Quantum constraint dynamics for two spinless particles under vector interaction

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Using Dirac's constraint mechanics we derive two-body Klein-Gordon equations for two spinless particles under mutual vector interaction. We construct generalized mass-shell constraints which incorporate the gauge structure of this interaction for the constituent particles. The resultant direct-interaction formalism does more than just dress static potentials with relativistic two-body kinematics. It includes dynamical recoil effects in the potential characteristic of those that appear in field theories. We demonstrate this classically by showing its canonical equivalence in the slow-motion, weak-potential domain (the semirelativistic approximation) to the Darwin Hamiltonian. We also show this quantum mechanically by demonstrating its equivalence (for weak potentials) to Todorov's homogeneous quasipotential equation (which in turn leads to the standard Breit results for perturbative QED). Not only is our one-body Schrödinger-type equation local and covariant, but also it leads to forms of interaction that make nonperturbative quantum-mechanical sense at short distances. Thus this constraint approach is ideally suited for use in phenomenological applications where a perturbative treatment may be inadequate (with no need for extra smoothing parameters or finite particle size).

I. INTRODUCTION

In this paper we derive a two-body Klein-Gordon equation for two spinless particles under mutual vector interaction. We shall accomplish this by quantization of generalized mass-shell constraints for the constituent particles, a procedure that has come to be known as relativistic quantum constraint mechanics. This method, based on Dirac's Hamiltonian constraint mechanics,[1–4] leads to a consistent description of particles that interact directly through effective mechanical potentials that are functions of the relativistic particle degrees of freedom. The resulting dynamical systems may be viewed either as purely phenomenological models of interaction or, if associated in any of a number of ways with the two-body problem of a quantum field theory, as an alternative version of the field-theoretic dynamics in which the field degrees of freedom have been eliminated from the effective particle mechanics. As yet, no one has systematically demonstrated how and to what degree reduction of the two-particle sector of field theory to the two-particle constraint approach takes place. We contend, however, that any such reduction of a gauge field theory ought to preserve the "memory" of the gauge structure underlying it by translating that structure into characteristic forms of the effective potentials. We conjecture that the two-body system with vector interaction looks just like that for two relativistic charged particles interacting with a classical field (except that the field seen by each particle has been replaced by an effective potential) introduced through minimal substitution on the constituent four-momenta, $p_i^\mu \rightarrow p_i^\mu - A_i^\mu$. We shall show that this structure is sufficient to reproduce results hitherto derived primarily from field theory, the only additional dynamical input being an invariant version of the static Coulomb potential $-\alpha/r$ with the interparticle separation, r, covariantly reinterpreted. The resulting formalism may be called relativistic quantum constraint electrodynamics or simply constraint electrodynamics. Starting from generalized mass-shell constraints which maintain the gauge structure of the vector interaction for the constituent particles through minimal substitution, we obtain several new results, the most important of which is a two-body Klein-Gordon equation that, for weak potentials, reduces to Todorov's homogeneous quasipotential equation for "stationary states" of two spinless particles.[5] This implies that the formalism gives results for spectral calculations that agree with field-theoretic predictions since the inhomogeneous form of Todorov's quasipotential equation generates an energy-dependent potential for his homogeneous equation from perturbative quantum field theory.

It has been known for some time that the generalized mass-shell conditions in the constraint approach are related to the quasipotential equation.[1] However, the explicit relationship between the constraint equations and quasipotential approach for constituent potentials with specific Lorentz transformation properties has not been demonstrated until recently. We have shown in a previous paper[6] that for world scalar constituent potentials (for particles either with spin or without spin) the constraint equations reduce to the quasipotential equation when the potentials are weak (potentials whose expectation values are small in magnitude compared with the rest mass of either of the constituent particles). The equivalence between the weak-potential form of the constraint equations and the quasipotential equation implies that the spectra predicted by the two methods are the same through order α^4. These perturbative results agree with those produced by the more conventional Fermi-Breit approximation to the Bethe-Salpeter equation.[7]

Todorov has pointed out that the simple covariant momentum dependence of the quasipotential equation gives it a technical advantage over the Breit equation in bound-state calculations. This is also true for the constraint equations we derive. To illustrate this point we present a semirelativistic treatment of the constraint equation which leads to a form that is canonically equivalent to the Breit equation for spinless particles. The canonical correspondence allows one to see why these two approaches produce the same spectral results from widely different starting points. Its singular nature demonstrates the advantage provided by the covariant nature of the constraint equation over the Breit approximation.

