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New approach to the quasinormal modes of a black hole

Valeria Ferrari
Istituto di Fisica G. Marconi, Università di Roma, Piazzale Aldo Moro 2, 00185 Roma, Italy

Bahram Mashhoon
Institut für Theoretische Physik, Universität zu Köln, D-5000 Cologne 41, Federal Republic of Germany
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We describe a new analytic approach to the problem of black-hole oscillations, which has been investigated numerically thus far. Our treatment is based on a connection between the quasinormal modes and the bound states of the inverted black-hole effective potentials. Approximate analytic formulas for the quasinormal frequencies of Schwarzschild, Reissner-Nordström, and slowly rotating Kerr black holes are provided. We find that a real quasinormal frequency for an extreme Kerr black hole has vanishing amplitude in the ordinary (i.e., nonsuperradiant) regime; therefore, extreme Kerr black holes are *not* marginally *unstable* in this case. These results are significant for the question of the stability of a black hole as well as for the late-time behavior of radiation from gravitationally collapsing configurations.

I. INTRODUCTION

The question of the stability of a black hole was first treated in the pioneering work of Regge and Wheeler,[1] who investigated the linear perturbations of the exterior Schwarzschild spacetime. Further work on this problem[2] led to the study of quasinormal modes (QNM's) and their role in the response of a black hole to external perturbations. Extensive numerical work over the past decade[3] has shown that the response of a black hole is generally dominated at late times by certain damped oscillations. External perturbations excite the QNM's which in turn appear as damped vibrations in the black-hole response. These oscillations are intrinsic characteristics of the exterior geometry and can, in principle, help uniquely identify a black hole. Thus the QNM's represent the imprint of a black hole on its response to external perturbations. Even for systems undergoing gravitational collapse, the gravitational radiation spectrum at the late stages of collapse may be dominated by the quasinormal oscillations.[4] Thus, in addition to their importance for the analysis of the stability of black holes, QNM's may play a significant role in the continuing search for gravitational radiation and black holes.

To study the QNM's analytically, a method has been developed[5] which establishes a link between the QNM's and the bound states of the inverted black-hole potentials. A description of the general properties of the QNM's on the basis of this method is presented in Sec. II. Analytical estimates for the bound states of the inverted Schwarzschild and Reissner-Nordström curvature potentials are used in Secs. III and IV to give approximate analytical formulas for the quasinormal frequencies and wave functions. The problem of QNM's for the exterior Kerr spacetime is discussed in Sec. V. Rotation removes the degeneracy of the modes of a spherical black hole. A formula is derived for the splitting of the quasinormal frequencies in the limit of slow rotation. Furthermore, we show that the quasinormal frequencies of a Kerr black hole cannot be real, except possibly in the superradiant case. Thus, the suggestion that an extreme Kerr black hole is, in some sense, marginally unstable[6] cannot be maintained. In fact, in Sec. VI we give a detailed analysis of certain null rays in the Kerr geometry to show that an extreme Kerr black hole is stable in the eikonal approximation.

II. GENERAL PROPERTIES OF QNM's

Small-amplitude perturbations of the exterior field of black-hole spacetimes lead to linear second-order partial differential equations which may be separated completely for the Schwarzschild, Reissner-Nordström, and Kerr geometries.[7] This means that the wave amplitude for a general perturbation of integer spin s ($s=0,1$, or 2, say) may be expressed as a sum of simple modes (of frequency ω and angular momentum parameters j and m) of the form

$$\Psi=e^{i(\omega t-m\varphi)}\mathscr{S}_{\omega jms}(\theta)\mathscr{R}_{\omega jms}(r) , \tag{1}$$

where $j\geq s$ and $-j\leq m\leq j$. Let us introduce the generalized Regge-Wheeler coordinate x, and set

$$\psi(x)=\mathscr{R}_{\omega jms}(r) . \tag{2}$$

Then ψ satisfies a "Schrödinger" equation with a *real* potential which depends in general on frequency[8]

$$\frac{d^2\psi}{dx^2}+[\omega^2-V(x;\omega)]\psi=\mathscr{T}(x;\omega) . \tag{3}$$

Let us first consider a spherical black hole. In this case the effective potential $U(x)$ is independent of frequency and $U\rightarrow 0$ as $x\rightarrow\pm\infty$. The quasinormal modes are defined to be the solutions of the homogeneous form of (3) with the boundary conditions

$$\psi\propto\exp(\mp i\omega x) \quad \text{as } x\rightarrow\pm\infty , \tag{4}$$

which correspond to outgoing waves at infinity and ingoing waves at the horizon. It follows from flux conservation that for a QNM ω must be complex, $\omega=\omega^0+i\Gamma$, with $\omega^0\neq 0$. Moreover, if the "average" value of the potential is positive for a QNM, i.e.,

$$\int_{-\infty}^{\infty} U(x)\,|\psi|^2 dx \Big/ \int_{-\infty}^{\infty} |\psi|^2 dx \geq 0 \,, \tag{5}$$

then $\Gamma>0$. This is simply shown by multiplying the source-free form of Eq. (3) by ψ^* and integrating the result from $x=-\infty$ to $+\infty$. A contradiction is encountered if Eq. (5) is satisfied and Γ is negative. For a spherical black hole condition (5) is always satisfied since $U\geq 0$; hence, the QNM's are confined to the upper half of the complex frequency plane. It is clear, therefore, that for a QNM, $|\psi|\rightarrow\infty$ as $x\rightarrow\pm\infty$, and so the QNM's cannot be considered *perturbations* of a black hole. A more general description of QNM's is obtained if we consider the problem of reflection and transmission of waves by a potential (Fig. 1). It is clear that the reflection amplitude $R(\omega)=R^*(-\omega)$ is always finite for real ω, $|R(\omega)|\leq 1$, by flux conservation. By extending the problem to the complex frequency domain, we find that the QNM's correspond to the singularities of the reflection amplitude $R(z)$,

$$R(z)=R^*(-z^*) \,, \tag{6}$$

such that $\mathrm{Re}(z)\neq 0$ and $T(z)/R(z)$ is regular (and nonzero). These constraints would ensure that the boundary conditions for the QNM's are satisfied. It follows from Eq. (6) that the quasinormal frequencies are symmetrically distributed with respect to the imaginary axis in the complex plane.

