Gauge couplings as local gauge and Poincaré transformations: Generalized Taub-Volkov solutions

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The single-particle wave equations for a spin ≤ 1 charged particle coupled covariantly to an external electromagnetic plane-wave potential are solved in a simple and unified manner. The vector solution is new and corresponds to a Yang-Mills coupling to the external field. It is found that the solutions for spin ≤ 1 are all generated from the free-field solutions in the same way by local gauge and Poincaré transformations yielding a structure that is a distinctive feature of local gauge symmetry and corresponding to a certain equivalence between the free-field action and the external-field action.

I. INTRODUCTION

Long ago Volkov[1] found that the Dirac equation for a charged spin-$\frac{1}{2}$ particle in an external plane-wave field $[A_\mu = A_\mu(n\cdot x),\ n^2=0]$ could be solved exactly, implying a similar solution for the Klein-Gordon equation. Since then these solutions have been rederived many times and widely exploited often in conjunction with investigations of their physical significance. In particular, Taub's studies[2] revealed a Lorentz structure of the Volkov solution much after which Kupersztych[3] elucidated further aspects of the solution as a transformation.[4] Recently, we have shown[5] that the full underlying structure of spin ≤ 1 solutions consists of a product, ULT, of local gauge (U), Lorentz (L), and space-time displacement (T) transformations acting on the free-field ($A_\mu=0$) wave function.

The preceding general structure is destroyed by anomalous-magnetic-moment terms ($g\neq 2$) even though an exact solution is still possible.[6] The distinctive behavior of classical charged-particle observables (four-velocity and four-spin vectors) as evolving according to the same Lorentz transformation when $g=2$ was emphasized by Brodsky and Primack.[7,8] This suggests a relationship between the symmetry-transformation realization of the Volkov solutions for the Klein-Gordon and Dirac equations and the minimal-gauge-coupling prescription. A crucial test of this conjecture involves a charged spin-1 particle in a plane electromagnetic field since in this instance the minimal-coupling prescription is decidedly nontrivial.[9] Indeed, one finds[5] that for a value $\kappa=1$ and only this value of the magnetic-moment parameter of the vector particle, a spin-1 version of the Volkov solution is realized that has the same symmetry-transformation identifications as in the Dirac and Klein-Gordon cases.

This association of ULT-type Volkov solutions of the wave equations for spin $\equiv s \leq 1$ particles *gauge-covariantly*[10] coupled to an external plane electromagnetic wave is intrinsically interesting as a new characteristic of locally gauge-invariant *renormalizable* field theories. The single-particle solutions themselves are relevant to the representation of modifications arising from a background plane wave of the charged-particle lines appearing in a multiparticle diagram. They are therefore relevant to the general structure of particle interactions in field theory, an important example being the plane-wave decoupling theorem for tree graphs given in Ref. 5.[11,12]

This paper is devoted to the detailed investigation of the $s\leq 1$ Volkov solutions with particular attention to the properties of the ULT transformation that serves as their distinguishing feature. The symmetry apparent in the motion of a classical charged particle in a plane wave is reviewed and extended in Sec. II as providing motivation for the U,L,T transformations introduced in Sec. III in order to effect a similarity transformation connecting the covariant and ordinary derivatives. With this transformation the derivation in Sec. IV of the Volkov solutions is almost trivial. This greatly simplifies previous derivations for $s=0,\frac{1}{2}$ and provides a new result for $s=1$ which emphasizes the intimate relationship between these solutions and gauge-covariant couplings. We also show in Sec. IV that the action for the coupling to external plane waves is turned into the free-field action under the ULT transformation; this action equivalence provides an elegant way of rephrasing the characteristic transformation properties of our solutions. Our results are summarized in Sec. V.

II. CLASSICAL CHARGED-PARTICLE MOTION IN A PLANE ELECTROMAGNETIC FIELD

We show in this section that the symmetries of the Volkov solutions are manifested at the classical level by reconsidering the problem[13–18] of a charged particle in a plane-wave field as governed by the Lorentz force equation

$$\frac{du^\mu}{d\tau} = \frac{Q}{m} F^\mu{}_\nu u^\nu\,, \tag{2.1}$$

for the covariant velocity and (under certain approximations) by the Bargman, Michel, and Telegdi[19] equation for the four-polarization vector s^μ,

$$\frac{ds^\mu}{d\tau} = \frac{Q}{m}\left[\delta^\mu_\sigma + \left(1-\frac{g}{2}\right) u^\mu u_\sigma\right] F^\sigma{}_\nu s^\nu\,, \tag{2.2}$$

where τ is the particle proper time. For $g=2$ Eqs. (2.1) and (2.2) have the same structure with their evolution governed by the local infinitesimal Lorentz transformations $(Q/m)F^{\mu}{}_{\nu}d\tau$ (Refs. 7 and 20) and it is this case that is of prime interest to us.