Although the Breit approximation, the quasipotential approach, and the constraint approach, all give the same spectral results through order α^4, they differ drastically in cases where perturbation theory is not applicable. This occurs often in phenomenological applications in QCD and nuclear physics. Unlike the other two approaches, the constraint equation makes quantum-mechanical sense in the nonperturbative regime. For example, Coulomb-type potentials produce contributions in the Breit equation and the quasipotential equation (like the delta function) that can only be treated perturbatively unless their effects are smoothed out by using additional parameters. We show that for vector interactions, the constraint approach has a built-in smoothing mechanism that eliminates the need for extra parameters. At the same time, it is important to verify that our new strong-potential form of the constraint equation produces the same results as its weak-potential approximation (the well-established Todorov equation). In particular, we show, using both a perturbative and a nonperturbative treatment of the exact constraint equation (with no prior weak-potential assumption imposed on the form of the equation) that the same spectral results are produced as with the perturbative treatment of the weak-potential form of the constraint equation.

The structure of this paper is as follows. Section I treats the classical constraint formalism and gives the relationship between the vector potentials associated with the constituent particles. We also use this section to review the relevant kinematical variables[8] for the relativistic two-body problem. The second section presents the quantization of the classical formalism and examines the properties of the resulting relativistic wave equations.

II. CLASSICAL CONSTRAINT MECHANICS

The mass-shell constraint on a free particle's four-momentum is

$$\mathscr{H}=p^2+m^2\approx 0\ , \tag{1}$$

into which one introduces interaction with an external vector potential through the modification

$$p^\mu\to\pi^\mu=p^\mu-A^\mu\ . \tag{2}$$

For a system of two interacting spinless particles we postulate that the corresponding generalized mass-shell constraints are

$$\mathscr{H}_1=\pi_1{}^2+m_1{}^2\approx 0,\ \ \mathscr{H}_2=\pi_2{}^2+m_2{}^2\approx 0\ , \tag{3}$$

where $\pi_i^\mu=p_i^\mu-A_i^\mu$, so that each particle appears to be in an external potential due to the presence of the other. The A_i^μ's are not fields but rather effective potentials that are point functions of the particles' coordinates and momenta.

Because the Dirac Hamiltonian for this system is $\mathscr{H}=\lambda_1\mathscr{H}_1+\lambda_2\mathscr{H}_2$, a sufficient condition for $\mathscr{H}_1$ and $\mathscr{H}_2$ to be conserved in τ is that the constraints be first class,

$$\{\mathscr{H}_1,\mathscr{H}_2\}\approx 0\ . \tag{4}$$

The weak equality signs in (3)—(4) mean that the constraints are to be imposed only after working out the Poisson brackets. Condition (4) then confines the motion to the constraint hypersurface defined by (3). The left-hand sides need only vanish on that surface but may vanish identically (strongly).

The fundamental brackets among the constituent variables are

$$\{x_i^\mu,p_j^\nu\}=g^{\mu\nu}\delta_{ij}\ . \tag{5}$$

As in nonrelativistic mechanics, we introduce canonical relative position and momentum variables. In order to ensure the correct relativistic kinematics, we require that the relative momentum variable collapse to the usual expression in the c.m. rest frame. Then

$$x=x_1-x_2,\ \ p=\frac{1}{w}(\epsilon_2p_1-\epsilon_1p_2)\ , \tag{6}$$

where

$$P=p_1+p_2,\ \ P^2=-w^2 \tag{7}$$

and

$$P\cdot p\approx 0\ . \tag{8}$$

The last equation just says that on the constraint hypersurface, p is the usual relative momentum. The constituent momenta are related to the total and relative momenta by $p_1=\epsilon_1\hat{P}+p$ and $p_2=\epsilon_2\hat{P}-p$, where $\hat{P}=P/w$ ($\hat{P}^2=-1$). The requirement $\{x^\mu,p^\nu\}=g^{\mu\nu}$ forces $\epsilon_i=\epsilon_i(P^2)$ and $\epsilon_1+\epsilon_2=w$. Since $\{x^\mu,P^\nu\}=0$ and the $\mathscr{H}_i$ depend only on relative x, the c.m. ($\vec{P}=0$) total energy w is a constant of the motion.