There exists a close connection between the singularities of the scattering amplitude and the bound states.[9] In fact, for a potential *well* it is immediately clear that the singularities of $R(\omega)$ for $\omega=\omega^0+i\Gamma$, $\omega^0=0$, and $\Gamma\leq 0$, correspond to the bound states of the potential *well.* For a potential *barrier,* however, the connection is not evident. We wish to show that the QNM's of a potential barrier are related to the bound states of the *inverted* potential. Let p be a set of parameters associated with the potential. These may belong to the potential already, or they may be simply introduced as scaling parameters. The

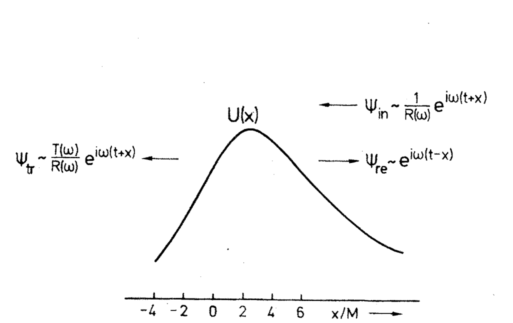

FIG. 1. The scattering and absorption of an incident gravitational wave from spatial infinity by a Schwarzschild black hole. The angular eigenfunctions are suppressed here. $R(\omega)$ and $T(\omega)$ are the reflection and transmission amplitudes, respectively.

parametrized potential is denoted by $U(x;p)$; the wave functions and the quasinormal frequencies are also functions of the parameters $p:\psi=\psi(x;p)$ and $\omega=\omega(p)$. Consider the formal transformations $x\rightarrow -ix$ and $p\rightarrow p'=\Pi(p)$ in such a way that the potential remains invariant,

$$U(-ix;p')=U(x;p) \,. \tag{7}$$

Let us define ϕ and Ω such that

$$\phi(x;p)=\psi(-ix;p') \tag{8}$$

and

$$\Omega(p)=\omega(p') \,. \tag{9}$$

Then ϕ satisfies the Schrödinger equation

$$\frac{d^2\phi}{dx^2}+(-\Omega^2+U)\phi=0 \,, \tag{10}$$

and the boundary conditions for the QNM's are reduced to

$$\phi(x;p)\propto \exp(\mp\Omega x) \,, \quad \text{as } x\rightarrow\pm\infty \,. \tag{11}$$

The $\Omega(p)$ are in general *complex;* however, it is clear that for a *real* $\Omega(p)\geq 0$, Eqs. (10) and (11) correspond to the Schrödinger equation with the potential $-U$ and the proper boundary conditions for bound states. Once $\Omega(p)$ is determined, the QNM's are found by the inverse transformation

$$\omega(p)=\Omega(\Pi^{-1}(p)) \text{ and } \psi(x;p)=\phi(ix;\Pi^{-1}(p)) \,. \tag{12}$$

Finally, the parameters p may be set equal to their original values; thereby, the QNM's associated with the potential $U(x)$ are determined. The QNM's which are directly related, through Eq. (12), to the true bound states of the inverted potential will be referred to as *proper* QNM's. The spectrum of the bound states of Eq. (10) is discrete; therefore, the proper QNM's form a discrete set in the complex frequency plane.

These results may be simply extended to potentials that do not vanish as $x\rightarrow\pm\infty$, such as the inverted harmonic-oscillator potential or the Eckart potential.[10]

Let us now consider a real effective potential which is explicitly dependent upon the frequency as in the case of a Kerr black hole.[8] Suppose that $V(x;\omega)\rightarrow 0$ as $x\rightarrow+\infty$ and $V(x;\omega)\rightarrow V_h(\omega)$ as $x\rightarrow-\infty$. Thus, for a wave incident from infinity with frequency ω the transmitted wave is of the form $\exp(i\omega t+ikx)$, where $k^2=\omega^2-V_h(\omega)$,

$$|R|^2+\frac{k}{\omega}|T|^2=1 \,, \tag{13}$$

so that $|R|\leq 1$ in the ordinary case ($k/\omega\geq 0$) and $|R|>1$ in the case of superradiance ($k/\omega<0$). There is complete reflection if k is imaginary. For a Kerr black hole k is simply given by $k=\omega-ma/(2Mr_+)$, where r_+ is the "radius" of the outer horizon, $r_+=M+(M^2-a^2)^{1/2}$. This follows from the fact that[8] $V_h(\omega)=2m\omega\Omega_h-m^2\Omega_h{}^2$, where $\Omega_h=a/(2Mr_+)$ is the horizon frequency,[11] and of the two possibilities $k=\pm(\omega-m\Omega_h)$ only $k=\omega-m\Omega_h$ has the correct limiting form for

$a \to 0$. The QNM's are defined to be the solutions of the homogeneous form of Eq. (3) which correspond to outgoing waves at spatial infinity as before, and to (physically) ingoing waves at the horizon,

$$\psi \propto \exp(ikx) \quad \text{as } x \to -\infty \ , \tag{14}$$

where $\mathrm{Re}(k)/\mathrm{Re}(\omega)$ is positive in the ordinary case and is negative in the superradiant case. Thus the QNM's correspond, again, to the singularities of the reflection amplitude $R_{jms}(z)$ such that $\mathrm{Re}(z) \neq 0$ and $T_{jms}(z)/R_{jms}(z)$ is regular (and nonzero). Since for a real frequency ω, $V_{jms}(x;\omega) = V_{j-ms}(x;-\omega)$ in general, $R_{jms}(z) = R^*_{j-ms}(-z^*)$ and a similar relation holds for $T_{jms}(z)$; therefore, if ω_{jms} is a quasinormal frequency, then so is $-\omega^*_{j-ms}$. It follows that the modes of a Kerr black hole are distributed symmetrically with respect to the imaginary axis in the complex ω plane (as before) since rotation removes the $(2j+1)$-fold degeneracy of the Schwarzschild modes. However, for a rotating black hole the possibility of occurrence of a real quasinormal frequency cannot be excluded: *It is clear from flux conservation [Eq. (13)] that a QNM with real ω cannot occur in the ordinary case, but it is in principle possible in the superradiant case*[12] *(cf. Sec. VI).*

