Quantized relativistic rotator

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The equations that describe the relativistic rotator classically are summarized to form a basis for the quantization. Strict adherence to the correspondence principle requires a Hamiltonian without any free parameters from which a linear wave equation is derived. The equations of motion of the various operators (in particular, those describing the *Zitterbewegung*) then depend only on the mass M and the spin s. The angular frequency ω of the *Zitterbewegung* is found to be related to M by $Mc^2=(s+\frac{1}{2})\hbar\omega$. The simplest constraint relation needed to preserve these equations of motion while defining a spectrum requires that M^2 should be a linear function of s, in excellent agreement with the data on a number of hadron towers.

I. THE CLASSICAL THEORY

A. Introduction

The classical relativistic theory of a point particle with spin was initiated in 1926–27 by Frenkel[1] and Thomas[2] and developed some ten to fifteen years later by Mathisson,[3] Lubanski,[4] Hönl and Papapetrou,[5] and Bhabha and Corben.[6] Since that time many papers have been written on the subject and a complete bibliography up to the time was published in 1968,[7] and later some papers on various aspects of the corresponding quantum theory have appeared.[8] Revived interest in the theory arises from the recent development of a very detailed quantized version which corresponds in many ways with the earlier work.[9,10] It is the purpose of this paper to offer an alternative quantized theory which is developed on the basis of a much stricter adherence to the correspondence principle. In particular, a linear Hamiltonian is postulated, and many results, such as the frequency of the *Zitterbewegung*, are derived without introducing any free parameters.

In this paper we consider only a free spinning particle. Nearly all of the results of Sec. I were published years ago for the more general case of a charged particle with an anomalous magnetic moment moving in an electromagnetic field.[7] Since Ref. 7 is out of print, it is useful to summarize the basic ideas and equations, at least for a free particle, in order to use them as a basis for the corresponding quantum theory described in Sec. II.

The fundamental property of a relativistic classical particle with intrinsic spin $\vec{\sigma}$ is that its momentum $\vec{p}$ is not necessarily in the direction of its velocity $\vec{v}$, even for a free particle.[3] Although known for close to half a century, this fact is not always recognized. The center of mass of a point particle with spin coincides with the position of the particle only in the system in which they are both at rest, or when $\vec{v}$ and $\vec{\sigma}$ are parallel. The radius $\vec{\rho}$ from the center of mass G to the particle P (and therefore to the center of charge, if it is charged) is given by

$$\vec{\rho}=(Mc^2)^{-1}\vec{\sigma}\times\vec{v} \tag{1.1}$$

for a particle of rest energy Mc^2, i.e., the energy in the system in which $\vec{p}$ is zero, but $\vec{v}$ is not.

To see this, we can imagine an observer watching an automobile being driven along a straight road with velocity $\vec{v}$. The average speed of the material in the top halves of the tires is obviously greater than that in the bottom halves. The center of mass G of each tire therefore appears to be displaced upward, i.e., in the direction of $\vec{v}\times\vec{\sigma}$, where $\vec{\sigma}$ is the angular momentum of the tire. Since the effect does not depend on the density of the tire and since, being relativistic, it must be inversely proportional to some power of c, expression (1.1) for the negative of this vector seems reasonable.

For a particle of radius less than σ/Mc, Eq. (1.1) implies that there is some observer relative to whom the center of mass lies *outside* of the particle. It is therefore impossible to construct a classical object with radius less than its "classical Compton wavelength" entirely from matter of positive energy density.[6] Some neutron stars come close to this limit, and with sufficient negative gravitational energy could even exceed it. For a point particle, the slightest motion in a direction other than that of $\vec{\sigma}$ moves the center of mass away from it.

This separation of the positions of G and P means that, if P is charged, an electrostatic field can produce a torque on the particle, causing it to precess. The separation also shows up in the *Zitterbewegung* of the Dirac and other relativistic wave equations—the velocity, proportional to $(\vec{x},H)$ is in general not proportional to $\vec{p}$.

Expression (1.1) vanishes in the nonrelativistic limit. Since the analysis that follows is relativistic, we shall choose units in which $c=1$ and, for the quantum theory, $\hbar=1$.

B. The equations of motion

Conservation of total angular momentum $\vec{J}=\vec{\sigma}+\vec{r}\times\vec{p}$ in a field of force $\vec{F}=d\vec{p}/dt$ can be written as

$$\frac{d\vec{J}}{dt}=\vec{r}\times\vec{F}=\frac{d\vec{\sigma}}{dt}+\vec{v}\times\vec{p}+\vec{r}\times\frac{d\vec{p}}{dt}$$

so that

$$\frac{d\vec{\sigma}}{dt}+\vec{v}\times\vec{p}=0\ .$$

Relativistically these equations become (with $\mu,\nu=0,1,2,3$)

$$J_{\mu\nu}=\sigma_{\mu\nu}+x_\mu p_\nu-x_\nu p_\mu\ , \tag{1.2}$$

$$\dot{\sigma}_{\mu\nu}+v_\mu p_\nu-v_\nu p_\mu=0\ \ (=\dot{J}_{\mu\nu}\text{ for a free particle})\ , \tag{1.3}$$

$$v_\mu=\dot{x}_\mu\ .$$

The overdot above v_μ, $\sigma_{\mu\nu}$, and $J_{\mu\nu}$ denotes differentiation with respect to the proper time along the path of the particle—not along the path of its center of mass. With $g_{\mu\nu}$ diagonal and equal to $(1,-1,-1,-1)$, it follows that

