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Extended model of the electron in general relativity

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A classical model of the spinning electron is proposed in which this particle is the source of the Kerr-Newman field. The electron is regarded as a charged rotating shell endowed with surface tension. It is the boundary where the exterior Kerr-Newman solution is matched to the interior flat spacetime metric. The shell is the surface of an oblate ellipsoid of revolution having a minor axis equal to the classical electron radius and a focal distance of the order of the corresponding Compton wavelength. This surface is undergoing rigid rotation with its equator at a velocity almost equal to the velocity of light. The arrangement of charges gives rise to a quadrupole electric moment, in addition to the magnetic dipole moment of the current distribution. The whole spacetime of this model is shown to be causally well behaved.

I. INTRODUCTION

Soon after the discovery of the Kerr-Newman metric,[1] it was realized that this solution implies a gyromagnetic ratio $g=2$, the same value given by Dirac's relativistic quantum equation.[2] This result suggested that the spinning electron might be classically visualized as a massive, charged source of the Kerr-Newman field. This idea was developed by Israel,[3] who identified the electron with the equatorial disk spanning the ring singularity of the Kerr-Newman geometry. However, as pointed out by the same author,[3] this is not a physically realistic model for several reasons. (1) The material on the disk rotates at velocities exceeding or at least equal to the velocity of light. (2) The magnetic moment produced by the current distribution cannot be properly defined in the curved background space. (3) The edge of the disk carries an infinite amount of mass, charge, and angular momentum. (4) The ring is in fact a naked curvature singularity where tidal gravitational forces grow unboundedly. (5) As shown by Carter,[2] the spacetime off the disk exhibits a gross violation of causality, namely, given two arbitrary events, they can be connected by both a future- and a past-directed timelike curve.

In this paper we introduce a new classical extended model of the spinning electron, derived from the Kerr-Newman solution, which is free from all the above-mentioned inconsistencies. The electron is assumed to be a rigidly rotating charged shell of zero thickness with surface tension. This shell is defined by the equation $r=e^2/2M$, where r is the affine parameter along a congruence of principal null geodesics. At this particular r all gravitational potentials vanish, allowing the Kerr-Newman metric to be matched to the flat vacuum solution inside the shell. On the other hand, since the electromagnetic potentials are not constant on its surface, the shell must contain not only a distribution of charges but also a distribution of electric dipoles. This dipole layer eliminates the electromagnetic field inside the surface, so that Einstein-Maxwell equations hold everywhere. The electron is thus a bubble of flat spacetime immersed in the Kerr-Newman geometry. Its wall is actually the surface of an ellipsoid of revolution with a minor axis equal to the classical electron radius and a focal distance of the order of the Compton wavelength of this particle. The rotational velocity at the equator happens to be numerically equal to the eccentricity of the ellipsoid; it is therefore less than the velocity of light ($c=1$), albeit very close to it, owing to the remarkable oblateness of the shell. With the aid of the theory of distributions, the material stress-energy tensor of the source is determined. It involves a surface tension of the bubble, which is required to compensate the repulsive forces of the electric charges. The sign of the charges is opposite to the electron charge, with the exception of a narrow ribbon surrounding the equator. The magnetic dipole moment, originated in the rotary motion of the bubble, may be defined much the same as in special relativity, because the metric is Minkowskian right on the surface of the shell. The result of the calculations is identical to the value obtained from the asymptotic expression of the magnetic potential. Also, the model gives the right values for the electron mass, charge, and angular momentum. The oblateness of the bubble produces, in addition, an electric quadrupole moment proportional to the square of the Compton wavelength. The presence of this moment does not contradict the known experimental facts, however, since in quantum mechanics a particle of spin $\frac{1}{2}$ does not give rise to static quadrupole interactions with an external electric field.

Although our model also contains a naked singularity, because the gravitational field has a finite jump across the shell, it is of a rather harmless nature. Owing to the fact that the metric tensor is bounded, we are not dealing here with a curvature singularity, so that tidal gravitational forces do not grow indefinitely in approaching it.

Perhaps the most interesting property of the model introduced in the present paper is that the whole spacetime is causally well behaved. This circumstance occurs because the flat spacetime metric inside the bubble replaces the unphysical region of the Kerr-Newman geometry responsible for the breakdown of the causality principle.