The problem of determination of the *proper* QNM's may be transformed into a bound-state problem for a rotating black hole as well. The transformation of the parameters must be so chosen that the transformed potential W,

$$W(x;\Omega;p) = V(-ix;\omega(p');p') \ , \tag{15}$$

is a real potential barrier. Once the bound states of $-W$ (i.e., $\Omega \geq 0$) are determined, the proper QNM's can be obtained from the inverse transformation, just as in Eq. (12). This method will be used in Sec. V to determine the QNM's of a slowly rotating Kerr black hole in the eikonal approximation.

Finally, it is important to illustrate how a QNM, which is not a proper perturbation of a black hole, can nevertheless be useful in characterizing the late-time behavior of its response to external excitations. Consider the perturbation of a black hole by incident matter or radiation. The response of the black hole as $x \to +\infty$ may be written as

$$\Psi_{\mathrm{re}} = \int_{-\infty}^{\infty} A(\omega) R(\omega) e^{i\omega u} d\omega \ , \tag{16}$$

where $u = t - x$ is the retarded null coordinate. For an incident wave packet $A(\omega)$ is the amplitude of radiation mode of frequency ω, and for incident matter

$$A(\omega) = \frac{i}{2\omega} \int_{-\infty}^{\infty} \chi(x';\omega) \mathscr{T}(x';\omega) dx' \ , \tag{17}$$

where χ is a solution of the homogeneous Eq. (3) with the boundary conditions

$$\chi \to e^{-i\omega x} + \frac{1}{R(\omega)} e^{i\omega x} \ , \quad \text{for } x \to +\infty \tag{18}$$

and

$$\chi \to \frac{T(\omega)}{R(\omega)} e^{ikx} \ , \quad \text{for } x \to -\infty \ . \tag{19}$$

Equation (17) is obtained from the general solution of Eq. (3) with the boundary conditions that the waves are outgoing at infinity and ingoing at the horizon.

To determine the behavior of the response at late time $u > 0$, let us assume that $F(z) = A(z) R(z)$ is a holomorphic function in the upper half-plane, except perhaps at a *finite* number of points not on the real axis and let $F \to 0$ as $|z| \to \infty$ in the upper half-plane, then[13]

$$\Psi_{\mathrm{re}} = 2\pi i \sum_S \hat{f}(z_S) e^{i z_S u} \ , \tag{20}$$

where $\hat{f}(z_S)$ is the residue of $F(z)$ at a singular point z_S. The damped oscillations characteristic of the QNM's (i.e., singularities of R) are contained in Eq. (20) with an amplitude determined by the relevant strength of the external perturbation and the intrinsic amplitude of the mode (i.e., the residue of R). Moreover, the response is generally dominated by terms with the least damping, and such terms could be due to the singularities of $A(z)$. For instance, an incident wave packet with wavelengths much larger than the mass of a Schwarzschild black hole is simply reflected with little change in shape, whereas the quasinormal modes dominate the response if the main wavelengths are comparable to (or smaller than) the size of the black hole.

The number of QNM's of a black hole for a given multipole order j is not known at present. The above argument is intended to show that although a QNM is not expected to be a proper perturbation of a black hole so far as its spatial behavior is concerned (ψ can diverge for $x \to \pm\infty$), nevertheless when the time dependence is properly taken into account the QNM appears as a propagating damped oscillation at late times as in Eq. (20).

III. QNM's OF A SCHWARZSCHILD BLACK HOLE

The method of Sec. II may be applied directly to the problem of determination of the QNM's of a Schwarzschild black hole. The effective potential is given by[14]

$$U(x) = \lambda \left[1 - \frac{2M}{r}\right] \left[\frac{j(j+1)}{r^2} + \frac{2\sigma M}{r^3}\right] , \tag{21}$$

where

$$x = r + 2M \ln\left[\frac{r}{2M} - 1\right] \tag{22}$$

and $\sigma = 1, 0, -3$, for the scalar, electromagnetic, and gravitational perturbations, respectively. The scaling parameter $\lambda > 0$, which is unity for black-hole perturbations, has been introduced such that under the transformations $x \to -ix$, $p = (M, \lambda) \to (-iM, -\lambda)$, the potential (21) remains invariant. Let $\Omega(M, \lambda)$ denote the bound states of the inverted potential, the proper quasinormal frequencies of a Schwarzschild black hole are then given by

$$\omega^0 + i\Gamma = \Omega(iM, -1) \ . \tag{23}$$

The analytical determination of $\Omega(M, \lambda)$ has not proved possible thus far; therefore, this function must be estimated by using simpler potentials that approximate (21) closely, especially near its maximum. For instance, the ground state plus the first few excited states may be well approximated by the bound states of[15] $-U_{\mathrm{PT}}(x)$,

$$U_{\mathrm{PT}}(x)=U_0/\cosh^2\alpha(x-x_0)\ , \tag{24}$$

since $U(x)$ drops exponentially to zero for $x\to-\infty$ (but falls off as x^{-2} for $x\to+\infty$). The quantities U_0 and $\alpha>0$ are given by the height and curvature of the potential at its maximum ($x=x_0$). Thus,

$$U_0=U(x_0)\text{ and }\alpha^2=-\frac{1}{2U_0}\left[\frac{d^2U}{dx^2}\right]_{x_0}. \tag{25}$$