Generally, $F_{\mu\nu}$ depends on both $x^{\mu}(\tau)$ and τ but when

$$F_{\mu\nu}=F_{\mu\nu}(\xi) , \tag{2.3}$$

where $\xi\equiv n\cdot x$ and n is a constant four-vector, the solution of (2.1) and (2.2) greatly simplifies provided that

$$n^{\mu}F_{\mu\nu}=0 , \tag{2.4}$$

corresponding to the propagation condition, $\partial^{\mu}F_{\mu\nu}=0$ which leads to

$$\xi=\left[\frac{n\cdot p}{m}\right]\tau+\xi_0 , \tag{2.5}$$

where $p(\tau)\equiv mu(\tau)$ and $n\cdot p$ is now a constant of the motion. The usual plane-wave situation is realized by (2.3)—(2.5) when n^{μ} is identified with the propagation vector ($n^2=0$); however, we suppose $n^2\neq 0$, until a certain point, as a classical analog of charged-particle interactions with virtual photons.

Given (2.5), $F_{\mu\nu}$ is an *explicit* function of τ and the solutions of (2.1) and (2.2) can be expressed in terms of the evolution matrix[3,6,15]

$$\mathscr{U}^{\mu}{}_{\nu}(\tau,\tau_0)=\delta^{\mu}_{\nu}+\int_{\tau_0}^{\tau}(d\tau')K^{\mu}{}_{\lambda}(\tau')\mathscr{U}^{\lambda}{}_{\nu}(\tau',\tau_0) , \tag{2.6}$$

where the kernels of (2.6) corresponding to (2.1) and (2.2) are identical only if $g=2$. If $K^{\mu\lambda}=-K^{\lambda\mu}$, then $\mathscr{U}^{\mu}{}_{\nu}$ is a τ-dependent Lorentz transformation.

If $F_{\mu\nu}$ satisfies (2.3) and (2.4), the kernel[21]

$$\begin{aligned}K_1^{\mu\nu}&\equiv\frac{Q}{m}F^{\mu\nu}(\xi)\\&=\frac{Q}{m}\frac{d}{d\xi}[n_{\mu}A_{\nu}(\xi)-A_{\mu}(\xi)n_{\nu}]\end{aligned} \tag{2.7}$$

generates via (2.6) the Lorentz transformation

$$\Lambda(\tau,\tau_0)=P\{\exp[\Omega(\xi)-\Omega(\xi_0)]\} , \tag{2.8}$$

where P denotes the (proper) time ordering of the exponential[22] and

$$\Omega^{\mu}{}_{\nu}(\xi)=\frac{Q}{n\cdot p}[n^{\mu}A_{\nu}(\xi)-A^{\mu}(\xi)n_{\nu}] . \tag{2.9}$$

Clearly $\Lambda^{\mu}{}_{\nu}$ is invariant under the class of gauge transformations $A_{\mu}(\xi)\rightarrow A_{\mu}(\xi)+\partial_{\mu}\chi(\xi)$.

We note that (2.8) is in general distinct from the unordered Lorentz transformation

$$[\Lambda(\tau,\tau_0)]_{\text{unordered}}=\exp[\Omega(\xi)-\Omega(\xi_0)] , \tag{2.10}$$

which can be expressed as

$$[\Lambda(\tau,\tau_0)]_{\text{unordered}}=I+\frac{\sinh\alpha}{\alpha}[\Omega(\xi)-\Omega(\xi_0)]+\left[\frac{\cosh\alpha-1}{\alpha^2}\right][\Omega(\xi)-\Omega(\xi_0)]^2 , \tag{2.11}$$

where the gauge-invariant quantity

$$\alpha(\tau,\tau_0)=[(n\cdot\hat{\epsilon})^2-\hat{\epsilon}^2n^2]^{1/2} \tag{2.12}$$

may be either purely real or purely imaginary and

$$\hat{\epsilon}_{\mu}(\tau,\tau_0)=\frac{Q}{n\cdot p}[A_{\mu}(\xi)-A_{\mu}(\xi_0)] . \tag{2.13}$$

The transformation (2.10)—(2.13) is relevant to arbitrary (virtual or real) photon attachments onto charged-particle lines.[23–25]

