d}$$

$$2J_{[\mu}\delta^{\nu}{}_{\rho]}=\tfrac{1}{2}D^{\nu}{}_{\gamma;\alpha}\epsilon^{\alpha\gamma}{}_{\mu\rho}-\tfrac{1}{2}Q_{\alpha}\epsilon^{\alpha\nu}{}_{\mu\rho}+C_{,[\mu}\delta^{\nu}{}_{\rho]}+P_{[\mu}\delta^{\nu}{}_{\rho]}+2ieCA_{[\rho}\delta^{\nu}{}_{\mu]}\ . \tag{2.12e}$$

It follows from these equations, after lengthy calculations, that

$$B_{(\alpha;\beta)}=-Mg_{\alpha\beta}\ , \tag{2.13a}$$

$$K_{\alpha}=-Q_{\alpha}+ieA_{\beta}D_{\alpha}{}^{\beta}\ , \tag{2.13b}$$

$$J_{\alpha}=\tfrac{1}{3}{}^{*}D^{\delta}{}_{\alpha;\delta}-ieCA_{\alpha}\ , \tag{2.13c}$$

$$S=-\tfrac{3}{4}E^{\delta}{}_{;\delta}+\frac{N}{2}\ , \tag{2.13d}$$

$$D_{\alpha\beta;\gamma}=D_{\gamma[\alpha;\beta]}+3Q_{[\alpha}g_{\beta]\gamma}\ , \tag{2.13e}$$

$$E_{\alpha\beta\gamma;\delta}=-E_{\delta[\alpha\beta;\gamma]}-4U_{[\alpha\beta}g_{\gamma]\delta}\ , \tag{2.13f}$$

where we have introduced the Hodge duals

$${}^{*}D_{\alpha\beta}=\tfrac{1}{2}\epsilon_{\alpha\beta\gamma\delta}D^{\gamma\delta},\quad E_{\alpha}=\tfrac{1}{6}\epsilon_{\alpha\beta\gamma\delta}E^{\beta\gamma\delta}\ . \tag{2.13g}$$

It should be noted that Eqs. (2.13e) and (2.13f) may be rewritten in the equivalent form

$$D_{\alpha(\beta;\gamma)}=Q_{\alpha}g_{\beta\gamma}-Q_{(\beta}g_{\gamma)\alpha}\ , \tag{2.14a}$$

$$E_{\alpha\beta(\gamma;\delta)}=-U_{\alpha\beta}g_{\gamma\delta}-U_{\beta(\gamma}g_{\delta)\alpha}+U_{\alpha(\gamma}g_{\delta)\beta}\ . \tag{2.14b}$$

We now see from Eqs. (2.13a), (2.14a), and (2.14b) that B_{α} satisfies the conformal Killing vector equation and that $D_{\alpha\beta}$ and $E_{\alpha\beta\gamma}$ satisfy conformally invariant generalizations of the generalized Killing equations of Yano and Bochner.[24] We are now left with Eq. (2.8c), which is satisfied if and only if

$$J^{\alpha}{}_{;\alpha}+ieCA_{\alpha}{}^{;\alpha}-ieA_{\alpha}P^{\alpha}+imM=0\ , \tag{2.15a}$$

$$K^{\alpha}{}_{;\alpha}-ieA_{\beta;\alpha}D^{\alpha\beta}-2ieA_{\alpha}K^{\alpha}+ieA_{\alpha}Q^{\alpha}-imN=0\ , \tag{2.15b}$$

$$2L^{\beta}{}_{\alpha;\beta}+\Phi_{,\alpha}-\tfrac{1}{2}B^{\beta}R_{\beta\alpha}+ieA_{\alpha;\beta}B^{\beta}-2ieA_{\gamma;\beta}E^{\beta\gamma}{}_{\alpha}-4ieA_{\beta}L^{\beta}{}_{\alpha}-2ieA_{\beta}U_{\alpha}{}^{\beta}-ieMA_{\alpha}+imP_{\alpha}=0\ , \tag{2.15c}$$