Before working out the restrictions that condition (4) imposes on our A's, we first point out that Poincaré invariance implies[9]

$$A_1^\mu=\alpha_1p_1^\mu+\beta_1p_2^\mu,\ \ A_2^\mu=\alpha_2p_2^\mu+\beta_2p_1^\mu\ . \tag{9}$$

Notice that we have omitted terms proportional to x^μ because they would produce unobservable gauge changes. We shall assume as a relativistic ansatz that the α's and β's are independent of the relative momentum p although they may depend on the total momentum P. Condition (4) then becomes

$$-\pi_1^\mu\pi_2^\nu(G_1\partial_\mu\pi_{2\nu}+G_2\partial_\nu\pi_{1\mu})\approx 0\ , \tag{10}$$

where $G_i\equiv 1-\alpha_i+\beta_i$. In the steps that follow it is helpful to note that the π_i^μ's can be written as

$$\begin{aligned}\pi_1^\mu&=\hat{P}^\mu(\epsilon_1G_1-w\beta_1)+G_1p^\mu\ ,\\ \pi_2^\mu&=\hat{P}^\mu(\epsilon_2G_2-w\beta_2)-G_2p^\mu\ .\end{aligned} \tag{11}$$

If the α's and β's depend on x only through its component $x_\perp$, perpendicular to P, that is,

$$\alpha_i=\alpha_i(x_\perp{}^2,w),\quad \beta_i=\beta_i(x_\perp{}^2,w)\ , \tag{12}$$

where

$$x_\perp^\mu+(g^{\mu\nu}+\hat{P}^\mu\hat{P}^\nu)x_\nu\ , \tag{13}$$

then condition (4) is satisfied strongly if

$$G_1{}^2\pi_1^\mu\partial_\nu\pi_{1\mu}-G_2{}^2\pi_2^\mu\partial_\nu\pi_{2\mu}=0\ . \tag{14}$$

The simplest solution to (14) comes from letting $G_1=G_2\equiv G$ which implies

$$\pi_1{}^2-\pi_2{}^2=\text{const}=p_1{}^2-p_2{}^2\ , \tag{15}$$

so that

$$-2p_1A_1+A_1{}^2=-2p_2A_2+A_2{}^2\equiv\Phi\ . \tag{16}$$

This serves as a relativistic counterpart of Newton's third law and leads to

$$\mathcal{H}_1-\mathcal{H}_2=2P\cdot p+(\epsilon_2-\epsilon_1)w+m_1{}^2-m_2{}^2\ . \tag{17}$$

Equations (17) and (8) can agree only if

$$\epsilon_1-\epsilon_2=\frac{1}{w}(m_1{}^2-m_2{}^2)\ . \tag{18}$$

Thus, our procedure completely determines the canonical variables. Because of Eq. (8) two of the variables, the relative energy and relative time in the c.m. system, have effectively disappeared. Equation (18) implies $\mathcal{H}_1-\mathcal{H}_2=2P\cdot p$. The remaining independent combination of the constraints then becomes

$$\mathcal{H}\equiv\frac{\epsilon_2}{w}\mathcal{H}_1+\frac{\epsilon_1}{w}\mathcal{H}_2=p^2-b^2(w)+\Phi\approx\mathcal{H}_1\approx\mathcal{H}_2\approx 0\ , \tag{19}$$

where the first two weak equalities result from $P\cdot p\approx 0$, Eq. (16), and the fact that

$$\epsilon_1{}^2-m_1{}^2=\epsilon_2{}^2-m_2{}^2\equiv b^2(w)\ .$$

In Eq. (19), b^2 (the on-shell value of the relative momentum) has the form

$$b^2(w)=\lambda(w^2,m_1{}^2,m_2{}^2)/4w^2\ , \tag{20}$$

where $\lambda(a,b,c)$ is the triangle function. Thus the form (19) (also that of Todorov's quasipotential Hamiltonian[1,5]) incorporates the correct relativistic two-body kinematics. Todorov defines the other useful variables

$$m_w=\frac{m_1m_2}{w}\ ,\qquad \epsilon_w=\frac{(w^2-m_1{}^2-m_2{}^2)}{2w}\ , \tag{21}$$

interpreted as the relativistic reduced mass and energy of a fictitious particle of relative motion. In terms of them,

$$b^2=\epsilon_w{}^2-m_w{}^2 \tag{22}$$

reinforcing this interpretation.