II. DESCRIPTION OF THE MODEL

We begin by considering the line element of the Kerr-Newman geometry in Kerr-Schild coordinates, namely,[4]

$$ds^2=dt^2-dx^2-dy^2-dz^2-(r^4+a^2z^2)^{-1}r^2(2Mr-e^2)\{(r^2+a^2)^{-1}[r(x\,dx+y\,dy)-a(x\,dy-y\,dx)]+r^{-1}z\,dz+dt\}^2\ , \tag{1}$$

where r is the affine parameter along a congruence of principal null geodesics and is given by the positive real root of the equation

$$r^4-(x^2+y^2+z^2-a^2)r^2-a^2z^2=0\ . \tag{2}$$

A glance at Eq. (1) shows that all gravitational potentials vanish when the affine parameter is equal to one half the classical electron radius e^2/M, i.e.,

$$r=r_0\equiv e^2/2M\ . \tag{3}$$

Hence, the metric (1) may be joined continuously to the flat Minkowskian metric across the two-surface $t=\text{constant}, r=r_0$. This fact motivates the building up of a model for the source of the Kerr-Newman field consisting of a massive, charged shell coincident with this surface. Inside the bubble the electromagnetic potentials should also vanish to guarantee that the Einstein-Maxwell equations hold. From Eq. (2) one discovers the two-surface $r=r_0$ is actually an ellipsoid of revolution having a major axis equal to $2(r_0{}^2+a^2)^{1/2}$ and a minor axis identical to the classical electron radius.

Since the metric tensor is continuous across the shell, the theory of distributions may be applied to the search of the stress-energy tensor upon the bubble.[5] In fact, as was shown by Taub,[6] because the affine connection has only a finite jump across the shell, the nonlinear terms in the Einstein equations are well defined as distributions. The second derivatives of the metric tensor, in turn, develop a Dirac δ singularity with support on the shell, which is physically identified as a stress-energy surface tensor. On the other hand, the electromagnetic potential vector is discontinuous across the shell. However, since the Maxwell equations are still linear in a curved spacetime, they are well defined in the sense of distributions.

The calculations are done more easily when we adopt Boyer-Lindquist coordinates,[7] in terms of which the line element takes the form

$$\begin{aligned}ds^2=dt^2-\rho^2(\Delta^{-1}dr^2+d\theta^2)-(r^2+a^2)\sin^2\theta\, d\phi^2\\ -(2Mr-e^2)\rho^{-2}(dt-a\sin^2\theta\, d\phi)^2\end{aligned} \tag{4}$$

with

$$\rho^2=r^2+a^2\cos^2\theta\ , \tag{5}$$

$$\Delta=r^2-2Mr+a^2+e^2\ . \tag{6}$$

The associated electromagnetic potential is

$$A_i=-er\rho^{-2}(1;0,0,-a\sin^2\theta)\ . \tag{7}$$

To find the material and electromagnetic sources on the shell, we follow the procedure of Ref. 5; the stress-energy tensor $T_i{}^k$ and the current vector j^a are *defined* on the shell by the distribution-valued left-hand sides of the coupled Einstein-Maxwell equations

$$R_i{}^k-\tfrac{1}{2}\delta_i{}^kR\equiv 8\pi T_i{}^k\ , \tag{8}$$

$$F^{ab}{}_{;b}\equiv 4\pi j^a\ . \tag{9}$$

When the gravitational and electromagnetic potentials given by Eqs. (4) and (7) are inserted into the left-hand sides of Eqs. (8) and (9), and the derivatives are evaluated in the sense of distributions, the following result is obtained:

$$T_a{}^b=-\sigma u_a u^b+\delta_a{}^b\sigma\quad (a,b=0,2,3)\ , \tag{10}$$

$$j^k=qu^k\quad (k=0,1,2,3)\ , \tag{11}$$

where the velocity vector is given by

$$u^k=(r_0{}^2+a^2\cos^2\theta)^{-1/2}(r_0{}^2+a^2)^{-1/2}(r_0{}^2+a^2;0,0,a)\ , \tag{12}$$

satisfying the condition for material particles

$$u_ku^k=1\ . \tag{13}$$

The stress-energy tensor (10) has the same structure found in Ref. 5 for the material source upon the equatorial disk. Hence, the simple model introduced in that reference applies as well to the present case. It consists of a mixture of two perfect surface fluids, gas and dust, spinning as a whole with constant angular velocity. The energy densities of both fluids must be of the same magnitude but of opposite sign. The "gas" has in fact a negative pressure (positive tension), which is equal to

$$\sigma=(8\pi)^{-1}M\rho_0{}^{-1}(r_0{}^2+a^2)^{-1/2}\delta_{\text{shell}}\ , \tag{14}$$

where $\rho_0{}^2=r_0{}^2+a^2\cos^2\theta$ and δ_{shell} is defined by its value on a test function $\psi(r,\theta,\phi)$,[8] namely, by

$$\langle\delta_{\text{shell}},\psi\rangle=\int_{\text{shell}}\psi(r_0,\theta,\phi)d\Sigma_{\text{shell}}\ . \tag{15}$$