$$v\cdot v\equiv v^\alpha v_\alpha=1\ . \tag{1.4}$$

We restrict attention to cases in which $\vec{\tau}$, the polar vector associated with $\vec{\sigma}$ vanishes in the system in which $\vec{v}=0$, so that a charged particle would have no electric moment in that system. A relativistically covariant equation that ensures this is

$$\sigma_{\mu\alpha}v^\alpha=0 \tag{1.5}$$

or

$$\vec{\tau}=\vec{v}\times\vec{\sigma}=-M\vec{\rho} \tag{1.5$'$}$$

[using Eq. (1.1)]. In particular,

$$\vec{\tau}\cdot\vec{\sigma}=0\ . \tag{1.6}$$

A parameter m, with the dimensions of a mass, may now be defined by

$$m=v^\alpha p_\alpha\equiv v\cdot p \tag{1.7}$$

the classical analog of the Dirac, Majorana, and other wave equations. Only in special circumstances is m equal to the mass M of the particle. A consequence of (1.3), (1.4), (1.5), and (1.7) is that

$$p_\mu=mv_\mu-\sigma_{\mu\alpha}\dot{v}^\alpha \tag{1.8}$$

so that, for a free particle

$$m\dot{v}_\mu=\sigma_{\mu\alpha}\ddot{v}^\alpha \tag{1.9}$$

and m is a constant of the motion.

For $\vec{p}=0$, Eqs. (1.8) and (1.9) have the solution

$$\begin{gathered}\vec{\sigma}=\text{const},\ \ \vec{v}=\vec{\omega}\times\vec{\rho},\ \ \vec{\omega}=-\frac{M}{\sigma^2}\vec{\sigma}\ ,\\ M=m\gamma^{-1}=-\vec{\sigma}\cdot\vec{\omega}\ .\end{gathered} \tag{1.10}$$

The particle P therefore moves in a circle of radius ρ with angular velocity ω where

$$M=\sigma\omega,\ \ \rho=\frac{\beta\sigma}{M}=(\gamma^2-1)^{1/2}\frac{\sigma}{m}\ . \tag{1.11}$$

The center of this circle is the fixed center of mass G, and the plane of the circle is normal to $\vec{\sigma}$. For $\vec{p}\neq0$ this circular path is drawn out into a helix. In the consequent free helical motion the energy W and momentum p are given by

$$W=M\sec\theta,\ \ p=M\tan\theta\ ,$$

where θ is the constant angle between $\vec{\sigma}$ and $\vec{p}$.

Equation (1.2) would seem to clearly separate the orbital and spin angular momenta, but it can also be written as

$$J_{\mu\nu}=\Sigma_{\mu\nu}+X_\mu p_\nu-X_\nu p_\mu\ , \tag{1.12}$$

where

$$X_\mu=x_\mu-\rho_\mu\ , \tag{1.13}$$

$$\Sigma_{\mu\nu}=\sigma_{\mu\nu}+\rho_\mu p_\nu-\rho_\nu p_\mu\ . \tag{1.14}$$

The choice of ρ_μ is arbitrary, but the particular choice

$$M^2\rho_\mu=-\sigma_{\mu\alpha}p^\alpha \tag{1.15}$$

with

$$p\cdot p\equiv p_\alpha p^\alpha=M^2 \tag{1.16}$$

is of special interest because it leads to the equation

$$\Sigma_{\mu\alpha}p^\alpha=0\ . \tag{1.17}$$

This relates $\Sigma_{\mu\nu}$ to the four-momentum in the same way that $\sigma_{\mu\nu}$ is related to the four-velocity by Eq. (1.5) and it ensures that in the center-of-mass rest system the polar components of $\Sigma_{\mu\nu}$ vanish.

From (1.15) and (1.5)

$$\rho_\alpha p^\alpha=0,\ \ \rho_\alpha v^\alpha=0\ , \tag{1.18}$$

so that

$$m\rho_\mu=\Sigma_{\mu\alpha}v^\alpha\ . \tag{1.19}$$

This reveals a symmetry between v_α and $\hat{p}_\alpha\equiv p_\alpha M^{-1}$, $\sigma_{\mu\nu}$ and $\Sigma_{\mu\nu}$, m and M:

$$\begin{gathered}\hat{p}_\alpha\hat{p}^\alpha=1,\ \ v_\alpha v^\alpha=1,\ \ M\rho_\mu=-\sigma_{\mu\alpha}\hat{p}^\alpha\ ,\\ m\rho_\mu=\Sigma_{\mu\alpha}v^\alpha,\ \ \Sigma_{\mu\alpha}p^\alpha=0,\ \ \sigma_{\mu\alpha}v^\alpha=0\ ,\\ J_{\mu\alpha}\hat{p}^\alpha=X_\mu M-(X_\alpha\hat{p}^\alpha)p_\mu,\ \ J_{\mu\alpha}v^\alpha=x_\mu m-(x_\alpha v^\alpha)p_\mu\ .\end{gathered} \tag{1.20}$$

From the last of these equations

$$(J_{\mu\alpha}\hat{p}_\nu-J_{\nu\alpha}\hat{p}_\mu)\hat{p}^\alpha=(X_\mu p_\nu-X_\nu p_\mu)$$

so that, for a free particle the spin and orbital terms of Eq. (1.12) are separately constant, and in particular $\Sigma_{\mu\nu}=$constant. However, in general the spin and orbital parts of (1.2) are not separately constant.