Equation (16) allows us to relate the constituent potentials to a single effective potential seen by this fictitious particle. First we observe that Eq. (11) has now become

$$\pi_1^\mu=G[\hat{P}^\mu(\epsilon_1-\mathcal{A}_1)+p^\mu]\ ,\qquad \pi_2^\mu=G[\hat{P}^\mu(\epsilon_2-\mathcal{A}_2)-p^\mu]\ , \tag{23}$$

where

$$\mathcal{A}_i=\frac{w\beta_i}{G},\quad i=1,2\ . \tag{24}$$

Equation (16) then implies

$$2\epsilon_1\left[\mathcal{A}_1-\frac{w}{2}(1-G^{-2})\right]-\mathcal{A}_1{}^2=2\epsilon_2\left[\mathcal{A}_2-\frac{w}{2}(1-G^{-2})\right]-\mathcal{A}_2{}^2\ . \tag{25}$$

The assumption that $G_1=G_2$ allows a choice of relations between the constituent vector potentials. For example, choosing $\alpha_i=\beta_i$ leads to time-like $A_i^\mu(=\mathcal{A}_i\hat{P}^\mu)$ since $G=1$. This "self-symmetric" ansatz gives a physical picture in which the two particles make equal contributions to each vector potential. We also point out that for this choice we have the separate Lorentz gauge conditions[10]

$$\partial_\mu A_1{}^\mu=\partial_\mu A_2^\mu=0\ . \tag{26}$$

Equation (25) implies that only one of the A_i's is independent in that case. A symmetrical choice for the independent scalar becomes apparent if we write $\mathcal{H}$ in an effective one-particle generalized mass-shell or Klein-Gordon form. Equation (22) suggests that we define

$$\mathcal{P}^\mu=p^\mu+\epsilon_w\hat{P}^\mu \tag{27}$$

since without interaction (19) is simply the free Klein-Gordon form $\mathcal{P}^2+m_w{}^2\approx 0$. In the c.m. system, $\mathcal{P}^\mu=(\epsilon_w,\vec{p})$ and, if we decide to introduce an effective time-like vector potential through $\mathcal{P}^\mu\to\pi^\mu=\mathcal{P}^\mu-\mathcal{V}\hat{P}^\mu$, then our system Hamiltonian (19) is

$$\mathcal{H}=\pi^2+m_w{}^2=p^2-(\epsilon_w-\mathcal{V})^2+m_w{}^2\approx 0\ , \tag{28}$$

if $\mathcal{V}$ is related to the $\mathcal{A}_i$'s in (25) by

$$\epsilon_1{}^2-2\epsilon_w\mathcal{V}+\mathcal{V}^2=(\epsilon_1-\mathcal{A}_1)^2\ , \tag{29}$$

$$\epsilon_2{}^2-2\epsilon_w\mathcal{V}+\mathcal{V}^2=(\epsilon_2-\mathcal{A}_2)^2\ . \tag{30}$$

Notice that the timelike vectors are the same ($\mathcal{A}_1=\mathcal{A}_2$) only for equal masses.

If $G\neq 1$, then the constituent A_i^μ's are not timelike, containing spacelike (or transverse) parts as well. The "cross-symmetric" choice $\alpha_1=\alpha_2$ and $\beta_1=\beta_2$ yields $\mathcal{A}_1=\mathcal{A}_2=\mathcal{A}$. Equation (25) then implies

$$G^2=1\Big/\left[1-\frac{2\mathcal{A}}{w}\right]\ . \tag{31}$$

We have then

$$A_1^\mu=[1-\tfrac{1}{2}(G+G^{-1})]p_1^\mu+\frac{G\mathscr{A}}{w}p_2^\mu\ ,$$
$$A_2^\mu=[1-\tfrac{1}{2}(G+G^{-1})]p_2^\mu+\frac{G\mathscr{A}}{w}p_1^\mu\ . \tag{32}$$