Here $d\Sigma_{\text{shell}}$ is the invariant element of the two-area of the shell in flat spheroidal coordinates, i.e.,

$$d\Sigma_{\text{shell}}=\rho_0(r_0{}^2+a^2)^{1/2}\sin\theta\, d\theta\, d\phi\ . \tag{16}$$

On the other hand, the surface charge density q has the form

$$\begin{aligned}q=(4\pi)^{-1}e\rho_0{}^4(r_0{}^2-a^2\cos^2\theta)\delta_{\text{shell}}\\ -(4\pi)^{-1}er_0\rho_0{}^{-2}\frac{\partial}{\partial r}\delta_{\text{shell}}\ ,\end{aligned} \tag{17}$$

where the derivative on the second term is defined as[8]

$$\left\langle \frac{\partial}{\partial r}\delta_{\text{shell}},\psi\right\rangle = -\int_{\text{shell}} \frac{\partial}{\partial r}\psi(r,\theta,\phi)\bigg|_{r=r_0} d\Sigma_{\text{shell}} . \tag{18}$$

According to Eq. (17), the density q consists of a surface distribution of charges and an outward-pointing dipole distribution. Owing to the fact that their moments point over all directions, these dipoles will essentially affect only a small neighborhood of the bubble. Referring to the charge distribution, we will see later on that $r_0^2 \ll a^2$, so that the factor $r_0^2 - a^2\cos^2\theta$ is negative throughout the surface, with the exception of a narrow ribbon embracing the equator.

From Eq. (12) we obtain the angular velocity of the bubble

$$\omega \equiv u^3/u^0 = a(r_0^2+a^2)^{-1} , \tag{19}$$

whence we infer the value of the linear velocity

$$v = (r_0^2+a^2)^{-1/2} a \sin\theta , \tag{20}$$

which is less than the velocity of light ($c=1$) all over the shell.

The total gravitational energy contained both on the shell and in the electromagnetic field is given by Tolman's formula[9]

$$U = \langle T_0{}^0 - T_1{}^1 - T_2{}^2 - T_3{}^3, \psi^* \rangle , \tag{21}$$

where ψ^* is a test function equal to 1 over the whole three-space, with the exception of an arbitrarily small neighborhood of infinity. Here, $T_i{}^k$ is given by Eq. (10) upon the bubble and by Maxwell's tensor in electrovacuum. A straightforward integration gives

$$U = M . \tag{22}$$

Following a similar procedure, the total angular momentum J is evaluated

$$J = \langle T_3{}^0, \psi^* \rangle = -Ma . \tag{23}$$

Similarly, the electric charge Q is obtained from the formula

$$Q = \langle j^0, \psi^* \rangle , \tag{24}$$

where, in this case, $\psi^* = 1$ in a neighborhood of the shell. Thus, the dipole term in Eq. (17) gives no contribution and the integral (24) leads to the result

$$Q = e . \tag{25}$$

Next, we calculate the magnetic moment produced by the distribution of currents (11). To this end we use the same expression valid in flat spacetime, namely,[10]

$$\mu = -\tfrac{1}{2}\langle j_3, \psi^* \rangle , \tag{26}$$

where the covariant component of the current is calculated from the metric (4) by putting $r = r_0$. We thus obtain

$$j_3 = -(r_0^2+a^2)\sin^2\theta\, j^3 . \tag{27}$$

Again the dipole layer does not contribute and the result of the integration is

$$\mu = ea , \tag{28}$$

in accord with the value derived in Ref. 2 from the asymptotic expression of A_3. As is well known, Eqs. (23) and (28) reproduce the gyromagnetic ratio given by Dirac's equation

$$\mu/|J| = e/M . \tag{29}$$

The flattening of the spheroid at the poles implies also the existence of an electric quadrupole moment D, which is defined by

$$D = \langle (2z^2 - x^2 - y^2) j^0, \psi^* \rangle , \tag{30}$$

where the Cartesian coordinates x,y,z are related to the flat spheroidal coordinates by the equations

$$z^2 = r^2\cos^2\theta, \quad x^2+y^2 = (r^2+a^2)\sin^2\theta . \tag{31}$$

When we insert these relations into Eq. (30) and perform the integration, we arrive at the result

$$D = -2ea^2 . \tag{32}$$

The presence of this quadrupole moment might seem to contradict the known atomic spectroscopical evidence. However, according to quantum mechanics, the quadrupole moment of a particle of spin $\frac{1}{2}$ does not show up in static interactions with an external electric field.[11] This property may be viewed as a result of the uncertainty principle, which prevents the precise location of the angular momentum vector. In fact, for a particle with $J_z = \frac{1}{2}\hbar$, the vector $\vec{J}$ is so slightly aligned with the z axis that the expectation value of the quadrupole moment vanishes.