The propagation condition (2.4) implies that $\alpha=0$ and therefore (2.8) and (2.11) are equivalent and reduce to

$$\Lambda(\tau,\tau_0)=I+[\Omega(\xi)-\Omega(\xi_0)]+\tfrac{1}{2}[\Omega(\xi)-\Omega(\xi_0)]^2 \tag{2.14}$$

and

$$\{[\Omega(\xi)-\Omega(\xi_0)]^2\}^{\mu}{}_{\nu}=n^2\hat{\epsilon}^{\mu}\hat{\epsilon}_{\nu}-\hat{\epsilon}^2n^{\mu}n_{\nu} , \tag{2.15}$$

which vanishes for $n^2\neq 0$.[26] Henceforth, we take $n^2=0$. Typically ξ_0 is identified with an asymptotic value for which $A_{\mu}(\xi_0)=0$, so then $n\cdot A(\xi)=0$. Specifically we choose $A_{\mu}(-\infty)=0$ as part of our choice of gauge.[27,28] Then (2.14) assumes the exact form of the local Lorentz transformation[29] employed in Ref. 5:

$$\Lambda_{\mu\nu}(\xi)=g_{\mu\nu}+\Omega_{\mu\nu}(\xi)-\frac{Q^2}{2(n\cdot p)^2}A^2(\xi)n_{\mu}n_{\nu} . \tag{2.16}$$

At a proper time τ the covariant particle velocity is

$$u^{\mu}(\tau)=\Lambda^{\mu}{}_{\nu}(\xi)u^{\nu}(-\infty) , \tag{2.17}$$

so that

$$p^{\mu}(\tau)=(p^{\mu}-QA^{\mu})+Q\left[\frac{p\cdot A}{n\cdot p}-\frac{Q}{2}\left[\frac{Q}{n\cdot p}\right]A^2\right]n^{\mu} . \tag{2.18}$$

The particle orbit is obtained as an (indefinite) quadrature of (2.18):

$$x^{\mu}(\tau)=\frac{1}{m}\int^{\tau}(d\tau')[p^{\mu}-QA^{\mu}(\xi')]+(p\cdot d-Q\theta)\frac{n^{\mu}}{n\cdot p} , \tag{2.19}$$

where the local displacement vector

$$d^{\mu}=\frac{Q}{n\cdot p}\int^{\xi}(d\xi')A^{\mu}(\xi') \tag{2.20}$$

and the local phase angle in charge space

$$\theta=\frac{Q}{2n\cdot p}\int^{\xi}(d\xi')A^2(\xi') \tag{2.21}$$

are the translation and gauge transformation parameters that are shown to enter into the spin ≤ 1 Volkov solutions in the next section.

Our principal interest in the particle orbit lies in its crucial role in the expression of the classical action

$$I(\xi)=-\int^{\tau}(d\tau')\left[m+\frac{Q}{m}A\cdot p\right], \tag{2.22}$$

which, if we use (2.18) and (2.19), can be placed into the form

$$I(\xi)=-p\cdot x-p\cdot d+Q\theta\,, \tag{2.23}$$

where x, d, and θ each depend upon ξ. The classical action appears as the phase of the Volkov solutions that evidently differs from its free values by contributions from a gauge transformation and a space-time translation.

The Lorentz symmetry aspect of the ULT transformation is reflected in (2.16)–(2.18), where here the vector representation is, of course, appropriate, and in the fact that when $g=2$, the polarization vector $s^{\mu}(\tau)$ evolves exactly as $u^{\mu}(\tau)$ does [cf. (2.17)]. We note that generally the kernel of (2.6) appropriate to (2.2) is

$$K_2^{\mu\nu}=\frac{Q}{m}F^{\mu\nu}(\xi)+\frac{Q}{m}\left[1-\frac{g}{2}\right]u^{\mu}(\xi)F^{\sigma}{}_{\nu}(\xi)\,; \tag{2.24}$$

however, $K_2^{\mu\nu}$ is not antisymmetric and thus $s^{\mu}(\tau)$ is not generated by a Lorentz transformation unless $g=2$. Equations (2.16)–(2.19) continue to hold and closed-form solutions for $s^{\mu}(\tau)$ are still possible if $g\neq 2$ when, for instance, $A_{\mu}(\xi)=\epsilon_{\mu}f(\xi)$ (Refs. 6 and 22) but the Lorentz symmetry interpretation is lost.

In summary, for a plane-wave external field, both of the kinematic variables $p^{\mu}(\tau)$ and $s^{\mu}(\tau)$ of a classical charged particle with $g=2$ evolve from their asymptotic values by the *same* local Lorentz transformation.[30] In the case of $x^{\mu}(\tau)$, the integral of this Lorentz transformation over the evolution of the orbit is required. This brings into play the generators of the local space-time translation and local gauge transformation that also enter into the classical action as well as into the phases of the Volkov solutions.