As opposed to what happens in the "self-symmetric" case, here the two particles do not contribute equally to each potential. In fact, for weak potentials ($G\sim 1$) one particle acts as the sole source for the other particle's vector potential. The "cross-symmetric" case shares this feature with fieldlike versions of dynamics like those of Feynman and Wheeler.[11] An interesting feature of this case is that the sum of the vector potentials is in the Lorentz gauge while the individual particle potentials are not. In particular,

$$\partial_\mu A_1^\mu=-\partial_\mu A_2^\mu=[(1-G^2)/G^2]\vec{\nabla}\ln G\cdot\vec{p}$$

(in the c.m. system), which demonstrates that the deviation from the Lorentz gauge condition[10] is a recoil effect ($G\neq 1$). We summarize for this case the various relations between the scalar $\mathscr{A}$ and the components of the constituent vector potentials in the c.m. system:

$$\pi_1^0=\epsilon_1-A_1^0=G(\epsilon_1-\mathscr{A})\ , \tag{33}$$

$$\pi_2^0=\epsilon_2-A_2^0=G(\epsilon_2-\mathscr{A})\ , \tag{34}$$

$$\vec{\pi}_1=\vec{p}-\vec{A}_1=G\vec{p}=-\vec{\pi}_2\ . \tag{35}$$

If we use the identities

$$\epsilon_1-\frac{m_1{}^2}{w}=\epsilon_2-\frac{m_2{}^2}{w}=\epsilon_w \tag{36}$$

or

$$\epsilon_1\epsilon_2+b^2=w\epsilon_w\ , \tag{37}$$

then our system Hamiltonian becomes

$$\mathscr{H}=G^2[p^2-(\epsilon_w-\mathscr{A})^2+m_w{}^2]\approx 0\ . \tag{38}$$

It is a striking result that (for $\mathscr{V}=\mathscr{A}$) the two Hamiltonians (28) and (38) produce equivalent classical dynamics since they merely differ by a multiplicative factor which can be absorbed in the arbitrariness of the evolution parameter. This equivalence occurs in spite of the very different physical pictures of vector interaction corresponding to the choices

$$\text{(I)}\ \ \mathscr{A}_i=\epsilon_i-(\epsilon_i{}^2-2\epsilon_w\mathscr{V}+\mathscr{V}^2)^{1/2},\ \ G=1 \tag{39}$$

and

$$\text{(II)}\ \ \mathscr{A}_i=\mathscr{A},G=\left[1-\frac{2\mathscr{A}}{w}\right]^{1/2}. \tag{40}$$

As we shall see in the next section, however, these two pictures are quantum mechanically inequivalent.

In each case the specification of just one invariant scalar function of the invariant $x_\perp{}^2\equiv r^2$ is sufficient to define the dynamics. (Note that r is the interparticle separation only in the c.m. system.) How one chooses this invariant function of r depends on the purposes to which the constraint mechanics is to be put. If one proposes to use these equations (or more likely their quantum counterparts) for phenomenological applications, the form of $\mathscr{A}$ need not be tied to any particular field theory.[12] On the other hand, if one wishes to use the constraint formalism to extract information from a quantum field theory, one needs at least a perturbative method to relate this invariant function to that field theory.[13] One could even match a semirelativistic expansion for the total c.m. w obtained from (28) or (38) to similiar expansions from a classical field theory.[14] Agreement of the nonrelativistic limit of either of these connections to field theory with the nonrelativistic limit of the constraint approach [see the first four terms of (49) below] requires that the expression for $\mathscr{A}$ be the Coulomb term $-\alpha/r$.