When we substitute in Eq. (23) the known value $\hbar[\frac{1}{2}(\frac{1}{2}+1)]^{1/2}$ for the electron spin, we obtain for the focal distance of the ellipsoid the following expression:

$$2a = \sqrt{3}\lambdabar_e , \tag{33}$$

where $\lambdabar_e$ is the electron Compton wavelength. In consequence of this relation, the quadrupole moment (32) can be written as

$$D = -6e\lambdabar_e^2 , \tag{34}$$

and the eccentricity ϵ of the ellipsoid becomes

$$\epsilon = [1+(r_0/a)^2]^{-1/2} = (1+\tfrac{1}{3}\alpha^2)^{-1/2} , \tag{35}$$

where $\alpha = e^2/\hbar$ is the fine-structure constant. The shape of the electron is thus independent of its mass. The bubble looks like a thin disk whose radius is nearly equal to the Compton wavelength. From Eq. (20) it follows that the eccentricity is identical to the rotational velocity of the bubble at the equator. Its numerical value is slightly less than the speed of light,[12] namely,

$$v_{\max} = \epsilon = 0.999\,991\ldots . \tag{36}$$

An interesting feature we want to stress here is that the classical radius, as well as the quantum radius (Compton wavelength), refer to magnitudes specifying the shape of the electron. Thus, it seems incorrect to state the classical radius is meaningless, based on the fact that it is more than one hundred times smaller than the region where quantum effects become important.[13]

The surface tension σ, given by Eq. (14), corresponds to the nonelectromagnetic stresses postulated *ad hoc* in 1905 by Poincaré[14] to compensate the Coulomb repulsion. This identification is easily understood in the particular case of a nonspinning charged particle, which occurs when $a=0$ in Eq. (14). The magnitude of the surface tension reduces then to the constant value

$$\sigma=M(8\pi r_0{}^2)^{-1} , \tag{37}$$

whence, on account of Eq. (3), it follows that

$$r_0{}^3=e^2/16\pi\sigma , \tag{38}$$

which is the well-known equilibrium condition for a charged bubble with surface tension in flat spacetime.[15]

III. DISCUSSION

The Kerr-Newman geometry (1) describes the field of a naked singularity when the following relation holds:

$$a^2+e^2>M^2 . \tag{39}$$

The numerical values of these quantities for the electron are

$$a^2\approx 10^{-22}\ \text{cm}^2,\ \ e^2\approx 10^{-68}\ \text{cm}^2, \\ M^2\approx 10^{-110}\ \text{cm}^2 , \tag{40}$$

so that the e^2 term in Eq. (39) exceeds M^2 for more than 40 orders of magnitude. Therefore, the intensity of the gravitational field is negligible compared to the electromagnetic field all the way down to the surface of the electron. Besides, the magnitude of the affine parameter r on the bubble is sufficiently high to ensure the validity of the causality principle in our model.[2] To prove this assertion, we consider the metric coefficient of $d\phi^2$ in the line element (4), namely,

$$g_{33}=-(r^2+a^2)\sin^2\theta-\rho^{-2}(2Mr-e^2)a^2\sin^4\theta . \tag{41}$$

Outside the bubble $r>e^2/2M$, so that g_{33} never changes its sign and, consequently, there are no closed timelike lines in the whole spacetime.[2]

We have not yet touched on the stability of the model under arbitrary small perturbations of the bubble. A similar problem in special relativity was considered by Gnädig, Kunszt, Hasenfratz, and Kuti.[15] They studied the stability of a classical electron model without spin, introduced by Dirac in 1962.[16] The electron was represented by a charged, conducting spherical bubble with surface tension. These authors found that Dirac's model is unstable under quadrupole deformations that change the sphere into a prolate spheroid. On the other hand, the sphere is stable under small perturbations leading to an oblate spheroid. In Ref. 15, it is suggested that the inclusion of spin might stabilize the model. This idea seems quite reasonable in light of the findings of our work, which shows that the presence of spin implies a remarkable flattening of the bubble at the poles. A rigorous proof of this conjecture is beyond the scope of the present work.

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