$$L_{\alpha\beta;\gamma}\epsilon^{\alpha\beta\gamma}{}_{\delta}+S_{,\delta}+\tfrac{1}{2}E^{\alpha}{}_{\beta\gamma}\epsilon^{\sigma\beta\gamma\rho}R_{\alpha\sigma\rho\delta}+ieA_{\beta;\alpha}E^{\alpha}{}_{\gamma\epsilon}\epsilon^{\gamma\epsilon\beta}{}_{\delta}-2ieSA_{\delta}+ieA_{\delta}N-ieA_{\alpha}U_{\beta\epsilon}\epsilon^{\beta\epsilon\alpha}{}_{\delta}-imQ_{\delta}=0\ , \tag{2.15d}$$

$$\tfrac{1}{2}K_{\alpha;\beta}\epsilon^{\beta\alpha}{}_{\gamma\delta}+J_{[\delta;\gamma]}-\tfrac{1}{2}D^{\alpha}{}_{\beta}\epsilon^{\mu\beta\nu}{}_{[\gamma|}R_{\alpha\mu\nu|\delta]}+ieCA_{[\delta;\gamma]}-\tfrac{1}{2}ieA_{\beta;\alpha}D^{\alpha}{}_{\epsilon}\epsilon^{\epsilon\beta}{}_{\gamma\delta}+2ieA_{[\delta}J_{\gamma]}-ieP_{[\gamma}A_{\delta]}+\tfrac{1}{2}ieA_{\alpha}Q_{\beta}\epsilon^{\beta\alpha}{}_{\gamma\delta}+imU_{\gamma\delta}=0\ . \tag{2.15e}$$

Using Eqs. (2.13), it may be shown after considerable calculations that Eqs. (2.15) reduce to

$$mM=mN=0\ , \tag{2.16a}$$

$$\Phi_{,\beta}+\tfrac{3}{2}M_{,\beta}+ie(A_{\beta;\alpha}B^{\alpha}+A_{\alpha}B^{\alpha}{}_{;\beta})+imP_{\beta}=0\ , \tag{2.16b}$$

$$N_{,\alpha}-2imQ_{\alpha}+4ieF_{\alpha\gamma}E^{\gamma}=0\ , \tag{2.16c}$$

$$eF_{\alpha}{}^{[\beta}D^{\gamma]\alpha}+m{}^{*}U^{\beta\gamma}=0\ , \tag{2.16d}$$

where we have introduced the electromagnetic field tensor $F_{\alpha\beta}\equiv 2A_{[\beta;\alpha]}$ and the Hodge dual ${}^{*}U_{\alpha\beta}=\tfrac{1}{2}\epsilon_{\alpha\beta\gamma\delta}U^{\gamma\delta}$. Equations (2.10), (2.11), (2.13), and (2.16) are equivalent to our starting conditions (2.8). They express in terms of C, N, Φ, B_{α}, $D_{\alpha\beta}$, $E_{\alpha\beta\gamma}$, and A_{α} necessary and sufficient conditions for L to be a symmetry operator.

It should be noted that $\mathscr{S}$ is completely determined in terms of L via Eqs. (2.13a), (2.13d), (2.12a), (2.13e), and (2.13f); explicitly we have

$$M=-\tfrac{1}{4}B_{\alpha}{}^{;\alpha},\quad Q_{\alpha}=\tfrac{1}{3}D_{\alpha}{}^{\beta}{}_{;\beta},\quad U_{\alpha\beta}=-\tfrac{1}{2}E_{\alpha\beta}{}^{\gamma}{}_{;\gamma}\ . \tag{2.17}$$

In view of Eqs. (2.15) we must, as one would have expected, distinguish between the massless and massive cases in writing the final expression of our symmetry operators. When $m=0$, we have proved by Eqs. (2.10), (2.11), (2.12a), (2.13), and (2.16), the following.