The transition from the potential barrier U_{PT} to the inverted potential is achieved by the transformations $x\to-ix$, $(U_0,\alpha)\to(U_0,i\alpha)$. The bound states of $-U_{\mathrm{PT}}$ are given by (cf. Appendix A)

$$\Omega_n(U_0,\alpha)=\alpha\left[-(n+\tfrac{1}{2})+\left[\frac{1}{4}+\frac{U_0}{\alpha^2}\right]^{1/2}\right], \tag{26}$$

for $n=0,1,2,\ldots,N-1$, where $\Omega_N<0$. The proper QNM's may be obtained from $\Omega_n(U_0,-i\alpha)$; the corresponding frequencies are given by

$$\omega^0=\pm(U_0-\alpha^2/4)^{1/2}\text{ and }\Gamma_n=\alpha(n+\tfrac{1}{2})\ . \tag{27}$$

Table I presents the values of quasinormal frequencies calculated using Eqs. (25)–(27). It is important to stress that only the proper modes are given in Table I, i.e., those for which a corresponding bound state exists. For the electromagnetic perturbations we find that Γ is independent of j; in fact, a simple calculation shows that $\alpha^{-1}=3\sqrt{3}M$ in this case. Moreover, ω^0 is independent of n, whereas numerical work[16] has indicated that for gravitational perturbations ω^0 decreases slightly with n. This is related to how precisely $\Omega(M,\lambda)$ approximates the bound states of the inverted potential. If the inverted harmonic-oscillator potential[5] is used to estimate $\Omega(M,\lambda)$, both ω^0 and Γ increase with n. The Pöschl-Teller potential provides a better estimate; moreover, a generalization of this potential may be used[10] to explain the decrease of ω^0 with n. The agreement between our results (Table I) with numerical work[4,16] is better than a few percent for the fundamental (i.e., least-damped) QNM's. The same holds for the damping factor for all the modes.

In the eikonal approximation $j\gg1$, the dominant term in the potential (21) is proportional to $j(j+1)$ so that the quasinormal frequencies are given—irrespective of the spin of the perturbing field—by (cf. Appendix B)

$$\omega_j^0\approx\pm\gamma_0(j+\tfrac{1}{2}) \tag{28}$$

and

$$\Gamma_n\approx\gamma_0(n+\tfrac{1}{2})\ , \tag{29}$$

where

$$\gamma_0=(3\sqrt{3}M)^{-1} \tag{30}$$

and $n=0,1,2,\ldots$ ($n\ll j$).

IV. QNM's OF A REISSNER-NORDSTRÖM BLACK HOLE

The perturbation equations for the exterior Reissner-Nordström geometry have been derived by a number of authors.[17] Let ψ_1 and ψ_2 represent the amplitudes of electromagnetic and gravitational perturbations, respectively. The black hole is charged; therefore, purely electromagnetic perturbations induce gravitational perturbations and vice versa. The perturbation equations decouple, however, for $\psi_\pm$, where

TABLE I. Quasinormal frequencies of the scalar, electromagnetic, and gravitational perturbations of a Schwarzschild black hole. ω is expressed in units of $(2M)^{-1}$.

j	n	Scalar modes	Electromagnetic modes	Gravitational modes
0	0	$0.230+0.230i$		
1	0	$0.597+0.201i$	$0.509+0.193i$	
	1	$0.597+0.604i$	$0.509+0.577i$	
2	0	$0.975+0.196i$	$0.923+0.193i$	$0.757+0.181i$
	1	$0.975+0.587i$	$0.923+0.577i$	$0.757+0.543i$
	2	$0.975+0.979i$	$0.923+0.962i$	
3	0	$1.356+0.194i$	$1.319+0.193i$	$1.205+0.187i$
	1	$1.356+0.583i$	$1.319+0.577i$	$1.205+0.560i$
	2	$1.356+0.971i$	$1.319+0.962i$	$1.705+0.934i$
	3	$1.356+1.359i$		
4	0	$1.739+0.194i$	$1.711+0.193i$	$1.623+0.189i$
	1	$1.739+0.581i$	$1.711+0.577i$	$1.623+0.567i$
	2	$1.739+0.968i$	$1.711+0.962i$	$1.623+0.946i$
	3	$1.739+1.355i$	$1.711+1.347i$	$1.623+1.324i$
	4	$1.739+1.742i$		
5	0	$2.123+0.193i$	$2.099+0.193i$	$2.028+0.190i$
	1	$2.123+0.580i$	$2.099+0.577i$	$2.028+0.571i$
	2	$2.123+0.966i$	$2.099+0.962i$	$2.028+0.951i$
	3	$2.723+1.352i$	$2.099+1.347i$	$2.028+1.332i$
	4	$2.123+1.738i$	$2.099+1.732i$	$2.078+1.712i$

$$\psi_+ = q_+\psi_1 + 2Q(j-1)(j+2)\psi_2 \,, \tag{31}$$

and

$$\psi_- = 2Q(j-1)(j+2)\psi_1 - q_+\psi_2 \,. \tag{32}$$

Here Q is the black-hole charge, and

$$q_\pm = 3M \pm [9M^2 + 4Q^2(j-1)(j+2)]^{1/2} \,. \tag{33}$$

The effective potentials associated with the decoupled equations for $\psi_\pm$ can be put in the form

$$U_\pm(x) = \left[1 - \frac{2M}{r} + \frac{Q^2}{r^2}\right]\left[\frac{j(j+1)}{r^2} - \frac{q_\mp}{r^3} + \frac{4Q^2}{r^4}\right] . \tag{34}$$