Because

$$p_\alpha\dot{X}^\alpha=p_\alpha v^\alpha=m$$

(since $p_\alpha\dot{\rho}^\alpha=0$) it follows that

$$p_\mu=\frac{M^2}{m}\dot{X}_\mu\ . \tag{1.21}$$

Thus the four-velocity $\dot{X}_\mu$ is in the direction of the momentum. However, as defined this way $\dot{X}_\mu$ does not have unit length because as noted earlier the overdot denotes differentiation along the path of the particle rather than along the path of its center of mass. We therefore define

$$\frac{dX_\mu}{dS}=\frac{M}{m}\dot{X}_\mu \tag{1.22}$$

so that

$$p_\mu=M\frac{dX_\mu}{dS}, \quad \frac{dX_\alpha}{dS}\frac{dX^\alpha}{dS}=1 . \tag{1.23}$$

From Eq. (1.10), for a particle "at rest" (i.e., $\vec{p}=0$, $\vec{v}\neq 0$)

$$m=\gamma M, \quad \frac{dX_\mu}{dS}=\gamma^{-1}\dot{X}_\mu=\frac{dX_\mu}{dt} ,$$

where t is the time in the center-of-mass rest system.

It is useful to define

$$l_{\mu\nu}=\rho_\mu p_\nu-\rho_\nu p_\mu \tag{1.24}$$

so that, from (1.14)

$$\Sigma_{\mu\nu}=\sigma_{\mu\nu}+l_{\mu\nu} . \tag{1.25}$$

Thus

$$l_0{}^2\equiv\tfrac{1}{2}l_{\alpha\beta}l^{\alpha\beta}=M^2\rho_\alpha\rho^\alpha \tag{1.26}$$

and we define

$$\sigma_0{}^2=\tfrac{1}{2}\sigma_{\alpha\beta}\sigma^{\alpha\beta}, \quad \Sigma_0{}^2=\tfrac{1}{2}\Sigma_{\alpha\beta}\Sigma^{\alpha\beta} . \tag{1.27}$$

From the definition (1.15) it follows that

$$\sigma_{\alpha\beta}l^{\alpha\beta}=-2l_0{}^2 \tag{1.28}$$

so that[11]

$$\Sigma_0{}^2=\sigma_0{}^2-l_0{}^2 . \tag{1.29}$$

From (1.18) and (1.20) in the rest system of the center of mass,

$$\rho_0=0, \quad \Sigma_{i0}=0, \quad \Sigma_0{}^2=\Sigma^2=\sigma^2 ,$$

$$l_0{}^2=-M^2\rho^2=-\beta^2\sigma^2 ,$$

so that from (1.29)

$$\Sigma_0{}^2=\sigma_0{}^2+\beta^2\Sigma_0{}^2=\frac{m^2}{M^2}\sigma_0{}^2 . \tag{1.30}$$

This relation between invariant quantities is quite general and is a consequence of the analysis developed in the next section.

C. The Hamiltonian formulation

For any antisymmetrical tensor, such as $\sigma_{\mu\nu}$, whose components satisfy Eq. (1.6) it is possible to write down a very useful identity, provided that, as is the case classically, the components commute:

$$\sigma_0{}^2\sigma_{\mu\nu}=-\sigma_{\mu\alpha}\sigma^{\alpha\beta}\sigma_{\beta\nu} . \tag{1.31}$$

Possibly the easiest way to verify this is to choose values for μ and ν and write out the various terms. By multiplying Eq. (1.8) by $\sigma_{\alpha\beta}\sigma^{\beta\mu}$ and using (1.5) and (1.31) it is possible to express the velocity in terms of the momentum and the spin:

$$mv_\mu=p_\mu+\sigma_0{}^{-2}\sigma_{\mu\alpha}\sigma^{\alpha\beta}p_\beta . \tag{1.32}$$

Thus Eq. (1.7) may be expressed in the classical Hamiltonian form:

$$H=\tfrac{1}{2}(v^\alpha p_\alpha-m)=0 \tag{1.33}$$

$$=\frac{1}{2m}(p_\alpha p^\alpha-m^2-\sigma_0{}^{-2}\sigma_{\mu\alpha}p^\alpha\sigma^{\mu\beta}p_\beta)$$

$$=\frac{1}{2m}(p_\alpha p^\alpha-m^2-M^2l_0{}^2\sigma_0{}^{-2}) \tag{1.34}$$

from (1.15) and (1.26). From (1.29) this leads immediately to (1.30).

Because of Eq. (1.17) the spin $\Sigma_{\mu\nu}$ also satisfies (1.6) and therefore (1.31). Since

$$m\rho_\mu=\Sigma_{\mu\alpha}v^\alpha \tag{1.35}$$

$$=m^{-1}\Sigma_{\mu\alpha}\Sigma^{\alpha\beta}\dot{v}_\beta \tag{1.36}$$

it follows after some analysis that

$$mp_\mu=M^2v_\mu+M^2\Sigma_0{}^{-2}\Sigma_{\mu\alpha}\Sigma^{\alpha\beta}v_\beta . \tag{1.37}$$

The relation $v^\alpha p_\alpha=m$ then becomes

$$M^2-M^2\Sigma_0{}^{-2}m^2\rho_\alpha\rho^\alpha=m^2$$

or

$$M^2-m^2\Sigma_0{}^{-2}(\sigma_0{}^2-\Sigma_0{}^2)=m^2$$

which reproduces Eq. (1.30).