Theorem I. The most-general first-order symmetry operator for the neutrino field operator, that is an operator $\widetilde{K}$ *satisfying*

$$[\widetilde{K},i\gamma^{\alpha}\nabla_{\alpha}]=\widetilde{\mathscr{R}}i\gamma^{\alpha}\nabla_{\alpha}\ , \tag{2.18a}$$

is given by

$$\widetilde{K}=(\widetilde{B}^{\alpha}I+\widetilde{C}\gamma^{\alpha}+\widetilde{D}^{\alpha}{}_{\beta}\gamma^{5}\gamma^{\beta}+\widetilde{E}^{\alpha}{}_{\beta\gamma}\gamma^{\beta\gamma})\nabla_{\alpha}+(\tfrac{3}{8}\widetilde{B}^{\alpha}{}_{;\alpha}+\widetilde{\Phi})I+\left[-\tfrac{3}{4}\widetilde{E}^{\delta}{}_{;\delta}+\frac{\widetilde{N}}{2}\right]\gamma^{5}+\tfrac{1}{3}{}^{*}\widetilde{D}^{\delta}{}_{\alpha;\delta}\gamma^{\alpha}+\tfrac{1}{3}\widetilde{D}^{\beta}{}_{\alpha;\beta}\gamma^{5}\gamma^{\alpha}+(\tfrac{1}{4}\widetilde{E}_{\alpha\beta}{}^{\gamma}{}_{;\gamma}-\tfrac{1}{4}\widetilde{B}_{\alpha;\beta})\gamma^{\alpha\beta}\ , \tag{2.18b}$$

where the tensor fields $\widetilde{B}_{\alpha}$, $\widetilde{D}_{\alpha\beta}$, *and* $\widetilde{E}_{\alpha\beta\gamma}$ *and the scalar fields* $\widetilde{N}$ *and* $\widetilde{\Phi}$ *satisfy*

$$\widetilde{D}_{\alpha\beta}=\widetilde{D}_{[\alpha\beta]},\quad \widetilde{E}_{\alpha\beta\gamma}=\widetilde{E}_{[\alpha\beta\gamma]}, \tag{2.18c}$$

$$\widetilde{N}_{,\alpha}=\widetilde{\Phi}_{,\alpha}=0\ , \tag{2.18d}$$

$$\widetilde{B}_{(\alpha;\beta)}=\tfrac{1}{4}\widetilde{B}^{\gamma}{}_{;\gamma}g_{\alpha\beta}\ , \tag{2.18e}$$

$$\widetilde{D}_{\alpha(\beta;\gamma)}=-\tfrac{1}{3}\widetilde{D}^{\delta}{}_{\alpha;\delta}g_{\beta\gamma}+\tfrac{1}{3}\widetilde{D}^{\delta}{}_{(\beta;|\delta|}g_{\gamma)\alpha}\ , \tag{2.18f}$$

$$\widetilde{E}_{\alpha\beta(\gamma;\delta)}=\tfrac{1}{2}\widetilde{E}_{\alpha\beta}{}^{\sigma}{}_{;\sigma}g_{\gamma\delta}+\tfrac{1}{2}\widetilde{E}_{\beta(\gamma}{}^{|\sigma|}{}_{;|\sigma|}g_{\delta)\alpha}-\tfrac{1}{2}\widetilde{E}_{\alpha(\gamma}{}^{|\sigma|}{}_{;|\sigma|}g_{\delta)\beta}\ , \tag{2.18g}$$

and where

$$\widetilde{\mathscr{R}}=-\tfrac{1}{4}\widetilde{B}_{\alpha}{}^{;\alpha}I+\widetilde{N}\gamma^{5}-\widetilde{C}_{,\alpha}\gamma^{\alpha}-\tfrac{1}{3}\widetilde{D}^{\beta}{}_{\alpha;\beta}\gamma^{5}\gamma^{\alpha}-\tfrac{1}{2}\widetilde{E}_{\alpha\beta}{}^{\gamma}{}_{;\gamma}\gamma^{\alpha\beta}\ . \tag{2.18h}$$

In the case $m\neq 0$, $e=0$ we have shown in view of Eqs. (2.10), (2.11), (2.12a), (2.13), and (2.16), the following.