The QNM's, which are intrinsic to the Reissner-Nordström geometry, are connected with the wave functions $\psi_\pm$, i.e., with the coupled electromagnetic and gravitational perturbations. Here the radial coordinate r is related to x via

$$\frac{dx}{dr} = \left[1 - \frac{2M}{r} + \frac{Q^2}{r^2}\right]^{-1} , \tag{35}$$

such that for $x \to -\infty$, $r \to r_+ = M + (M^2 - Q^2)^{1/2}$, which is the radius of the outer horizon. The potentials $U_\pm(x)$ have much the same mathematical behavior as in the Schwarzschild case, i.e., they are positive everywhere and go to zero exponentially for $x \to -\infty$ and fall off as x^{-2} for $x \to +\infty$, with a single maximum in between. Therefore, the QNM's associated with $\psi_\pm$ may be determined using the same method as in the Schwarzschild case: we scale the potentials by a parameter $\lambda > 0$ as before, and consider the parameter transformation $(M,Q,\lambda) \to (-iM, -iQ, -\lambda)$. Once the bound states $\Omega_\pm(M,Q,\lambda)$ have been determined, the quasinormal frequencies follow from

$$\omega_\pm^0 + i\Gamma_\pm = \Omega_\pm(iM, iQ, -1) \,, \tag{36}$$

and the mode wave functions can be obtained in a similar way (see Appendix A). Figures 2—4 present the results of our calculations of quasinormal frequencies using formulas (25)—(27). Our results may be compared with the extensive numerical calculations of Gunter;[18] the agreement is better than a few percent. Since for $j=1$ only spin-one perturbations have any physical significance, only ψ_+ is relevant for the QNM problem in this case as it reduces to a purely electromagnetic perturbation.[18] It is clear from Fig. 4 that $\Gamma_\pm$ vary only slightly with Q.

In the eikonal approximation $j \gg 1$, the quasinormal frequencies become independent of the spin of the perturbing field and are given by

$$\omega_j^0 \approx \pm \left[\frac{M}{r_0{}^3} - \frac{Q^2}{r_0{}^4}\right]^{1/2} (j + \tfrac{1}{2}) \tag{37}$$

and

$$\Gamma_n \approx \left[\frac{M}{r_0{}^3} - \frac{Q^2}{r_0{}^4}\right]^{1/2} \left[2 - 3\frac{M}{r_0}\right]^{1/2} (n + \tfrac{1}{2}) \,, \tag{38}$$

for $n = 0, 1, 2, \ldots$ ($n \ll j$). Here r_0,

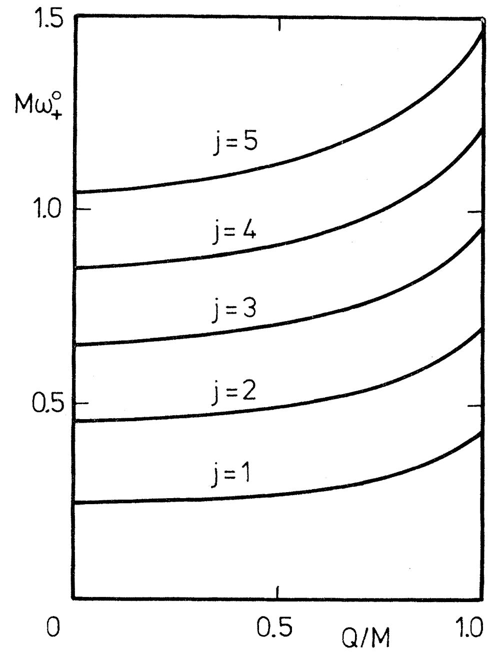

FIG. 2. The real part of the quasinormal frequency of a Reissner-Nordström black hole corresponding to the ψ_+ combination of electromagnetic and gravitational perturbations for several values of $j \geq 1$. Note that in our approximate analysis ω_+^0 does not depend on n.

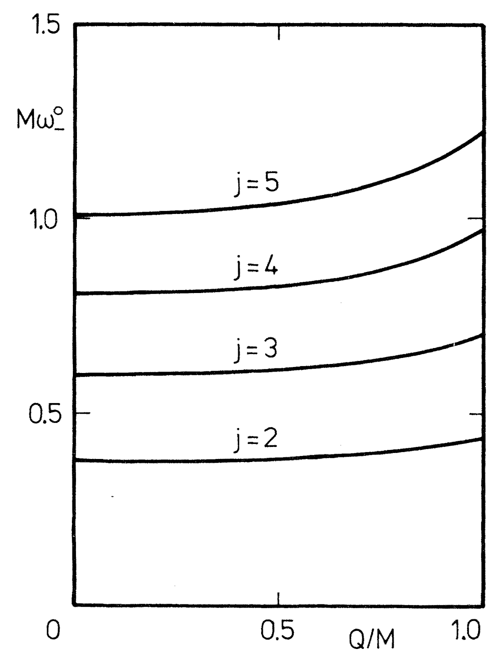

FIG. 3. The real part of the quasinormal frequency of a Reissner-Nordström black hole corresponding to the ψ_- combination of gravitational and electromagnetic perturbations for several values of $j \geq 2$. Note that in our approximate analysis ω_-^0 does not depend on n.

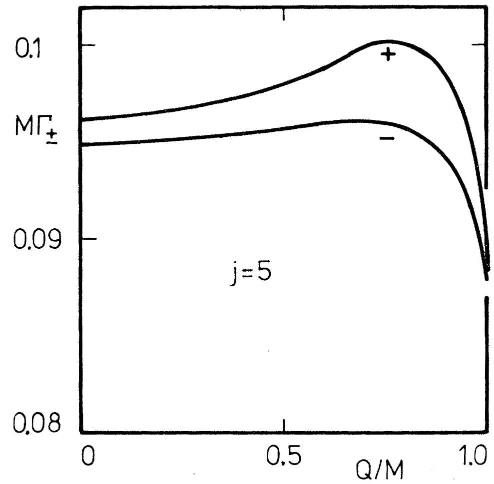

FIG. 4. The imaginary parts of the quasinormal frequencies corresponding to the $\psi_\pm$ combinations of electromagnetic and gravitational perturbations of a Reissner-Nordström black hole for $n=0$ and $j=5$.

$$2r_0=3M+(9M^2-8Q^2)^{1/2}\,, \tag{39}$$

is the radius of the unstable null circular orbit in the exterior Reissner-Nordström geometry.