The Hamiltonian (1.34) is easily seen to reproduce Eq. (1.32) from $v^\mu=\partial H/\partial p_\mu$ or by using the Poisson brackets

$$\begin{aligned}(x_\mu,p_\nu)&=g_{\mu\nu} ,\\ (\sigma_{\mu\nu},p_\sigma)&=0 ,\\ (\sigma_{\mu\nu},x_\sigma)&=0 .\end{aligned} \tag{1.38}$$

Equation (1.3) also follows from the Hamiltonian if the Poisson bracket relation

$$(\sigma_{\mu\nu},\sigma_{\rho\tau})=(\sigma_{\mu\rho}g_{\nu\tau}+\sigma_{\nu\tau}g_{\mu\rho}-\sigma_{\mu\tau}g_{\nu\rho}-\sigma_{\nu\rho}g_{\mu\tau}) \tag{1.39}$$

is postulated. Similarly $\dot{\rho}_\mu$ is given by

$$\begin{aligned}\dot{\rho}_\mu&=(\rho_\mu,H)=v_\mu-mM^{-2}p_\mu\\ &=M^2\sigma_0{}^{-2}m^{-1}(-\sigma_{\mu\alpha}\rho^\alpha+p_\mu\rho_\alpha\rho^\alpha) .\end{aligned} \tag{1.40}$$

Thus $\rho^\alpha\rho_\alpha$ is a constant and for the case $\vec{p}=0$ (so that $\rho_0=0$)

$$\frac{d\vec{\rho}}{ds}=\frac{M^2}{m\sigma_0{}^2}(\vec{\rho}\times\vec{\sigma}) \tag{1.41}$$

as compared with Eq. (1.10)

$$\frac{d\vec{\rho}}{dt}=\frac{M}{\sigma^2}(\vec{\rho}\times\vec{\sigma}) . \tag{1.10$'$}$$

With $M\gamma=m$, $\sigma_0\gamma=\sigma$, $\gamma ds=dt$ the two equations are seen to be identical, i.e., the frequency of this circular motion as measured in the center-of-mass system is determined entirely by the mass and the spin. We therefore expect this to hold in the quantum theory for the *Zitterbewegung* frequency [see Eq. (2.47)].

II. THE QUANTUM THEORY

A. Commutation relations

As we move to the quantum theory, the Poisson bracket relations (1.38) and (1.39) become commutation relations

$$(x_\mu,p_\nu)=-ig_{\mu\nu}1\ ,$$
$$(x_\mu,\sigma_{\nu\rho})=0,\ \ (p_\mu,\sigma_{\nu\rho})=0\ , \tag{2.1}$$
$$(\sigma_{\mu\nu},\sigma_{\rho\tau})=-i(\sigma_{\mu\rho}g_{\nu\tau}+\sigma_{\nu\tau}g_{\mu\rho}-\sigma_{\mu\tau}g_{\nu\rho}-\sigma_{\nu\rho}g_{\mu\tau})\ ,$$

together with

$$(x_\mu,x_\nu)=0,\ \ (p_\mu,p_\nu)=0\ . \tag{2.2}$$

These relations imply from (1.2) that

$$(x_\mu,J_{\nu\rho})=-i(g_{\mu\rho}x_\nu-g_{\mu\nu}x_\rho)\ ,$$
$$(p_\mu,J_{\nu\rho})=-i(g_{\mu\rho}p_\nu-g_{\mu\nu}p_\rho)\ , \tag{2.3}$$
$$(J_{\mu\nu},J_{\rho\tau})=-i(J_{\mu\rho}g_{\nu\tau}+J_{\nu\tau}g_{\mu\rho}-J_{\mu\tau}g_{\nu\rho}-J_{\nu\rho}g_{\mu\tau})\ ,$$

as required.

From the definition (1.15) of ρ_μ, it follows that

$$(\rho_\mu,\sigma_{\nu\tau})=iM^{-2}(\sigma_{\mu\nu}p_\tau-\sigma_{\mu\tau}p_\nu)+i(g_{\mu\nu}\rho_\tau-g_{\mu\tau}\rho_\nu)\ , \tag{2.4}$$
$$(\rho_\mu,\rho_\nu)=-iM^{-2}\Sigma_{\mu\nu}\ , \tag{2.5}$$

and from (1.16)

$$(x_\mu,M^2)=-2ip_\mu \tag{2.6}$$

so that

$$\begin{aligned}(x_\mu,M^2\rho_\nu)&=-i\sigma_{\mu\nu}\\ &=M^2(x_\mu,\rho_\nu)-2ip_\mu\rho_\nu\ ,\end{aligned} \tag{2.7}$$

or

$$(x_\mu,\rho_\nu)=iM^{-2}(-\sigma_{\mu\nu}+2p_\mu\rho_\nu)\ .$$

Thus

$$(x_\mu,\rho_\nu)+(\rho_\mu,x_\nu)=-2iM^{-2}\Sigma_{\mu\nu}$$

and, from (1.13),

$$(X_\mu,X_\nu)=+iM^{-2}\Sigma_{\mu\nu}\ . \tag{2.8}$$

It also follows from (2.5) that

$$(\rho_\mu,\rho^\alpha\rho_\alpha)=-iM^{-2}(\sigma_{\mu\alpha}\rho^\alpha+\rho^\alpha\sigma_{\mu\alpha}-2\rho_\alpha\rho^\alpha p_\mu)\ , \tag{2.9}$$
$$\begin{aligned}(\sigma_{\mu\nu},\rho^\alpha\rho_\alpha)=-iM^{-2}[&(\rho^\alpha\sigma_{\alpha\mu}+\sigma_{\alpha\mu}\rho^\alpha)p_\nu\\ &-(\rho^\alpha\sigma_{\alpha\nu}+\sigma_{\alpha\nu}\rho^\alpha)p_\mu]\ ,\end{aligned} \tag{2.10}$$