Theorem II. The most-general first-order symmetry operator for the massive Dirac operator in the absence of electromagnetic interactions, that is an operator K satisfying

$$[K,i\gamma^\alpha\nabla_\alpha-mI]=\mathscr{R}(i\gamma^\alpha\nabla_\alpha-mI)\ , \tag{2.19a}$$

is given by

$$K=(B^\alpha I+C\gamma^\alpha+D^\alpha{}_\beta\gamma^5\gamma^\beta+E^\alpha{}_{\beta\gamma}\gamma^{\beta\gamma})\nabla_\alpha$$
$$+\Phi I-\tfrac{3}{4}E^\delta{}_{;\delta}\gamma^5+\tfrac{1}{3}{}^*D^\delta{}_{\alpha;\delta}\gamma^\alpha-\tfrac{1}{4}B_{\alpha;\beta}\gamma^{\alpha\beta}\ , \tag{2.19b}$$

where the tensor fields B_α, $D_{\alpha\beta}$, and $E_{\alpha\beta\gamma}$ and the scalar fields C and Φ satisfy

$$D_{\alpha\beta}=D_{[\alpha\beta]}\ ,\ \ E_{\alpha\beta\gamma}=E_{[\alpha\beta\gamma]}\ , \tag{2.19c}$$

$$B_{(\alpha;\beta)}=D_{\alpha(\beta;\gamma)}=E_{\alpha\beta(\gamma;\delta)}=0\ , \tag{2.19d}$$

$$imC_{,\beta}=\Phi_{,\beta}\ , \tag{2.19e}$$

and where

$$\mathscr{R}=-C_{,\alpha}\gamma^\alpha\ . \tag{2.19f}$$

We remark that we can rewrite, modulo an obvious redefinition, the symmetry operator K given in (2.19b) as follows:

$$K=K_1+K_2\ , \tag{2.20a}$$

where K_2 is given by

$$K_2=C\gamma^\alpha\nabla_\alpha+\Phi I\ ,\ \ imC_{,\beta}=\Phi_{,\beta}\ , \tag{2.20b}$$

and hence satisfies

$$[K_2,H-mI]=-C_{,\alpha}\gamma^\alpha(H-mI)\ , \tag{2.20c}$$

and where K_1 is the most-general first-order operator *commuting* with the massive Dirac operator in the absence of electromagnetic interactions whose expression has been given by McLenaghan and Spindel.[25] This implies that working in the quotient space of symmetry operators modulo operators of the form (2.20b), every symmetry operator for the massive Dirac operator in the absence of electromagnetic interaction is a commuting operator (an analogous property is well known to hold for the Helmholtz equation in Euclidean space[26]).

Finally, in the case $m\neq 0\neq e$, we have shown, in view of Eqs. (2.10), (2.11), (2.12a), (2.13), and (2.16), the following.

Theorem III. The most-general first-order symmetry operator for the massive Dirac operator with electromagnetic interactions, that is an operator L satisfying

$$[L,i\gamma^\alpha(\nabla_\alpha-ieA_\alpha)-mI]=\mathscr{S}[i\gamma^\alpha(\nabla_\alpha-ieA_\alpha)-mI]\ , \tag{2.21a}$$

is given by

$$L=(B^\alpha I+C\gamma^\alpha+D^\alpha{}_\beta\gamma^5\gamma^\beta+E^\alpha{}_{\beta\gamma}\gamma^{\beta\gamma})\nabla_\alpha$$
$$+\Phi I-\tfrac{3}{4}E^\delta{}_{;\delta}\gamma^5+(\tfrac{1}{3}{}^*D^\delta{}_{\alpha;\delta}-ieCA_\alpha)\gamma^\alpha$$
$$+(\tfrac{1}{3}D^\beta{}_{\alpha;\beta}+ieA_\beta D_\alpha{}^\beta)\gamma^5\gamma^\alpha$$
$$+(\tfrac{1}{4}E_{\alpha\beta}{}^\gamma{}_{;\gamma}-\tfrac{1}{4}B_{\alpha;\beta}+ieA_\gamma E_\alpha{}^\gamma{}_\beta)\gamma^{\alpha\beta}\ , \tag{2.21b}$$

where the tensor fields B_α, $D_{\alpha\beta}$, $E_{\alpha\beta\gamma}$, and A_α satisfy

$$D_{\alpha\beta}=D_{[\alpha\beta]}\ ,\ \ E_{\alpha\beta\gamma}=E_{[\alpha\beta\gamma]}\ , \tag{2.21c}$$