V. QNM's OF A KERR BLACK HOLE

The general problem of the QNM's of a Kerr black hole is complicated by the fact that the effective potential in this case depends in an intricate manner on the frequency of radiation ω and the angular momentum of the black hole $J=aM$. It is possible, however, to treat the problem using our method since one can turn the QNM problem into a bound-state problem by an appropriate choice of the parameters. In this connection we simply note that Teukolsky's effective potential,[7] which is in general complex, may be transformed into a real inverted potential involving the bound-state eigenvalue Ω. Rather than treating the general case, we shall restrict our attention to a simple case in which (i) $a \ll M$, so that only linear terms in a/M are considered, and (ii) the eikonal approximation is valid, i.e., $j \geq |m| \gg 1$ (or $j \gg 1$ and $m=0$). Under these conditions, and using the fact that for a Schwarzschild black hole ω^0 is proportional to j for $j \gg 1$, the effective potential given by Teukolsky reduces to the simple form

$$V(x;\omega)\approx\lambda\left[\left[1-\frac{2M}{r}\right]\frac{j(j+1)}{r^2}+4\frac{am\omega M}{r^3}\right]\,, \tag{40}$$

where the scaling parameter $\lambda>0$ has also been introduced. This is a real potential independent of the spin of the perturbing field and the radial coordinates x and r are related as in the Schwarzschild case. The parameter set here is $p=(M,am,\omega,\lambda)$, which is transformed into $p'=(-iM,-am,\Omega,-\lambda)$ so that

$$W(x;\Omega)=\lambda\left[\left[1-\frac{2M}{r}\right]\frac{j(j+1)}{r^2}-4\frac{am\Omega M}{r^3}\right]\,. \tag{41}$$

We are interested in the bound states $\Omega\geq 0$ of $-W(x;\Omega)$; therefore, we use the analytic approximation formulas (24)–(26). Once $\Omega(M,am,\lambda)$ is determined, the quasinormal frequencies are obtained from $\omega^0+i\Gamma=\Omega(iM,-am,-1)$. The maximum of W occurs at

$$r_0=3M\left[1+\frac{2am\Omega}{j(j+1)}\right]\,, \tag{42}$$

the height of the potential at maximum is

$$W_0=\lambda\gamma_0{}^2[j(j+1)-4am\Omega]\,, \tag{43}$$

and the curvature parameter α is given by

$$\alpha=\frac{r_0}{3M}\gamma_0\,, \tag{44}$$

where γ_0 is given by Eq. (30). Inserting these expressions in Eq. (26) and solving the resulting second-order equation for Ω_n, we find (for $n \ll j$)

$$\Omega_n=[\lambda\gamma_0{}^2 j(j+1)]^{1/2}-2\lambda am\gamma_0{}^2-\gamma_0(n+\tfrac{1}{2})\,. \tag{45}$$

The proper quasinormal frequencies are then given by[19]

$$\omega^0_{jm}=\pm\gamma_0(j+\tfrac{1}{2})+2am\gamma_0{}^2 \tag{46}$$

and

$$\Gamma_n=\gamma_0(n+\tfrac{1}{2})\,,\quad n=0,1,2,\ldots\ (n\ll j)\,. \tag{47}$$

The quasinormal wave functions can be obtained in an analogous manner (see Appendix A).

These results may be compared and contrasted with the frequencies of oscillation of a rotating fluid mass.[20] The Kelvin modes of oscillation of a spherical fluid mass of uniform density μ is given by

$$\omega_j\approx(\tfrac{4}{3}\pi\mu)^{1/2}\sqrt{j}\quad\text{for } j\gg 1\,. \tag{48}$$

The $(2j+1)$-fold degeneracy of these modes may be removed by a slow rotation of frequency ω_*,

$$\omega_{jm}=\omega_j+m\omega_*\,. \tag{49}$$

For a black hole, ω^0_j varies linearly with j in contrast to Eq. (48), and ω_* is given by

$$\omega_*=2\frac{J}{(3M)^3}\,, \tag{50}$$

which is of the same order of magnitude as the dragging frequency of the inertial frames[21] at $r=3M$. For a *slowly* rotating black hole, the damping of the oscillations is not affected by rotation and is independent of the angular momentum parameters of the perturbation[22] for $j\gg 1$. This remaining degeneracy is, however, removed by the addition of electric charge.[23]

VI. STABILITY OF THE EXTREME KERR BLACK HOLE

The quasinormal frequencies of a rotating black hole were studied numerically by Detweiler,[6] who found resonant oscillations for rapidly rotating black holes with decay factor Γ approaching zero in the extreme Kerr limit ($a=M$). This led to the suggestion that extreme Kerr black holes are, in some sense, marginally unstable since excitations could lead to the emission of radiation without damping.[24] We show, however, that the response of the black hole at late times contains no such resonant oscillations. That is, the contribution of such modes to the black-hole response goes to zero as $\Gamma\to 0$. Hence, in any realistic situation the late-time behavior of the black-hole response is expected to be generally dominated by QNM's which are significantly damped.

To begin with, we recall from our general discussion of Sec. II that for a rotating black hole a QNM with *real* frequency cannot exist in the *ordinary* (i.e., nonsuperradiant) case. Thus, the above suggestion cannot be maintained since all QNM's determined numerically fall in the ordinary regime. The rest of this section is devoted to a physical explanation of this result. Consider the reflection amplitude extended to the complex domain. This function is *bounded* on the real axis in the ordinary regime; therefore, if a QNM approaches this part of the real axis, the corresponding residue must approach zero: The *amplitude of the excitation of this QNM goes to zero as* $\Gamma\to 0$ according to Eq. (20). To illustrate this point by a specific example, consider the reflection amplitude for an inverted harmonic-oscillator potential,

$$U(x)=U_0-\tfrac{1}{2}\kappa(x-x_0)^2 \tag{51}$$

with $U_0>0$ and $\kappa>0$; then

$$R(z)=\frac{i}{\sqrt{2\pi}}e^{-\pi\delta/2}\Gamma(\tfrac{1}{2}+i\delta)\ , \tag{52}$$

where δ is given by

$$\delta=\frac{z^2-U_0}{(2\kappa)^{1/2}}\ . \tag{53}$$

The reflection amplitude has poles at $\delta_n=i(n+\frac{1}{2})$ for $n=0,1,2,\ldots$, corresponding to $z_n=\omega_n^0+i\Gamma_n$, with residues $\hat{r}_n$,

$$|\hat{r}_n|=[(2\pi)^{1/2}(n+\tfrac{1}{2})n!]^{-1}\frac{\omega_n^0\Gamma_n}{|z_n|}\ . \tag{54}$$

Therefore, $\hat{r}_n$ is proportional to Γ_n, as expected.

