# Einstein-Weyl field equations in a Bianchi type-IX space-time

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It is proved that there exists no solution of the combined gravitational-neutrino field equations in general relativity if the space-time metric admits a group of isometries of Bianchi type IX and the neutrino field has geodesic and shearfree rays.

## I. INTRODUCTION

In general relativity, the interaction of a classical spin-$\frac{1}{2}$ Dirac field with a space-time admitting a three-parameter group of isometries was investigated by Michalik and Melvin,[1] Ray,[2] Henneaux,[3] Obregon and Ryan,[4] and Jantzen.[5] In Refs. 3—5 the nonzero-rest-mass Dirac field case is studied with the use of Hamiltonian techniques. In particular, in Ref. 3 the general solution of the Einstein-Dirac field equations for a Bianchi type-I metric is obtained. In Refs. 1 and 2 the neutrino field case is studied and exact (but not general) solutions of Bianchi types V and $V_h$ are obtained. The stationary axially symmetric solution of the Einstein-Weyl field equations, presented recently by Kolassis,[6] has a four-parameter group of isometries which contains a subgroup of Bianchi type II.

Here we consider a neutrino field with geodesic and shearfree rays coupled via the Einstein field equations to a Bianchi type-IX metric. From the mathematical point of view the hypothesis of geodesic and shearfree rays constitutes a restriction on the Einstein-Weyl field equations so that our investigations seem to be not quite general. However, this hypothesis results from physical considerations. Indeed, Wainwright[7] has proved that a neutrino field satisfying the weak energy condition $E_2$ necessarily has geodesic and shearfree rays. The weak energy condition $E_2$ is stated as follows: A field is said to satisfy the weak energy condition $E_2$ if its energy flow vector $T_{\mu\nu}u^\nu$ is timelike or null for all observers with four-velocity vector $u^\nu$ at each event for which $T_{\mu\nu}\neq 0$. The Newman-Penrose formalism is used throughout.

## II. THE FIELD EQUATIONS

The Einstein-Weyl field equations governing the interaction of a neutrino field with a gravitational field read

$$\sigma^{\mu}{}_{A\dot{X}}\xi^{A}{}_{;\mu}=0\ , \tag{2.1}$$

$$R_{\mu\nu}=-T_{\mu\nu}\ , \tag{2.2}$$

where

$$T_{\mu\nu}=\frac{i}{4}(\sigma_{\mu}^{A\dot{X}}\xi_A\bar{\xi}_{\dot{X};\nu}+\sigma_{\nu}^{A\dot{X}}\xi_A\bar{\xi}_{\dot{X};\mu}-\text{c.c.}) \tag{2.3}$$

is the symmetrized energy-momentum tensor of the neutrino field $\xi^A$. Here, capital latin indices are two-spinor indices. To treat the above equations it is convenient to introduce a two-spinor $\chi^A$ so that with $\xi^A$ it forms a spinor frame. This spinor frame gives rise to the adapted null tetrad

$$l^{\mu}=\sigma^{\mu}{}_{A\dot{X}}\xi^A\bar{\xi}^{\dot{X}}\ , \tag{2.4a}$$

$$k^{\mu}=\sigma^{\mu}{}_{A\dot{X}}\chi^A\bar{\chi}^{\dot{X}}\ , \tag{2.4b}$$

$$m^{\mu}=\sigma^{\mu}{}_{A\dot{X}}\xi^A\bar{\chi}^{\dot{X}}\ , \tag{2.4c}$$

$$\bar{m}^{\mu}=\sigma^{\mu}{}_{A\dot{X}}\chi^A\bar{\xi}^{\dot{X}}\ , \tag{2.4d}$$

where $l^\mu$ is the neutrino flux vector. The adapted null tetrad is determined up to the so-called null rotations about $l^\mu$ given by