$$B_{(\alpha;\beta)}=0\ , \tag{2.21d}$$

$$D_{\alpha(\beta;\gamma)}=-\tfrac{1}{3}D^\delta{}_{\alpha;\delta}g_{\beta\gamma}+\tfrac{1}{3}D^\delta{}_{(\beta;|\delta|}g_{\gamma)\alpha}\ , \tag{2.21e}$$

$$E_{\alpha\beta(\gamma;\delta)}=\tfrac{1}{2}E_{\alpha\beta}{}^\sigma{}_{;\sigma}g_{\gamma\delta}+\tfrac{1}{2}E_{\beta(\gamma}{}^{|\sigma|}{}_{;|\sigma|}g_{\delta)\alpha}$$
$$-\tfrac{1}{2}E_{\alpha(\gamma}{}^\sigma{}_{;|\sigma|}g_{\delta)\beta}\ , \tag{2.21f}$$

$$\Phi_{,\beta}+ie(A_{\beta;\alpha}B^\alpha+A_\alpha B^\alpha{}_{;\beta})-imC_{,\beta}=0\ , \tag{2.21g}$$

$$mQ_\alpha-2eF_{\alpha\gamma}E^\gamma=0\ , \tag{2.21h}$$

$$eF_\alpha{}^{[\beta}D^{\gamma]\alpha}+m^*U^{\beta\gamma}=0\ , \tag{2.21i}$$

and where

$$\mathscr{S}=-C_{,\alpha}\gamma^\alpha-\tfrac{1}{3}D^\beta{}_{\alpha;\beta}\gamma^5\gamma^\alpha-\tfrac{1}{2}E_{\alpha\beta}{}^\gamma{}_{;\gamma}\gamma^{\alpha\beta}\ . \tag{2.21j}$$

Let us finally note that if we require $\widetilde{K}$ (or K or L) to formally self-adjoint, we must have, by an argument identical to that given by McLenaghan and Spindel,[27]

$$\widetilde{C}=-\overline{\widetilde{C}}\ ,\ \ \widetilde{\Phi}=\overline{\widetilde{\Phi}}\ ,\ \ \widetilde{B}_\alpha=-\overline{\widetilde{B}}_\alpha\ ,$$
$$\widetilde{D}_{\alpha\beta}=\overline{\widetilde{D}}_{\alpha\beta}\ ,\ \ \widetilde{E}_{\alpha\beta\gamma}=\overline{\widetilde{E}}_{\alpha\beta\gamma}\ . \tag{2.22}$$

III. DISCUSSION

The conformal Penrose-Floyd equation possesses integrability conditions which put significant restrictions on the backgrounds in which symmetry operators for the neutrino operator may exist. Indeed, every solution to Eq. (2.14a) must satisfy

$$D_{\alpha\beta;\delta\gamma}=\tfrac{3}{2}R^\rho{}_{\gamma[\alpha\beta}D_{\delta]\rho}+2g_{\delta[\beta}Q_{\alpha];\gamma}+g_{\beta\gamma}Q_{[\alpha;\delta]}$$
$$+g_{\alpha\gamma}Q_{[\delta;\beta]}+g_{\gamma\delta}Q_{[\beta;\alpha]}\ , \tag{3.1a}$$

which reduces when $Q_\alpha\equiv 0$ to the integrability condition given by Carter and McLenaghan[28] for the full Penrose-Floyd tensor, and generalizes the well-known condition satisfied by any solution of the conformal Killing equation.[29] From Eq. (3.1a), it may be deduced that every conformal Penrose-Floyd tensor must satisfy

$$D_{\rho[\alpha}R^\rho{}_{\beta]\delta\gamma}+D_{\rho[\delta}R^\rho{}_{\gamma]\alpha\beta}+g_{\beta\gamma}Q_{(\alpha;\delta)}$$
$$-g_{\alpha\gamma}Q_{(\beta;\delta)}-g_{\beta\delta}Q_{(\alpha;\gamma)}+g_{\alpha\delta}Q_{(\beta;\gamma)}=0\ , \tag{3.1b}$$