III. U,L,T AND COVARIANT DERIVATIVES

The wave equations for particles with spins ≤ 1, charge Q, mass m, and gauge-covariantly[10] coupled to an external plane wave $A_{\mu}=A_{\mu}(\xi),\xi=n\cdot x,n^2=0$, in the Lorentz gauge $n\cdot A=0$ (cf. Sec. II), are

$$(D^2+m^2)\Psi=0\quad(\text{scalar})\,, \tag{3.1a}$$

$$(i\not{D}-m)\Psi=0\quad(\text{Dirac})\,, \tag{3.1b}$$

$$(D^2+m^2)\Psi_{\mu}+2iQF_{\mu\nu}\Psi^{\nu}=0$$
with (3.1c)
$$D\cdot\Psi=0\quad(\text{vector})\,.$$

The covariant derivative is

$$D_{\mu}=\partial_{\mu}+iQA_{\mu}\,. \tag{3.2}$$

We seek solutions of (3.1) of the form

$$\Psi(x)=V(\xi)\chi(x)\,, \tag{3.3}$$

where $\chi(x)$ represents solutions of (3.1) for $A_{\mu}=0$ subject, e.g., to the initial condition $\Psi(x)\to\chi(x)$ for $\xi\to-\infty$ [recall $A_{\mu}(-\infty)=0$].

The development in Sec. II suggests that the structure of $V(\xi)$ depends on a form of (2.14) appropriate to the present situation:

$$\Lambda_{\mu\nu}=(e^{\Omega})_{\mu\nu}=g_{\mu\nu}+\Omega_{\mu\nu}-\tfrac{1}{2}f^2A^2n_{\mu}n_{\nu}\,, \tag{3.4}$$

where

$$\Omega_{\mu\nu}=f\int_{-\infty}^{\xi}(d\xi')F_{\mu\nu}(\xi')=f(n_{\mu}A_{\nu}-A_{\mu}n_{\nu})\,, \tag{3.5}$$

and

$$f\equiv\frac{Q}{in\cdot\partial} \tag{3.6}$$

generalizes the role of the radiation factor $Q/p\cdot n$ appropriate to photon attachments onto lines with well-defined particle momenta. We note that since $(n\cdot\partial)\xi=(A\cdot\partial)\xi=0$, if $G(\xi)$ is any function of ξ,

$$[n\cdot\partial,G(\xi)]=0\,, \tag{3.7}$$

$$[A\cdot\partial,G(\xi)]=0\,, \tag{3.8}$$

and consequently

$$\left[\frac{1}{n\cdot\partial},G(\xi)\right]=0\,. \tag{3.9}$$

Also

$$[n\cdot\partial,A\cdot\partial]=0\,. \tag{3.10}$$

The inverse operator $(n\cdot\partial)^{-1}$ is well defined on the space of plane-wave solutions of the Klein-Gordon equation $(m\neq 0)$. We expect that $f^{\dagger}=f$ under suitable stipulations about a scalar product; however, we have no need to define the adjoint of f in the present section. All differential operators or their inverses are taken to operate to the right.

Since

$$\Lambda^{\mu}{}_{\nu}n^{\nu}=n^{\mu}\,, \tag{3.11}$$

Λ is an element of the (local) little group $E_2(\xi)$. Also, Λ generates a gauge transformation on A_{μ}:

$$\Lambda_{\mu}{}^{\nu}A_{\nu}=A_{\mu}+\partial_{\mu}(2\theta)\,, \tag{3.12}$$

which is consistent with the invariance[3,31]

$$\Lambda^{\mu}{}_{\alpha}\Lambda^{\nu}{}_{\beta}F^{\alpha\beta}=F^{\mu\nu}\,. \tag{3.13}$$

In (3.12) we have[32]

$$\theta\equiv\frac{f}{2}\int_{-\infty}^{\xi}(d\xi')A^2(\xi')\,, \tag{3.14}$$

which generalizes (2.21). A similar generalization of (2.20),

$$d^{\mu}=f\int_{-\infty}^{\xi}(d\xi')A^{\mu}(\xi')\,, \tag{3.15}$$

also enters into the following alternative expression for the infinitesimal generator of Λ:

$$\Omega_{\mu\nu}=(\partial_{\mu}d_{\nu}-\partial_{\nu}d_{\mu})\,. \tag{3.16}$$

Corresponding to (3.14) and (3.15) we have the gauge transformation

$$U=e^{iQ\theta} \tag{3.17}$$

and the space-time displacement transformation

$$T=e^{-id(i\partial)}\ , \tag{3.18}$$

respectively. U and T are unitary with an appropriate definition of the operator adjoints.