$$l^{\mu}=l'^{\mu}\ , \tag{2.5a}$$

$$k^{\mu}=k'^{\mu}+\psi m'^{\mu}+\bar{\psi}\bar{m}'^{\mu}+\psi\bar{\psi}l'^{\mu}\ , \tag{2.5b}$$

$$m^{\mu}=m'^{\mu}+\bar{\psi}l'^{\mu} \tag{2.5c}$$

with $\psi$ any complex function of the coordinates. The hypothesis that the neutrino field has geodesic and shearfree rays implies that the spin coefficients $\kappa$ and $\sigma$ associated with the adapted null tetrad vanish,

$$\kappa=0\ , \tag{2.6}$$

$$\sigma=0\ . \tag{2.7}$$

In the Newman-Penrose notation the spinor-frame components of the Einstein-Weyl field equations (2.1) and (2.2) are written

$$\rho=\epsilon\ , \tag{2.8}$$

$$\beta=\tau\ , \tag{2.9}$$

and

$$\Phi_{00}=0\ , \tag{2.10a}$$

$$\Phi_{01}=0\ , \tag{2.10b}$$

$$\Phi_{02}=0\ , \tag{2.10c}$$

$$\Phi_{11}=(i/8)(\bar{\rho}-\rho)\ , \tag{2.10d}$$

$$\Phi_{12}=(i/8)(\bar{\alpha}-2\tau)\ , \tag{2.10e}$$

$$\Phi_{22}=(i/4)(\bar{\gamma}-\gamma)\ . \tag{2.10f}$$

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The space-time is assumed to be of Bianchi type IX. The canonical basis elements of the Lie algebra of the transitive group of isometries are denoted by $\{n_i^\mu, i=1,2,3\}$ and satisfy the relation

$$\mathscr{L}_{n_i} n_j^\mu = \epsilon^k{}_{ij} n_k^\mu \ , \tag{2.11}$$

where $\epsilon^k{}_{ij}$ is the Levi-Civita symbol.

From a theorem proved recently[8] which concerns the inheritance of space-time symmetries by the neutrino field it follows that the adapted null tetrad obeys the equations

$$\mathscr{L}_{n_i} l^\mu = 0 \ , \tag{2.12a}$$

$$\mathscr{L}_{n_i} k^\mu = -r_i m^\mu - \bar{r}_i \bar{m}^\mu \ , \tag{2.12b}$$

$$\mathscr{L}_{n_i} m^\mu = -\bar{r}_i l^\mu - i s_i m^\mu \ , \tag{2.12c}$$

where $s_i$ are real constants and $r_i$ are complex functions of the coordinates. Equation (2.11) can also be written in the operator form

$$\mathscr{L}_{n_i}\mathscr{L}_{n_j} - \mathscr{L}_{n_j}\mathscr{L}_{n_i} = \epsilon^k{}_{ij}\mathscr{L}_{n_k} \ .$$

Acting on $m^\mu$ by means of this equation and using Eqs. (2.12a) and (2.12c), we obtain

$$s_i = 0 \ . \tag{2.13}$$

The vector fields $n_i^\mu$ can be expanded in terms of the adapted null tetrad as

$$n_i^\mu = a_i l^\mu + b_i k^\mu - c_i m^\mu - \bar{c}_i \bar{m}^\mu \ . \tag{2.14}$$

Inserting Eq. (2.14) into Eqs. (2.11) and (2.12a)–(2.12c) and taking into account Eq. (2.13), we obtain

$$\mathscr{L}_{n_i} a_j + c_j \bar{r}_i + \bar{c}_j r_i = \epsilon^k{}_{ij} a_k \ , \tag{2.15a}$$

$$\mathscr{L}_{n_i} b_j = \epsilon^k{}_{ij} b_k \ , \tag{2.15b}$$

$$\mathscr{L}_{n_i} c_j + b_j r_i = \epsilon^k{}_{ij} c_k \tag{2.15c}$$

and

$$b_{i,\mu} = n_i^\nu (l_{\nu;\mu} - l_{\mu;\nu}) \ , \tag{2.16a}$$

$$a_{i,\mu} = n_i^\nu (k_{\nu;\mu} - k_{\mu;\nu}) - r_i m_\mu - \bar{r}_i \bar{m}_\mu \ , \tag{2.16b}$$

$$\bar{c}_{i,\mu} = n_i^\nu (m_{\nu;\mu} - m_{\mu;\nu}) - \bar{r}_i l_\mu \ . \tag{2.16c}$$