so that

$$(\Sigma_{\mu\nu},\rho^\alpha\rho_\alpha)=0\ . \tag{2.11}$$

We now introduce the operators σ_μ, which transform as vectors with respect to the $\sigma_{\nu\rho}$:

$$(\sigma_\mu,\sigma_{\nu\rho})=-i(g_{\mu\rho}\sigma_\nu-g_{\mu\nu}\sigma_\rho)\ . \tag{2.12}$$

Thus

$$(\sigma_\mu,\rho_\nu)=iM^{-2}(p_\mu\sigma_\nu-g_{\mu\nu}(\sigma\cdot p))\ , \tag{2.13}$$
$$(\sigma\cdot p,\rho_\mu)=i\Gamma_\mu\ , \tag{2.14}$$

where [cf. (1.40)]

$$\begin{aligned}&\Gamma_\mu\equiv(\sigma_\mu-M^{-2}(\sigma\cdot p)p_\mu)\ ,\\ &\Gamma\cdot p\equiv\Gamma^\alpha p_\alpha=0\ ,\\ &\sigma\cdot p\equiv\sigma^\alpha p_\alpha\ .\end{aligned} \tag{2.15}$$

Also

$$(\sigma_\mu,\Sigma_{\alpha\beta})=iM^{-2}(\sigma_\alpha p_\beta-\sigma_\beta p_\alpha)p_\mu+i(g_{\mu\alpha}\Gamma_\beta-g_{\mu\beta}\Gamma_\alpha)\ , \tag{2.16}$$
$$(\sigma\cdot p,\Sigma_{\alpha\beta})=0\ . \tag{2.17}$$

This is significant, since $\sigma\cdot p$ will appear explicitly in the Hamiltonian.

We now postulate that

$$(\sigma_\mu,\sigma_\nu)=-i\sigma_{\mu\nu} \tag{2.18}$$

with the condition

$$(\sigma_\mu,\sigma^\alpha\sigma_\alpha)=0\ . \tag{2.19}$$

This is a necessary and sufficient condition that

$$\{\sigma_{\mu\alpha},\sigma^\alpha\}=0 \tag{2.20}$$

which also guarantees that

$$(\sigma_\mu,{\sigma_0}^2)=0\ . \tag{2.21}$$

Equation (2.20) would be the quantum generalization of the classical supplementary condition (1.5) if σ^α were the quantum generalization of the four-velocity v^α. This in its turn suggests the equation

$$(\sigma^\alpha p_\alpha-m)\psi=0 \tag{2.22}$$

as a generalization of Eq. (1.7).

We note that

$$(x_\mu,\sigma\cdot p)=-i\sigma_\mu,\ \ (\sigma_\mu,\sigma\cdot p)=iM^2\rho_\mu\ , \tag{2.23}$$

so that

$$(\sigma_\mu,(\sigma\cdot p)^2)=iM^2[\rho_\mu(\sigma\cdot p)+(\sigma\cdot p)\rho_\mu]\ . \tag{2.24}$$

Also

$$(\sigma_\mu,{\Sigma_0}^2)=i[\rho_\mu(\sigma\cdot p)+(\sigma\cdot p)\rho_\mu] \tag{2.25}$$

so that [Eq. (1.20)]

$$(\sigma_\mu,(\sigma\cdot\hat{p})^2-{\Sigma_0}^2)=0\ . \tag{2.26}$$

B. The Hamiltonian formulation

By analogy with Eq. (1.33) we postulate the Hamiltonian [cf. Eq. (2.22)]

$$H=\tfrac{1}{2}(\sigma^\alpha p_\alpha-m)\ . \tag{2.27}$$

For σ^α to correspond to v^α it is required that

$$\sigma_\mu=\dot{x}_\mu=i(x_\mu,H)\ . \tag{2.28}$$

Since $(x_\mu,\sigma_\nu)=0$ it follows that m must be an operator that satisfies the relation

$$(x_\mu, m) = i\sigma_\mu \tag{2.29}$$

so that

$$(\sigma_\mu, m) = i\dot{\sigma}_\mu \tag{2.30}$$

if also $(m, \sigma\cdot p) = 0$. The physical significance of the operator m has been determined for a more general classical theory of the symmetric top.[12] The result reduces to the well-known nonrelativistic expression for the energy which quantum mechanically is

$$m = \frac{1}{2I_2}[s(s+1) - \eta\epsilon^2] + \text{const} , \tag{2.31}$$

where $\eta = (I_1 - I_2)/I_1$, $\epsilon = s, s-1, \ldots -s$, and I_1, I_2, I_2 are the moments of inertia. Thus for a sphere with $I_1 = I_2 = I$,

$$m = \frac{1}{2I}s(s+1) + \text{const.} \tag{2.31$'$}$$

Since m is invariant, these values hold when the particle is executing *Zitterbewegung*, but they are then no longer equal to the energy M [see Eqs. (2.53) and (2.65)].