The Dirac realization of the Lorentz subgroup generated by Λ, viz.,

$$L^{-1}\gamma^{\mu}L=\Lambda^{\mu}{}_{\nu}\gamma^{\nu}\ , \tag{3.19}$$

is given by

$$L=1+\mathscr{F}\ , \tag{3.20}$$

where

$$\mathscr{F}\equiv\tfrac{1}{2}f\not{n}\not{A}\ . \tag{3.21}$$

Evidently

$$L^{-1}=1-\mathscr{F}\ . \tag{3.22}$$

Thus for the respective representations (scalar; Dirac; vector); we have

$$L=e^{S}=\{1;1+\mathscr{F};\Lambda\}\ , \tag{3.23}$$

$$S=\{0;\mathscr{F};\Omega\}\ . \tag{3.24}$$

The transformations U, L, and T all commute with each other but not with ∂_μ or D_μ since

$$[D_\mu,U]=[\partial_\mu,U]=\frac{i}{2}n_\mu QfA^2U\ , \tag{3.25a}$$

$$[D_\mu,T]=[\partial_\mu,T]=n_\mu Tf(A\cdot\partial)\ . \tag{3.25b}$$

Although $[D_\mu,L]=[\partial_\mu,L]=0$ in the scalar and Dirac cases, we note that the local Lorentz transformation given by (3.4) satisfies

$$\Lambda_\mu{}^{\nu}\partial_\nu=D_\mu+[D_\mu,U]+[D_\mu,T]\ , \tag{3.26}$$

so

$$[D_\mu,UT]=UT(\Lambda_\mu{}^{\nu}\partial_\nu-D_\mu)\ , \tag{3.27}$$

from which we obtain[33]

$$(UT)^{-1}D_\mu(UT)=\Lambda_\mu{}^{\nu}\partial_\nu\ . \tag{3.28}$$

Two other useful identities that follow from (3.4) are

$$\Lambda_\mu{}^{\nu}\partial_\nu\Lambda^{\mu}{}_{\lambda}=\partial_\lambda \tag{3.29}$$

and

$$[\partial^2,\Lambda^{\nu}{}_{\alpha}]=-2iQ\Lambda^{\nu}{}_{\lambda}F^{\lambda}{}_{\alpha}=2iQF^{\nu}{}_{\lambda}\Lambda^{\lambda}{}_{\alpha}\ . \tag{3.30}$$

From (3.28) and (3.29) we obtain

$$(UT)^{-1}D^2UT=\partial^2\ , \tag{3.31}$$

which can be used with (3.30) to prove

$$(UT)^{-1}(g_{\mu\nu}D^2+2iQF_{\mu\nu})(UT)\Lambda^{\nu}{}_{\alpha}=\Lambda_{\mu\alpha}\partial^2\ . \tag{3.32}$$

The identities (3.28)—(3.32) are utilized in the next section to solve Eqs. (3.1).

IV. VOLKOV SOLUTIONS AND EQUIVALENT ACTIONS

The identities (3.28), (3.31), and (3.32) imply that the solutions of (3.1) have the form [cf. (3.3)]

$$\Psi(x)=ULT\chi(x)\ , \tag{4.1}$$

where L is given by (3.23) for each of the three cases. In this regard, we note that with (3.19) we obtain from (3.28)

$$(UL_{\mathrm{Dirac}}T)^{-1}\not{D}(UL_{\mathrm{Dirac}}T)=\not{\partial}\ , \tag{4.2}$$

which immediately establishes (4.1) in the Dirac case. In the vector case one finds using (3.30) and (3.32) that

$$(g_{\mu\nu}D^2+2iQF_{\mu\nu})(ULT)^{\nu}{}_{\alpha}=UT\Lambda_{\mu\alpha}\partial^2\ , \tag{4.3}$$

which again leads to (4.1), where we also see, with the aid of (3.29), that

$$D_\mu\Psi^{\mu}=UT\partial_\mu\chi^{\mu}=0\ . \tag{4.4}$$

In analogy with (3.31) and (4.2) we can restate (4.3) as a similarity transformation

$$(ULT)_{\beta}{}^{-1\mu}(g_{\mu\nu}D^2+2iQF_{\mu\nu})(ULT)^{\nu}{}_{\alpha}=g_{\beta\alpha}\partial^2\ . \tag{4.5}$$

Next we consider the action

$$A=\int(d^4x)\mathscr{L}(x)\ , \tag{4.6}$$

where $\mathscr{L}(x)$ is the Lagrangian density corresponding to the equations of motion (3.1). Even though we are working with classical fields it is convenient to express the real, quadratic combinations of the complex (charged) fields in terms of a formal adjoint operation that avoids the notational complication of explicitly denoting the equivalent left and right differentiations. Total derivatives are involved either way in the integrations by parts involved in switching these operations and, as usual, their contributions to A are ignored.