With the help of Eqs. (2.16a)–(2.16c), (2.6)–(2.9), and the expansions of $l_{\mu;\nu}, k_{\mu;\nu}, m_{\mu;\nu}$ on the adapted null tetrad,[9] Eqs. (2.15a)–(2.15c) can be written

$$\vec{a} = \gamma \vec{a}\wedge\vec{b} + (\bar{\alpha}+\beta-\bar{\pi})\vec{c}\wedge\vec{a} + \nu\vec{b}\wedge\vec{\bar{c}} + \mu\vec{\bar{c}}\wedge\vec{c}$$

$$-\vec{c}\wedge\vec{\bar{r}} + \text{c.c.} \ , \tag{2.17a}$$

$$\vec{b} = \rho\vec{a}\wedge\vec{b} + \alpha\vec{b}\wedge\vec{\bar{c}} + \rho\vec{\bar{c}}\wedge\vec{c} + \text{c.c.} \ , \tag{2.17b}$$

$$\vec{c} = (\pi+\bar{\tau})\vec{a}\wedge\vec{b} + \rho\vec{c}\wedge\vec{a} + \lambda\vec{b}\wedge\vec{\bar{c}}$$

$$+(\bar{\gamma}-\gamma+\mu)\vec{b}\wedge\vec{c} + (\bar{\beta}-\alpha)\vec{c}\wedge\vec{\bar{c}} + \vec{r}\wedge\vec{b} \ . \tag{2.17c}$$

The arrow indicates three-tuplets, e.g., $\vec{c} \equiv (c_1, c_2, c_3)$. The symbol $\wedge$ denotes the exterior product of two three-tuplets. Using the intrinsic derivatives $D, \Delta, \delta, \bar{\delta}$ associated with the adapted null tetrad, Eqs. (2.16a)–(2.16c) can be written equivalently,

$$D\vec{b} = (\rho+\bar{\rho})\vec{b} \ , \tag{2.18a}$$

$$\Delta\vec{b} = -(\rho+\bar{\rho})\vec{a} + \bar{\alpha}\vec{c} + \alpha\vec{\bar{c}} \ , \tag{2.18b}$$

$$\delta\vec{b} = \bar{\alpha}\vec{b} + (\rho-\bar{\rho})\vec{\bar{c}} \ ; \tag{2.18c}$$

$$D\vec{a} = (\gamma+\bar{\gamma})\vec{b} + (\bar{\pi}-\bar{\alpha}-\beta)\vec{c} + (\pi-\alpha-\bar{\beta})\vec{\bar{c}} \ , \tag{2.19a}$$

$$\Delta\vec{a} = -(\gamma+\bar{\gamma})\vec{a} + \bar{\nu}\vec{c} + \nu\vec{\bar{c}} \ , \tag{2.19b}$$

$$\delta\vec{a} = (\bar{\pi}-\bar{\alpha}-\beta)\vec{a} + \bar{\nu}\vec{b} + (\mu-\bar{\mu})\vec{\bar{c}} + \vec{\bar{r}} \ ; \tag{2.19c}$$

$$D\vec{c} = (\pi+\bar{\tau})\vec{b} - \rho\vec{c} \ , \tag{2.20a}$$

$$\Delta\vec{c} = -(\pi+\bar{\tau})\vec{a} + (\bar{\gamma}-\gamma+\mu)\vec{c} + \lambda\vec{\bar{c}} - \vec{r} \ , \tag{2.20b}$$

$$\delta\vec{c} = -\rho\vec{a} + (\bar{\gamma}-\gamma+\mu)\vec{b} + (\alpha-\bar{\beta})\vec{\bar{c}} \ , \tag{2.20c}$$

$$\bar{\delta}\vec{c} = \lambda\vec{b} + (\bar{\beta}-\alpha)\vec{c} \ . \tag{2.20d}$$