From (1.15), (2.18), (2.27), and (2.30)

$$\begin{aligned}\dot{\sigma}_\mu &= i(\sigma_\mu, H) \\ &= \tfrac{1}{2}\sigma_{\mu\alpha}p^\alpha - \frac{i}{2}(\sigma_\mu, m) \\ &= \sigma_{\mu\alpha}p^\alpha = -M^2\rho_\mu .\end{aligned} \tag{2.32}$$

Similarly

$$(\sigma_{\mu\nu}, \sigma\cdot p) = i(\sigma_\mu p_\nu - \sigma_\nu p_\mu) = -(\sigma_{\mu\nu}, m) \tag{2.33}$$

from (2.13) so that

$$\dot{\sigma}_{\mu\nu} = -(\sigma_\mu p_\nu - \sigma_\nu p_\mu)$$

which is the quantum generalization of the conservation equation (1.3).

It also follows that

$$(\rho_\mu, m) = i\Gamma_\mu = -(\rho_\mu, \sigma\cdot p) \tag{2.34}$$

so that

$$\dot{\rho}_\mu \equiv i(\rho_\mu, H) = \Gamma_\mu . \tag{2.35}$$

Since Eq. (2.32) may be written as

$$\dot{\Gamma}_\mu = -M^2\rho_\mu \tag{2.32$'$}$$

we have

$$\ddot{\rho}_\mu + M^2\rho_\mu = 0, \quad \ddot{\Gamma}_\mu + M^2\Gamma_\mu = 0 , \tag{2.36}$$

equations which describe the internal oscillations of the *Zitterbewegung*. We define

$$q_\mu^\pm = \Gamma_\mu \pm iM\rho_\mu, \quad q_\pm^\mu = \Gamma^\mu \pm iM\rho^\mu , \tag{2.37}$$

so that

$$(q_\mu^\pm, \sigma\cdot\hat{p}) = \pm q_\mu^\pm , \tag{2.38}$$

i.e., the $q_\mu^\pm$ are lowering and raising operators on $\sigma\cdot\hat{p}$.

From (2.26) and (2.27)

$$(\sigma\cdot\hat{p})^2 = \Sigma_0{}^2 + \epsilon , \tag{2.39}$$

where ϵ is a constant, so that, from (2.38)

$$(q_\mu^\pm, \Sigma_0{}^2) = (1 \pm 2\sigma\cdot\hat{p})q_\mu^\pm . \tag{2.40}$$

Thus if

$$\sigma\cdot\hat{p}\psi_0 = a\psi_0 , \Sigma_0{}^2\psi_0 = s(s+1)\psi_0 , \tag{2.41}$$

and

$$\psi_\mu^\pm \equiv q_\mu^\pm \psi_0 ,$$

then

$$\Sigma_0{}^2\psi_\mu^\pm = (s^2 + s + 1 \mp 2a)\psi_\mu^\pm, \quad \sigma\cdot\hat{p}\psi_\mu^\pm = (a \mp 1)\psi_\mu^\pm . \tag{2.42}$$

With

$$\begin{aligned}\Sigma_0{}^2\psi_\mu^+ &= (s+1)(s+2)\psi_\mu^+ , \\ \Sigma_0{}^2\psi_\mu^- &= (s-1)s\psi_\mu^- ,\end{aligned} \tag{2.43}$$

it follows that

$$a = -(s + \tfrac{1}{2}) , \tag{2.44}$$

i.e., the eigenvalue of $\sigma\cdot\hat{p}$ operating on a state of spin s is $-(s+\frac{1}{2})$, the quantum form of Eq. (1.30). The value of ϵ in Eq. (2.39) is therefore $\frac{1}{4}$.

From the definition of X_μ [Eq. (1.13)] we now have [cf. (1.21)]

$$\dot{X}_\mu = \sigma_\mu - \Gamma_\mu = M^{-1}\sigma\cdot\hat{p}p_\mu = -(s+\tfrac{1}{2})\hat{p}_\mu \tag{2.45}$$

so that

$$\dot{X}_\alpha \dot{X}^\alpha = (s + \tfrac{1}{2})^2 .$$

As before [Eq. (1.22)] we define[13]

$$\frac{dX_\mu}{dS} = -\frac{1}{s+\frac{1}{2}}\dot{X}_\mu = \hat{p}_\mu \tag{2.46}$$

the quantum generalization of the four-velocity of the center of mass. Thus, although the angular velocity of the *Zitterbewegung* along the path of the particle is M [Eq. (2.36)] its angular velocity ω in the center-of-mass system is $M/(s+\frac{1}{2})$, or with $\hbar$ and c no longer set equal to unity [cf. (1.11)]:

$$Mc^2 = (s + \tfrac{1}{2})\hbar\omega . \tag{2.47}$$

This simple and convincing equation relates the mass spectrum to the modes of oscillation of the *Zitterbewegung* and is in disagreement with Eq. (3.24) of Ref. 9, which relates ω to the spin and mass and an arbitrary parameter λ. With different Hamiltonians postulated, it is natural that these results need not agree.