which yields upon contraction

$$D_{\rho(\alpha}R_{\delta)}{}^\rho+2Q_{(\alpha;\delta)}=0\ . \tag{3.1c}$$

It can easily be obtained from Eqs. (3.1b) and (3.1c) that $D_{\alpha\beta}$ must satisfy

$$D_{\rho[\gamma}C^\rho{}_{\delta]\alpha\beta}+D_{\rho[\alpha}C^\rho{}_{\beta]\gamma\delta}=0\ , \tag{3.1d}$$

which is the full integrability condition for the conformal Penrose-Floyd tensor. It should be noted that the conformally invariant condition (3.1d) holds also for the full Penrose-Floyd tensor. However, in that case there is an extra integrability condition, which turns out to be given by Eq. (3.1c), where Q_α is set equal to zero, which involves the Ricci tensor instead of the Weyl tensor, and which is, as one could expect, not conformally invariant.

The integrability condition (3.1d) has been studied by Spindel and McLenaghan in the more restricted context of full Penrose-Floyd tensors. However, since their discussion is purely algebraic, it is easily seen that their conclusions also hold for conformal Penrose-Floyd tensors. We may therefore conclude that those nonconformally flat spacetimes admitting a conformal Penrose-Floyd tensor are either Petrov type D with an algebraically general $D_{\alpha\beta}$ whose two distinct null eigenvectors are aligned with the two repeated principal null directions of the Petrov type-D Weyl tensor, or Petrov type N with an algebraically special $D_{\alpha\beta}$ whose repeated null eigenvector is aligned with the quadruply repeated principal null direction of the Weyl tensor.

An example of a noncommuting symmetry operator for the neutrino equation constructed on a conformal Penrose-Floyd tensor $D_{\alpha\beta}$ arises from the separation of variables[30] for the neutrino equation in the class $\mathscr{D}$ of solutions of Einstein's vacuum and electrovac field equations with cosmological constant for a nonsingular aligned Maxwell field. The symmetric two-index spinor associated to the conformal Penrose-Floyd tensor $D_{\alpha\beta}$ by the canonical correspondence

$$D_{\alpha\beta} \leftrightarrow \epsilon_{AB} \bar{K}_{\dot{A}\dot{B}} + \epsilon_{\dot{A}\dot{B}} K_{AB} , \tag{3.2a}$$

is of the form

$$K_{AB} = \rho o_{(A} \iota_{B)} , \tag{3.2b}$$

where o_A and ι_B are the repeated eigenspinors of the Petrov type-D Weyl tensor. Moreover, from Eqs. (2.14a) and (3.2), it can be deduced that K_{AB} satisfies the twistor equation

$$\nabla_{\dot{A}(A} K_{BC)} = 0 , \tag{3.2c}$$

and is consequently a *Killing spinor*.[31] We are thus able to explain the existence for every solution in $\mathscr{D}$ of a two-index Killing spinor K_{AB} of the form given in (3.2b), from the separability of the neutrino equation therein, via the construction of the symmetry operator underlying this separability property.

In contrast, the separation of variables for the massive charged Dirac equation within the class $\mathscr{D}$ is only possible in the subclass of Carter's $[\tilde{A}]$ solutions[32] and the null orbit solution A_0 found by Debever and McLenaghan[33] in which case the symmetry operator reduces to a commuting operator (first introduced by Carter and McLenaghan[34] in the special case of the Kerr-Newman solution) constructed on a *full* Penrose-Floyd tensor whose associated two-index spinor K_{AB} satisfies the twistor equation (3.2c) and the following skew-Hermiticity condition:[35]

$$\nabla_{\dot{A}C} K^C{}_B + \bar{\nabla}_{B\dot{C}} \bar{K}^{\dot{C}}{}_{\dot{A}} = 0 , \tag{3.2d}$$

which expresses the vanishing of Q_α in terms of two-spinors.

It finally should be mentioned that the symmetry operators $\tilde{K}$, K, and L introduced in theorems I, II, and III generalize the Lie-derivative operator on spinors which was introduced by Kossmann[36] and which corresponds to the parts depending on $\tilde{B}_\alpha$ and B_α in $\tilde{K}$, K, and L.

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