From the linear independence of the Killing vector fields $n_i^\mu$ it follows that

$$\text{rank}\{\vec{a}, \vec{b}, (\tfrac{1}{2})(\vec{c}+\vec{\bar{c}}), (i/2)(\vec{\bar{c}}-\vec{c})\} = 3 \ . \tag{2.21}$$

In order that the neutrino field be compatible with a Bianchi type-IX metric the solution of the differential Eqs. (2.18a)–(2.20d) must satisfy conditions (2.17a)–(2.17c) and (2.21). In the next section we give the proof of the following theorem.

"*There exist no solutions of the Einstein-Weyl field equations in a Bianchi type-IX space-time for a neutrino field with geodesic and shearfree rays.*"

## III. PROOF

In the sequel the various restrictions satisfied by the spin coefficients, the Einstein field equations (2.10a)–(2.10f), and the equations (2.18a)–(2.20d) are often used without explicit reference. In particular, the spin coefficients $\epsilon$ and $\beta$ are replaced in all equations by their equals $\rho$ and $\tau$, respectively. For the Ricci and Bianchi identities we use as in Refs. 6 and 8 the following notation: by (R.$i$), where $i=1,\ldots,18$ [(B.$j$) where $j=1,\ldots,11$], we denote the $i$th Ricci identity [the $j$th Bianchi identity] in the order of the listing given, e.g., by Pirani.[10]

At first we observe that necessarily

$$\rho - \bar{\rho} \neq 0 \ , \tag{3.1}$$

because, if $\rho-\bar{\rho}=0$, the scalar multiplication of Eq. (2.17b) by $\vec{b}$ gives $\vec{b}=\vec{0}$, and then, by scalar multiplication of Eq. (2.17c) by $\vec{\bar{c}}$, we obtain $\vec{c}=\vec{0}$; finally, Eq. (2.17a) yields $\vec{a}=\vec{0}$ so that $n_i^\mu=0$.

By virtue of Eq. (3.1) the adapted null tetrad can now be chosen so that

$$\alpha - 2\bar{\tau} = 0 \ . \tag{3.2}$$

This choice of the null tetrad implies[11]

$$\vec{r} = \vec{0} \ . \tag{3.3}$$

From Eqs. (R.5) and (R.11) it follows that

$$D\tau=(2\rho-\bar{\rho})\tau+\bar{\pi}\rho \ . \tag{3.4}$$

By scalar multiplication of Eq. (2.17b) by $\vec{c}$ and Eq. (2.17c) by $\vec{b}$ and comparison between the resulting equations, we obtain

$$\bar{\rho}(\vec{a}\wedge\vec{b})\vec{c}=\bar{\tau}(\vec{c}\wedge\bar{\vec{c}})\vec{b} \ . \tag{3.5}$$

By virtue of Eqs. (R.1), (3.3), and (3.4) and the conditions (2.21) and (3.1), the $D$ derivative of Eq. (3.5) yields

$$\vec{a}=B\vec{b} \ , \tag{3.6}$$

$$\tau=0 \ , \tag{3.7}$$

$$\pi=0 \ , \tag{3.8}$$

where $B$ is a real function of the coordinates. Using Eq. (3.7) and the condition (3.1), we get from Eq. (R.16)

$$\lambda=0 \ . \tag{3.9}$$

Inserting Eq. (3.6) into Eq. (2.17a) and taking into account Eqs. (2.17b), (3.3), and (2.21) we obtain