Equation (2.39) (with $\epsilon = \frac{1}{4}$) is derived in Ref. 9 [Eq. (2.50)] and Ref. 10 [Eq. (A 19)] for the Majorana representation. The above analysis implies this representation. To see this, we note that from (2.5), after some algebra [cf. (2.9)]

$$\begin{aligned}M^2(\rho_\mu, \rho_\alpha\rho^\alpha) = {}& iM^{-2}(\sigma\cdot p\sigma_\mu + \sigma_\mu\sigma\cdot p) \\ &+ 2ip_\mu(\rho_\alpha\rho^\alpha - M^{-2}\sigma_\alpha\sigma^\alpha) .\end{aligned} \tag{2.48}$$

On contracting with p^μ we conclude that

$$\sigma_\alpha\sigma^\alpha=(\sigma\cdot\hat{p})^2+l_0^2 \quad (2.49)$$

$$=\sigma_0^2+\tfrac{1}{4} \quad (2.49')$$

by (1.29). However, from (2.12) and (2.18)

$$2(\sigma_\alpha\sigma_\mu\sigma^\alpha-\sigma_\alpha\sigma^\alpha\sigma_\mu)=-3\sigma_\mu \quad (2.50)$$

so that, on multiplying by σ^μ,

$$\sigma_\alpha\sigma^\alpha=\tfrac{2}{3}\sigma_0^2\ . \quad (2.51)$$

Hence, from (2.49)

$$\sigma_0^2=-\tfrac{3}{4},\ \ \sigma_\alpha\sigma^\alpha=-\tfrac{1}{2} \quad (2.52)$$

characteristic of the Majorana representation. The latter relation is a contracted form of the representation relation postulated in Ref. 9 [Eq. (2.41)]. In fact, the basic Majorana equation derived from (2.27) implies that

$$M=\pm\frac{m}{s+\frac{1}{2}} \quad (2.53)$$

the well-known down-going mass spectrum for constant m. Here, however, it is turned around to give Eq. (2.44):

$$\sigma\cdot\hat{p}=-(s+\tfrac{1}{2})\ . \quad (2.44')$$

In the center-of-mass system $\rho_0=0$, so that the radius ρ of the *Zitterbewegung* motion is given by[14]

$$\rho^2=-\rho_\alpha\rho^\alpha=\left[\frac{s^2+s+\frac{3}{4}}{M^2}\right]\ . \quad (2.54)$$

When this is compared with (2.47) we find that the speed $v\equiv\omega\rho$ of the motion is determined entirely by the spin:

$$v^2=1+\frac{1}{2(s+\frac{1}{2})^2}\ . \quad (2.55)$$

The speed is therefore greater than that of light, but since no energy is being transported at that rate, the result does not present any difficulty. In fact, from (2.53) this leads to the relation

$$M=m[2(v^2-1)]^{1/2}$$

which apart from a factor $\sqrt{2}$ reproduces (1.10).

We complete this section by listing a number of commutation relations between the Γ_μ, ρ_ν, and hence between the $q_\mu^\pm$:

$$(\Gamma_\mu,\Gamma_\nu)=-i\Sigma_{\mu\nu}\ ,$$

$$(\Gamma_\mu,\rho_\nu)=-iM^{-2}(g_{\mu\nu}-M^{-2}p_\mu p_\nu)(\sigma\cdot p)\ , \quad (2.56)$$

$$(\Gamma_\mu,\Gamma_\alpha\Gamma^\alpha)=-i(\sigma\cdot p\rho_\mu+\rho_\mu\sigma\cdot p)\ ,$$

$$(\rho_\mu,\Gamma_\alpha\Gamma^\alpha)=iM^{-2}(\sigma\cdot p\Gamma_\mu+\Gamma_\mu\sigma\cdot p)\ , \quad (2.57)$$

$$\begin{aligned}&(q_\mu^+,q_\nu^+)=0=(q_\mu^-,q_\nu^-)\ ,\\ &(q_\mu^+,q_\nu^-)=-2i\Sigma_{\mu\nu}-2M^{-1}(g_{\mu\nu}-M^{-2}p_\mu p_\nu)(\sigma\cdot p)\ ,\\ &q_\alpha^+q_+^\alpha=q_\alpha^-q_-^\alpha=0\ ,\\ &\dot{q}_\mu^\pm=\pm iMq_\mu^\pm\ ,\\ &\tfrac{1}{2}(q_\alpha^+q_-^\alpha+q_\alpha^-q_+^\alpha)=\Gamma_\alpha\Gamma^\alpha+M^2\rho_\alpha\rho^\alpha=2l_0^2\ ,\end{aligned} \quad (2.58)$$

since from (2.49)

$$\Gamma_\alpha\Gamma^\alpha=M^2\rho_\alpha\rho^\alpha=l_0^2\ . \quad (2.59)$$

Thus the q_μ^+ are commuting null vectors, so also are the q_μ^-. Each time that a wave function is multiplied by q_μ^+ the spin of the state increases by one unit, and, as in atomic physics, the parity is changed. In the center-of-mass rest system, $q_0^\pm=0$ and

$$q_i^\pm=\sigma_i\pm iM\rho_i\ .$$

Thus the $q_i^\pm$ are null vector operators in three dimensions, since $\sigma_i\sigma^i=M^2\rho_i\rho^i=s^2+s+\frac{3}{4}$ and $\rho_0=0$.

Most of the results described so far also apply to the Dirac equation, those results that are valid being trivial but instructive. Equation (2.48) vanishes so that, although (2.51) is valid, (2.49′) and (2.52) are not. Hence the $q_\mu^\pm$ are no longer null vectors. It is found that

$$\Gamma_\alpha\Gamma^\alpha=l_0^2=\tfrac{3}{4},\ \ \sigma_\alpha=\tfrac{1}{2}\gamma_\alpha\ ,$$

$$q_\mu^\pm=\frac{1}{2M}(\gamma_\mu\mp\hat{p}_\mu)(M\pm m_0)\ ,$$

where

$$(\gamma^\alpha p_\alpha-m_0)\psi=0\ .$$

Thus if $M=m_0>0$ then $q_\mu^-=0$ and q_μ^+ lowers $\gamma\cdot p$ by $2m_0$ to give the corresponding negative energy state; if $M=-m_0<0$ then $q_\mu^+=0$ and q_μ^- raises $\gamma\cdot p$ by $2m_0$. While the radii ρ of the hadrons [Eq. (2.54)] are real, the "radius" ρ of a Dirac particle is given by $\rho^2=-3/4M^2$ and therefore is not observable. An arbitrary parameter is clearly not needed or admissible, at least in this case.