We shall now present a different approach based on an explicit perturbation of the Kerr black hole which is studied in the eikonal approximation using null rays in the Kerr geometry.[25] This problem was partially analyzed by Goebel;[26] however, our treatment and interpretation are different from his. The advantage of this method is that the QNM's investigated numerically by Detweiler are explicitly exhibited and it is shown that in the extreme Kerr limit these modes have vanishing amplitude.

Consider null rays orbiting a Kerr black hole in the unstable circular orbit in the equatorial plane. The orbital equations can be written as

$$(t,r,\theta,\varphi)=(\eta,r_0,\pi/2,\omega_\pm\eta)\ , \tag{55}$$

where η is the affine parameter along the ray,

$$r_0=3M\mp 2a(M/r_0)^{1/2}\ , \tag{56}$$

and

$$\omega_\pm=[a\pm(r_0{}^3/M)^{1/2}]^{-1}\ . \tag{57}$$

The upper sign in Eqs. (56) and (57) refers to an orbit corotating with the black hole and the lower sign refers to a counterrotating orbit. Note that when $a=0$, $r_0=3M$ and $\omega_\pm=\pm\gamma_0$.

Suppose that at $t=0$ we perturb the orbit slightly; the null rays diverge away from the unstable orbit, some falling into the black hole and some escaping to infinity, thereby simulating the boundary conditions corresponding to the QNM problem. This perturbation of the black hole corresponds to a superposition of eigenmodes of high frequency $\omega M\gg 1$, so that $j\gg 1$ (eikonal approximation). We expect the QNM's to dominate the characteristics of the outgoing radiation; therefore, information about the QNM's in the high-frequency limit may be obtained from studying the characteristics of the diverging null rays.

Let $|\epsilon|\ll 1$ represent the strength of the orbital perturbation. We compare the perturbed orbit in the equatorial plane with the original unperturbed orbit to linear order in ϵ for $t\geq 0$; therefore, the perturbed orbit may be expressed as

$$r=r_0[1+\epsilon f(t)+\cdots]\ , \tag{58}$$

$\theta=\pi/2$ and

$$\varphi=\omega_\pm[t+\epsilon g(t)+\cdots] \tag{59}$$

with the affine parameter η given by

$$\eta=t+\epsilon h(t)+\cdots\ . \tag{60}$$

The functions f, g, and h are to be determined from the requirements that the perturbed orbit is a null geodesic of the Kerr field together with the boundary conditions that $f(0)=g(0)=h(0)=0$. The results are

$$f(t)=\sinh\gamma t\ , \tag{61}$$

$$g(t)=0\ , \tag{62}$$

and

$$h(t)=2\frac{|\omega_\pm|}{\gamma^2}\left[\frac{3M}{r_0}\right]^{1/2}(\cosh\gamma t-1)\ , \tag{63}$$

where $\gamma>0$ is given by

$$\gamma^2=3\omega_\pm{}^2\left[1-\frac{2M}{r_0}+\frac{a^2}{r_0{}^2}\right]\ . \tag{64}$$

The null rays consist of a superposition of eigenmodes of the form (1) with $m=\pm j$ and with frequency

$$\omega^0 = m\frac{d\varphi}{dt} = \pm j\omega_{\pm} , \qquad (65)$$

which corresponds to the *real* part of the frequency of the QNM's that dominate the late-time behavior of the outgoing radiation. It is interesting to note that to first order in a/M, Eq. (65) agrees with Eq. (46) for $m=\pm j$.

The outgoing ($\epsilon>0$) and ingoing ($\epsilon<0$) null rays deplete the energy of the field concentrated at the null circular orbit at $t=0$. Let ρ be the density of geodesic rays k^μ per unit proper volume. The conservation law for the number of null geodesic rays implies

$$(\rho k^\mu)_{;\mu} = 0 , \qquad (66)$$

which may be written as

$$\frac{1}{\rho}\frac{d\rho}{d\eta} = -\frac{1}{r^2}\frac{\partial}{\partial r}\left[r^2\frac{dr}{d\eta}\right] . \qquad (67)$$

It is possible to show using Eqs. (58)—(64) that to lowest order in ϵ,

$$\rho(t) = \rho_0/\cosh\gamma t , \qquad (68)$$

where ρ is the constant density of rays along the unstable circular orbit. For $t>0$, Eq. (68) can be expressed as

$$\rho = 2\rho_0(e^{-\gamma t} - e^{-3\gamma t} + e^{-5\gamma t} - \cdots) , \qquad (69)$$

which indicates that the decay factors for the outgoing waves, corresponding to the imaginary parts of the quasinormal frequencies, are given by

$$\Gamma = \gamma(n+\tfrac{1}{2}) , \quad n=0,1,2,\ldots , \qquad (70)$$

since $\rho(t)$ indicates the *density of rays along the outgoing (or ingoing) null orbit.* For a slowly rotating black hole, $\gamma-\gamma_0$ is second order in a/M and Eq. (70) is consistent with Eq. (47) and the Schwarzschild limit.

Consider the outgoing null rays that rotate in the same sense as the black hole ($m=j$). In this case Γ decreases monotonically with a/M and in the extreme Kerr limit $a\to M$, $r_0\to r_+=M$ and $\Gamma\to 0$. But this also indicates that there is no outgoing (or ingoing) radiation (i.e., $f\to 0$), so that the energy is trapped (i.e., $\rho\to\rho_0$) in the extreme Kerr null circular orbit coincident with the horizon. We conclude that QNM's with real frequency in the ordinary regime have vanishing amplitude for the extreme Kerr black hole, which is therefore stable in the eikonal approximation.