$$\nu=0 \ , \tag{3.10}$$

$$\mu-\bar{\mu}=B(\rho-\bar{\rho}) \ . \tag{3.11}$$

By scalar multiplication of Eq. (2.17c) by $\bar{\vec{c}}$ and taking the purely imaginary part of the resulting equation, we obtain

$$\mu+\bar{\mu}=B(\rho+\bar{\rho}) \ ,$$

which, with Eq. (3.11), yields

$$\mu=B\rho \ . \tag{3.12}$$

The $D$ derivative and $\Delta$ derivative of Eq. (3.6), respectively, yield

$$DB=\gamma+\bar{\gamma}-B(\rho+\bar{\rho}) \ , \tag{3.13}$$

$$\Delta B=B^2(\rho+\bar{\rho})-B(\gamma+\bar{\gamma}) \ . \tag{3.14}$$

From Eqs. (R.8), (R.12), (R.14), and (R.17), we obtain

$$D\mu=\rho(\mu-\bar{\mu})+\gamma(\rho-\bar{\rho})+\Phi_{11} \ , \tag{3.15}$$

$$\Delta\mu=-\mu(\gamma+\bar{\gamma}+\mu)-\Phi_{22} \ , \tag{3.16}$$

$$\Delta\rho=\gamma\bar{\rho}+\bar{\gamma}\rho-2\rho\mu-\Phi_{11} \ . \tag{3.17}$$

By using Eqs. (R.1), (3.13), (3.15), and the $D$ derivative of Eq. (3.12), we obtain

$$\Phi_{11}=\gamma\bar{\rho}+\bar{\gamma}\rho+\rho\bar{\mu} \ . \tag{3.18}$$

By using Eqs. (3.14), (3.16), (3.17), (3.18), and the $\Delta$ derivative of Eq. (3.12), we obtain

$$\gamma=\bar{\gamma} \ . \tag{3.19}$$

By virtue of this last equation, Eqs. (2.17c), (3.3), (3.6), (3.12), and the other restrictions satisfied by the spin coefficients, it follows that

$$\vec{c}=\vec{0} \ . \tag{3.20}$$

Clearly, this equation is in contradiction to (2.21) and so the proof of the theorem is now completed.

## IV. DISCUSSION

The nonexistence of Bianchi type-IX solutions of the Einstein-Weyl field equations is a generalization of similar results concerning spherical symmetry which were obtained by Trim and Wainwright,[12] Audretsch,[13] and Griffiths.[14] In particular, Griffiths, without the use of the assumption of geodesic and shearfree rays, proved the nonexistence of nonsingular solutions of the Einstein-Weyl field equations in a spherically symmetric space-time.

It is worthwhile to note that the theorem which is proved here holds even if a cosmological term is introduced in the Einstein field equations. The proof of this is similar to that given in Sec. III and is left to the interested reader.

The experimental observation seems to show that only left-handed neutrinos exist and this is in accordance with Weyl's equation which describes fields with definite helicity. However, it is interesting to note that one of the main reasons for the nonexistence of Bianchi type-IX solutions of the Einstein-Weyl equations seems to be this restriction on the helicity. In fact, Soares and Novello[15] proved that a spherically symmetric space-time is compatible only with a neutrino field with nondefinite helicity.

A corollary of the theorem we proved here is that the $k=1$ Friedmann-Robertson-Walker metric is incompatible with a neutrino field with geodesic and shearfree rays. This is in accordance with a result obtained by Isham and Nelson[16] and later in a more general context also by Obregon and Ryan.[4] It must be pointed out that although the above authors consider the general case of a nonzero-rest-mass Dirac field, they made the assumption for the Dirac field to be homogeneous, i.e., to inherit space-time symmetries. But it is not at all clear if this assumption is a consequence of the Einstein-Dirac field equations or if it constitutes a restriction so strong that the solution of the Dirac field equation collapses to zero.

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