In the scalar case we have

$$\mathscr{L}_{\mathrm{scalar}}(x)=(D^{\mu}\Psi)^{\dagger}(D_\mu\Psi)+m^2\Psi^{\dagger}\Psi\ , \tag{4.7}$$

with Ψ given by (4.1). Our adjoint convention is such that if $G(i\partial)$ is any function of $i\partial$ then[34]

$$G(i\partial)^{\dagger}=G(-i\partial)^{*}$$

and

$$\Psi^{\dagger}=\chi^{*}(UT)^{\dagger}=\chi^{*}(UT)^{-1}\ . \tag{4.8}$$

Clearly $\Psi^{\dagger}(x)$ and $\Psi^{*}(x)$ differ only by surface terms. It is simple to show using (3.28) that

$$(D^{\mu}\Psi)^{\dagger}(D_\mu\Psi)=(\partial^{\mu}\chi)^{\dagger}(\partial_\mu\chi) \tag{4.9}$$

so

$$\mathscr{L}_{\mathrm{scalar}}(\Psi^{\dagger},\Psi)=\mathscr{L}^{0}_{\mathrm{scalar}}(\chi^{\dagger},\chi)\ , \tag{4.10}$$

where $\mathscr{L}^{0}_{\mathrm{scalar}}$ is given by (4.7) with $D_\mu\to\partial_\mu,\Psi\to\chi$.

It easily follows from (4.2) that for Dirac fields

$$\mathscr{L}_{\mathrm{Dirac}}(\overline{\Psi},\Psi)=\mathscr{L}^{0}_{\mathrm{Dirac}}(\overline{\chi},\chi)\ , \tag{4.11}$$

where

$$\mathscr{L}_{\mathrm{Dirac}}(\overline{\Psi},\Psi)=i\overline{\Psi}\not{D}\Psi-m\overline{\Psi}\Psi \tag{4.12}$$

and $\mathscr{L}^0_{\rm Dirac}$ is interpreted as before.

The corresponding calculation for the vector Lagrangian

$$\mathscr{L}_{\rm vector}(\Psi_\mu,\Psi_\mu{}^\dagger)=-\tfrac{1}{2}(D_\mu\Psi_\nu-D_\nu\Psi_\mu)^\dagger(D^\mu\Psi^\nu-D^\nu\Psi^\mu)+(\Psi_\mu{}^\dagger\Psi_\nu)iQF^{\mu\nu}+m^2\Psi_\mu{}^\dagger\Psi^\mu\ , \tag{4.13}$$

is considerably more tedious. Note, e.g., that

$$\Psi_\mu{}^\dagger\Psi^\mu=(\Lambda_\mu{}^\gamma UT\chi_\gamma)^\dagger(\Lambda^\mu{}_\lambda UT\chi^\lambda)=\chi_\gamma{}^\dagger\Lambda_\mu{}^\gamma\Lambda^\mu{}_\lambda\chi^\lambda=\chi_\mu{}^\dagger\chi^\mu\ , \tag{4.14}$$

to within surface terms. One finds that

$$\mathscr{L}_{\rm vector}(\Psi_\mu,\Psi_\mu^\dagger)=\mathscr{L}^0_{\rm vector}(\chi_\mu,\chi_\mu^\dagger)-\Delta\ , \tag{4.15}$$

where

$$\Delta=[(\bar{A}\cdot\partial)(n\cdot\chi)]^\dagger f(n\cdot\chi)+(n\cdot\chi)^\dagger f(\bar{A}\cdot\partial)(n\cdot\chi) \tag{4.16}$$

and $\bar{A}_\mu(\xi)=dA_\mu(\xi)/d\xi$. Since by our adjoint convention

$$[(\bar{A}\cdot\partial)(n\cdot\chi)]^\dagger=(n\cdot\chi)^\dagger(\bar{A}\cdot\partial)^\dagger=-(n\cdot\chi)^\dagger(\bar{A}\cdot\partial)\ ,$$

it is easily seen that $\Delta=0$.

The preceding *formal* derivation is independent of whether or not $\chi(x)$ satisfies the appropriate free-particle wave equation. Thus, e.g., given (4.1), the following two self-interacting scalar Lagrangians have the same action:

$$\mathscr{L}_{\rm scalar}(\chi,\chi^\dagger)=(\partial^\mu\chi)^\dagger(\partial_\mu\chi)+V(\chi^\dagger\chi)\ ,$$

$$\mathscr{L}_{\rm scalar}(\Psi^\dagger,\Psi)=(D^\mu\Psi)^\dagger(D_\mu\Psi)+V(\Psi^\dagger\Psi)\ ,$$

where the potential $V(\chi^\dagger\chi)$ is a function of $\chi^\dagger\chi$. The solutions of the corresponding inhomogeneous forms of the wave equations (3.1) are also related by (4.1). What prevents these (self-) interacting versions of the Volkov solutions from being valid without further qualifications is the crucial assumption concerning the existence of the inverse operator $(n\cdot\partial)^{-1}$. This inverse is guaranteed to exist if $\chi(x)$ corresponds to (a superposition of) free states $(m\neq 0)$.