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APPENDIX A

The purpose of this appendix is to present the QNM eigenfunctions of a black hole on the basis of the analytic approximation scheme presented in Secs. III—V. It is therefore sufficient to determine explicitly the QNM's of the Pöschl-Teller potential (24). The solution of the wave equation with the boundary condition that for $x\to-\infty$, $\psi\to\exp(i\omega x)$ can be written as

$$\psi = [\xi(1-\xi)]^{i\omega/2\alpha} F\left[1+\beta+i\frac{\omega}{\alpha}, -\beta+i\frac{\omega}{\alpha}; 1+i\frac{\omega}{\alpha}; \xi\right] , \qquad (\text{A1})$$

where ξ and β are defined by

$$\xi^{-1} = 1+\exp[-2\alpha(x-x_0)] \qquad (\text{A2})$$

and

$$\beta = -\tfrac{1}{2} + \left[\frac{1}{4} - \frac{U_0}{\alpha^2}\right]^{1/2} . \qquad (\text{A3})$$

For $x\to+\infty$, $\xi\to 1$, ψ is given by

$$\psi \to \frac{R}{T}e^{-i\omega x} + \frac{1}{T}e^{i\omega x} , \qquad (\text{A4})$$

where the reflection and transmission amplitudes are

$$R(\omega) = \frac{\Gamma(-i\omega/\alpha)\Gamma(1+\beta+i\omega/\alpha)\Gamma(-\beta+i\omega/\alpha)}{\Gamma(i\omega/\alpha)\Gamma(1+\beta)\Gamma(-\beta)} \qquad (\text{A5})$$

and

$$T(\omega) = \frac{\Gamma(1+\beta+i\omega/\alpha)\Gamma(-\beta+i\omega/\alpha)}{\Gamma(1+i\omega/\alpha)\Gamma(i\omega/\alpha)} . \qquad (\text{A6})$$

The QNM's of the Pöschl-Teller potential correspond to simple poles common to R and T, i.e., either $1+\beta+i\omega/\alpha$ or $-\beta+i\omega/\alpha$ must be equal to $-n$ ($n=0,1,2,\ldots$), so that

$$\omega = i\alpha(n+\tfrac{1}{2}) \pm i\alpha\left[\frac{1}{4}-\frac{U_0}{\alpha^2}\right]^{1/2} , \quad n=0,1,2,\ldots . \qquad (\text{A7})$$

If $4U_0\le\alpha^2$, there is no QNM since no *propagating* solution exists. For $4U_0>\alpha^2$, which is in fact the case for black-hole perturbations, the quasinormal frequencies are located on lines parallel to the imaginary axis in the upper half-plane. The corresponding wave functions are given in Eq. (A1), where the hypergeometric function now reduces to a polynomial of degree n.

The general "bound" states of the inverted Pöschl-Teller potential are given by transformations (8) and (9), i.e., $\phi=\psi(-ix;i\alpha)$ and $\Omega=\omega(i\alpha)$, whereas the true bound states correspond to $\Omega\ge 0$ [cf. Eq. (26)].

APPENDIX B

In this appendix the quasinormal frequencies of a Schwarzschild black hole are given explicitly by using the QNM's of the Pöschl-Teller potential (cf. Sec. III and Appendix A).

The effective potential (21) has a single maximum at x_0 determined by

$$2y_0 = 3(1-\zeta) + (9+14\zeta+9\zeta^2)^{1/2} , \qquad (\text{B1})$$

where x_0 and $r = My_0$ are related as in Eq. (22) and

$$\zeta = \frac{\sigma}{j(j+1)} . \tag{B2}$$

The maximum of the potential, U_0, and the curvature parameter α are given by [cf. Eq. (25)]

$$U_0 = \frac{j(j+1)}{M^2} \frac{(1-\zeta)y_0 + 4\zeta}{{y_0}^4} \tag{B3}$$

and

$$\alpha = \frac{1}{M} \frac{y_0 - 2}{{y_0}^2} \left[\frac{3(1-\zeta)y_0 + 16\zeta}{(1-\zeta)y_0 + 4\zeta} \right]^{1/2} . \tag{B4}$$

The real and imaginary parts of the quasinormal frequencies are then obtained from Eq. (27). Let us now consider two special cases. First, for $j = 0$, which is relevant only for scalar perturbations,

$$2M\omega^0 = M\alpha = (\tfrac{3}{8})^{3/2} . \tag{B5}$$

The inverted potential has only a single bound state; therefore, there is only one proper QNM in this case with its frequency having equal real and imaginary parts. Next, for electromagnetic perturbations (with any $j \geq 1$) we find

$$\omega_j^0 = \gamma_0 (j^2 + j - \tfrac{1}{4})^{1/2} \text{ and } \Gamma_n = \gamma_0 (n + \tfrac{1}{2}) , \tag{B6}$$

where γ_0 is defined by Eq. (30).

In the general case, i.e., scalar waves with $j \geq 1$ and gravitational waves ($j \geq 2$), $|\zeta| \leq \frac{1}{2}$ and y_0 can be expressed as a power series in ζ:

$$y_0 = 3(1 - \tfrac{1}{9}\zeta + \tfrac{8}{81}\zeta^2 + \cdots) , \tag{B7}$$

so that

$$U_0 = \gamma_0 j(j+1)(1 + \tfrac{2}{3}\zeta + \tfrac{1}{27}\zeta^2 + \cdots) \tag{B8}$$

and

$$\alpha = \gamma_0 (1 + \tfrac{1}{9}\zeta - \tfrac{1}{27}\zeta^2 + \cdots) . \tag{B9}$$

It follows that

$$\omega_j^0 = \gamma_0 \left[j(j+1) + \tfrac{2}{3}\sigma - \tfrac{1}{4} + \frac{1}{54} \frac{\sigma(2\sigma - 3)}{j(j+1)} + \ldots \right]^{1/2} \tag{B10}$$

and

$$\Gamma_{jn} = \gamma_0 \left[1 + \frac{1}{9} \frac{\sigma}{j(j+1)} - \frac{1}{27} \frac{\sigma^2}{j^2(j+1)^2} + \ldots \right] \times (n + \tfrac{1}{2}) , \tag{B11}$$

where $\sigma = 1$ for scalar perturbations and $\sigma = -3$ for gravitational perturbations.

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