C. The mass spectrum

It has been suggested many times[15] that some of the excited states of hadrons may be rotational levels. As with molecules and atomic nuclei, it should be possible to determine these levels, at least approximately, without any knowledge of the internal structure, except for the values of one or two parameters. The Hamiltonian (2.27) relates the eigenvalues of M and m, but it does not provide enough information to determine the mass spectrum. However, it leads to the conclusion that $p_\alpha p^\alpha$, $(\sigma\cdot\hat{p})$, and $\sigma_0^2=-\frac{3}{4}$ are constants for a free particle. It is necessary to postulate a constraint relation between them and examine the consequences.

In Ref. 9, Eq. (2.56), the postulated relation is in the form of the Hamiltonian

$$\mathscr{H}=\phi\{p_\alpha p^\alpha-\lambda^2[(\sigma\cdot\hat{p})^2+\alpha^2-\tfrac{5}{2}]\}=0\ , \quad (2.60)$$

where λ and α are constants. The equations of motion are derived from this Hamiltonian, or from various equivalent forms that it can assume. Here, however, the equations of motion for the *Zitterbewegung* have already been derived from the Hamiltonian (2.27). Since they also describe the *Zitterbewegung* of the Dirac equation and correspond very closely to the relativistic classical theory of Sec. I, we would like them to remain unchanged for any modified form of the Hamiltonian that is proposed. Since the Hamiltonian (2.27) is linear in $(\sigma\cdot\hat{p})$, we postulate that, instead of (2.58), the relation between $p_\alpha p^\alpha$ and $(\sigma\cdot\hat{p})$ should also be linear in $(\sigma\cdot\hat{p})$:

$$H=M(\lambda^{-2}p_\alpha p^\alpha+\sigma\cdot\hat{p}-b)=0\ , \tag{2.61}$$

where λ and b are arbitrary constants. The only equation of motion that is modified by this Hamiltonian, as opposed to (2.27) is Eq. (2.28) which becomes

$$\dot{x}_\mu=\sigma_\mu+2M\lambda^{-2}p_\mu \tag{2.62}$$

which does not affect the internal motion. Note that the sign of σ_μ is not determined by the defining Eqs. (2.12) and (2.18)—rather it is prescribed by the assumption (2.43) that ψ_μ^+ represents a state with spin $(s+1)$ and ψ_μ^- a state with spin $(s-1)$. If this identification is reversed, so also is the sign of σ_μ, and Eq. (2.61) is replaced by

$$H'=M(\lambda^{-2}p_\alpha p^\alpha-\sigma\cdot\hat{p}-b)=0\ . \tag{2.61$'$}$$

It is a short step from Eqs. (2.60), (2.61), and (2.61′) to the corresponding mass spectra. From (2.43), (2.44), and (2.61) we obtain the spectrum

$$M^2=\lambda^2(s+\tfrac{1}{2}+b)\ . \tag{2.63}$$

This linear relation between M^2 and s is remarkably accurate for many $C=0$ hadron towers, with λ^2 assuming approximately the same value for more than 50 hadron states that form a wide variety of types:

$$\lambda^2=1.1\pm0.06\ (\mathrm{GeV}/c^2)^2\ . \tag{2.64}$$

The parameter b is essentially constant for a given tower and for a number of towers it is close to an integer or half-integer. Equation (2.63) has been studied from the basis of Regge poles[16] supplemented either by the principle of maximum strength[17] or by the three-quark model for baryons.[18] The group structure of Eq. (2.61′) has been examined in detail[19] and the equation has been studied as a consequence of a generalization[20] of the 1971 Dirac equation,[21] and Eq. (2.63) has been examined simply in terms of the data.[22]

It may be noted that from (2.31′) and (2.53)

$$M^2=\frac{1}{2R^2}(s+\tfrac{1}{2})+\text{const}\ (s+\tfrac{1}{2})^{-1}\ , \tag{2.65}$$

where R is the radius of gyration. Comparison with Eq. (2.63) suggests a possible physical model for the parameter λ, i.e., $\lambda=2^{-1/2}R^{-1}$ for a spherical top.

The analysis of this paper does not distinguish between the masses of states of different parity, isospin, baryon number, and strangeness, and there is considerable evidence that for the higher states Nature does not appear to distinguish between them either. For example, there are at least 14 mesons and baryons with masses reported[23] to range between 1.71 and 1.65 GeV/c^2 and with mass uncertainties less than ±30 MeV/c^2. Their spins range from $\frac{1}{2}$ to 3, with $|S|=0,1,3$ and $I=0,\frac{1}{2},1,\frac{3}{2}$. With the value (2.64) and $s+b=2$, Eq. (2.63) gives $M=1.66$ GeV/c^2. Variations between the masses of the ground states are of course due to the internal SU(3) structure not considered here, but these variations, while presumably still present, play a relatively less important role for the excited states.

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