The transformation (4.1), which depends upon the external-field interactions from $\xi=-\infty$ to $\xi=n\cdot x$, connects the interacting and noninteracting Lagrangians and therefore functions as an evolution operator. The equivalences (4.10), (4.11), and (4.15) are precursors of the decoupling behavior[5,12] of a self-interacting system (in the tree approximation) with N well-defined external momenta from a background electromagnetic plane-wave field which has been characterized as *radiation symmetry*.[35]

V. REMARKS AND CONCLUSIONS

1. We have shown that the problem of a classical spinning charged particle in a plane electromagnetic field reflects the symmetry properties of the spin ≤ 1 Volkov solutions to the corresponding single-particle wave equations. In particular, the classical action, which is shown to be a sum of contributions from the generators of local gauge and space-time translations, appears in the phase of the Volkov solutions. The Lorentz transformation, that for $g=2$ describes the evolution of the classical kinematic variables p^μ and s^μ is also the same one that enters into the Volkov solutions.

2. A new Volkov-type solution is found for charged spin-unity particles, where the counterpart of classical minimal coupling with $g=2$ is the stipulation of a Yang-Mills coupling to the external electromagnetic field. The vector Volkov solution requires the full trilinear coupling corresponding to a magnetic moment parameter $\kappa=1$.

3. The Volkov solutions for spins ≤ 1 are obtained in a straightforward way by the use of identities connecting the covariant and ordinary derivatives.[36] These identities involve the local gauge and Poincaré transformations (U,L,T) as distinctive features both of the Volkov solutions and of the underlying gauge symmetry.

4. The L and T transformations are the respective finite forms [appropriate to external (multiphoton) fields, Refs. 5, 12, and 24] of the infinitesimal Lorentz and space-time translations that characterize single-photon attachments onto charged-particle lines.[24,37] The finite gauge transformation U, with an $O(Q^2)$ infinitesimal generator is relevant to n-photon attachments with $n\geq 2$.

5. The field Lagrangians for charged spin ≤ 1 particles minimally coupled to a background electromagnetic plane-wave field are shown to be equal (to within surface terms) to their respective free Lagrangians when the interacting and free classical fields are related by the ULT transformation.

6. The fact that the preceding results seem to have no immediate higher-spin counterparts supports the observed correlation between Volkov solutions and renormalizable electromagnetic couplings. For example, in the spin-$\frac{3}{2}$ version of (4.1) the vector-indexed spinor wave function ψ_μ would be expected to have [cf. (3.23)] the Lorentz part $L_{3/2}=L_{\rm vector}L_{\rm Dirac}$ if the Volkov pattern were extended to this case. Nonetheless, a wave function with this structure is not a solution of any of the obvious possibilities for minimally coupled extensions of the Rarita-Schwinger equations for the case of a charged spin-$\frac{3}{2}$ particle in a plane electromagnetic field. We conjecture that the nonrenormalizability of spin-$\frac{3}{2}$ electrodynamics is reflected in the absence of Volkov solutions for this case.

7. Higher-spin boson and fermion gauge fields such as the graviton (spin 2) and the gravitino (spin $\frac{3}{2}$) may generate their own distinctive versions of the "Volkov solutions" given suitable restrictions on their couplings. Also, we have recently shown[38] that the theorem for radiation zeros, which provides a signature for renormalizable photon couplings also applies, given supersymmetry, to spin-$\frac{1}{2}$ photino emission and absorption. This suggests the possibility of supersymmetric analogs of the photon Volkov solutions, only now with an external plane-wave "photino field."

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[8] See also Refs. 3.

[9] A particularly lucid discussion is given by I. J. R. Aitchison and A. J. G. Hey, *Gauge Theories in Particle Physics* (Adam Hilger, Bristol, 1982), pp. 228—232.

[10] Generally, this means that the following kinds of (covariant) derivatives are allowed in interaction Lagrangians: none for Dirac, single for scalar, single for vector, and double (Higgs-type) for scalar as arise in Yang-Mills trilinear forms, and products of these forms. In the present context this is consistent with the application of the usual minimal-coupling prescriptions, but with no anomalous magnetic-moment term for $s=\frac{1}{2}$ and with $\kappa=1$ for $s=1$.

[11] See Ref. 12, which is the sequel to this paper, for further details of the proof of this theorem.

[12] R. W. Brown and K. L. Kowalski (unpublished).

[13] This problem has been studied many times. See, e.g., Refs. 3, 6, 7, and 14—18. Nevertheless, the present treatment contains several features both in technical detail as well as interpretation that do not seem to have been discussed previously.

[14] J. Frenkel, Z. Phys. **32**, 27 (1925).

[15] A. H. Taub, Phys. Rev. **73**, 786 (1948).

[16] L. S. Brown and T. W. B. Kibble, Phys. Rev. **133**, A705 (1964).

[17] L. S. Brown and R. L. Goble, Phys. Rev. **173**, 1505 (1968).

[18] C. Itzykson and J. B. Zuber, *Quantum Field Theory* (McGraw-Hill, New York, 1980), pp. 18 and 19.

[19] V. Bargman, L. Michel, and V. L. Telegdi, Phys. Rev. Lett. **2**, 435 (1959).

[20] The association of $F^{\mu}{}_{\nu}$ with infinitesimal generators of Lorentz transformations seems to have been first made (and exploited) by Taub (Ref. 15).

[21] We fix the class of gauges to those where A_μ is only a function of ξ. No further gauge fixing is required if (2.4) is satisfied.

[22] It is pointed out in Refs. 3, 6, and 15 that the solution (2.8) of (2.6) is an ordinary exponential provided the matrices $F(\tau)$ and $\int^{\tau}(d\tau')F(\tau')$ commute. This condition is satisfied independently of (2.4), and whether or not $n^2=0$, if $A_\mu(\xi)=\epsilon_\mu f(\xi)$, where ϵ_μ is a constant polarization vector. The development in Ref. 18 is confined only to this case. Generally, we do not restrict ourselves to constant polarizations.

[23] See Refs. 24 and 25 where photon attachments with momenta $q^2\neq 0$ and the infinitesimal generators of (2.11) are considered. Note that (2.11), which to our knowledge has not appeared in the literature previously, holds without any restrictions upon the polarization thus allowing longitudinal and scalar polarizations for off-mass-shell photons.

[24] R. W. Brown, K. L. Kowalski, and S. J. Brodsky, Phys. Rev. D **28**, 624 (1983).

[25] R. W. Brown and K. L. Kowalski, Phys. Rev. D **29**, 2100 (1984).

[26] For $\alpha=0$ we have $F(\xi)F(\xi')F(\xi'')=0$, while $F(\xi)F(\xi')=0$ as well when $n^2\neq 0$. Note also that $\hat{\epsilon}\cdot n=0$ when $n^2=0$.

[27] This does not imply $A_\mu(+\infty)=0$, however, as pointed out in Ref. 28.

[28] T. W. B. Kibble, Phys. Rev. **138**, B740 (1965).

[29] See also Ref. 24. This Lorentz transformation seems to have been first recognized as a part of the plane-wave problem by Taub (Refs. 2 and 15) and was exploited extensively by Kupersztych (Ref. 3). The properties of (2.14) are recognized and utilized by Brown and Goble (Ref. 17) in an auxiliary calculation.

[30] This is similar to the generation of the dynamics by the type of canonical transformations characteristic of Hamilton-Jacobi theory (cf. Ref. 16).

[31] The close relationship of $F_{\mu\nu}$ and $\Lambda_{\mu\nu}$ is reflected in the divergenceless properties $\partial^\mu\Lambda_{\mu\nu}=\partial^\nu\Lambda_{\mu\nu}=0$ and the fact that $\Lambda_{\mu\nu}$ itself satisfies the wave equation $\partial^2\Lambda_{\mu\nu}=0$.

[32] We have chosen a fixed lower limit on our ξ integrals consistent with our stipulation $A_\mu(-\infty)=0$. The essential features of the analysis carry through with the lower limits left indefinite.

[33] The similarity transformation (3.28) for D_μ, in which the roles of the symmetries are exploited and most expedient, should be compared with the similarity transformation for ∂_μ alone, given by Eq. (2.13) of Ref. 6.

[34] For the formally Hermitian momentum operator $i\partial$ we have $(i\overrightarrow{\partial})^\dagger=(-i\overleftarrow{\partial})=i\overrightarrow{\partial}$, where the arrows refer to the direction of the differentiation operator. Our adjoint convention is similar to that used in the coordinate representation of ordinary quantum mechanics.

[35] A review and discussion of radiation symmetry is found in R. W. Brown, in *Electroweak Effects at High Energies,* proceedings of the Europhysics Study Conference, Erice, Italy, 1983, edited by H. Newman (Plenum, New York, 1984).

[36] This solution strategy may be generalizable to other potentials.

[37] S. J. Brodsky and R. W. Brown, Phys. Rev. Lett. **49**, 966 (1982).

[38] R. W. Brown and K. L. Kowalski, Phys. Lett. **144B**, 235 (1984).