We now explore the consequences of the gaugelike structure of the interaction Φ through a study of the semirelativistic limit of the constraint mechanics. To begin with, we discuss the relation of our covariant classical Hamiltonian to the total naive (i.e., no momentum-dependent terms in the potential) c.m. ($\vec{p}_1=-\vec{p}_2$) energy form for two relativistic particles under mutual Coulomb-type interaction:

$$w=(\vec{p}^{\,2}+m_1{}^2)^{1/2}+(\vec{p}^{\,2}+m_2{}^2)^{1/2}+\mathscr{A}\ . \tag{41}$$

In our comparison we assume that both approaches have the same momentum-independent $\mathscr{A}$ so that both automatically have the same nonrelativistic limit. Using (20), (22), and (38), we can write the following implicit solution for w:

$$w=[\vec{p}^{\,2}+m_1{}^2+\Phi(\mathscr{A},w)]^{1/2}+[\vec{p}^{\,2}+m_2{}^2+\Phi(\mathscr{A},w)]^{1/2}\ . \tag{42}$$

The above two forms are obviously inequivalent for the same $\mathscr{A}$. Another way to view the difference is to manipulate (41) into Todorov's form $\vec{p}^{\,2}+\Phi'=b^2$. One finds[15]

$$\Phi'=\frac{2w\mathscr{A}-\mathscr{A}^2}{4}\left[1-\frac{(m_1{}^2-m_2{}^2)^2}{w^2(w-\mathscr{A})^2}\right]. \tag{43}$$

If $|\mathscr{A}|\ll w$ and $|w-m_1-m_2|\ll m_1,m_2$, then both Φ and Φ' are equal to $2\mu\mathscr{A}$. As expected, they both have the same nonrelativistic limit. But for the same $\mathscr{A}$, their relativistic corrections differ. In order to contrast them we derive semirelativistic expansions of (41) and (42). We define the "semirelativistic" expansion as one that includes the lowest-order relativistic corrections. For a weak potential this approximation includes terms like $(\vec{p}^{\,2})^2,\vec{p}^{\,2}\mathscr{A}$, and $\mathscr{A}^2$. To this order (41) reduces to

$$w=m_1+m_2+\frac{\vec{p}^{\,2}}{2\mu}+\mathscr{A}-\frac{(\vec{p}^{\,2})^2}{8}\left[\frac{1}{m_1{}^3}+\frac{1}{m_2{}^3}\right], \tag{44}$$

where $\mu=m_1m_2/(m_1+m_2)$. To find the semirelativistic expansion of the constraint result (42) we proceed in two steps. First we perform a weak-potential expansion, one that includes terms up to second order in Φ. This makes necessary a first-order expansion of the w dependence of Φ itself. Defining

$$w=w_0+\Delta w\ , \tag{45}$$

with

$$w_0=\epsilon_1^0(\vec{p})+\epsilon_2^0(\vec{p})$$
$$\equiv(\vec{p}^{\,2}+m_1{}^2)^{1/2}+(\vec{p}^{\,2}+m_2{}^2)^{1/2}\ , \tag{46}$$

we find

$$w=w_0\left[1+\tfrac{1}{2}\left[\Phi(\mathscr{A},w_0)+\frac{\partial}{\partial w}\Phi(\mathscr{A},w_0)\Delta w_0\right]\Big/\epsilon_1^0(\vec{p})\epsilon_2^0(\vec{p})\right]-\tfrac{1}{8}\Phi^2(\mathscr{A},w_0)\left[\frac{1}{\epsilon_1^0(\vec{p})^3}+\frac{1}{\epsilon_2^0(\vec{p})^3}\right], \tag{47}$$

where to lowest order

$$\Delta w\equiv\Delta w_0=\frac{w_0\Phi(\mathscr{A},w_0)}{2\epsilon_1^0(\vec{p})\epsilon_2^0(\vec{p})}. \tag{48}$$

The second step of the semirelativistic expansion is a slow motion expansion of $\epsilon_1^0(\vec{p})$ and $\epsilon_2^0(\vec{p})$. Doing this in Eq. (47) yields

$$w=m_1+m_2+\frac{\vec{p}^{\,2}}{2\mu}+\mathscr{A}-\frac{(\vec{p}^{\,2})^2}{8}\left[\frac{1}{m_1{}^3}+\frac{1}{m_2{}^3}\right]+\frac{1}{m_1m_2}\vec{p}^{\,2}\mathscr{A}+\frac{1}{2(m_2+m_2)}\mathscr{A}^2. \tag{49}$$

The difference between (49) and (44) involves recoil-dependent terms, ones that vanish when one of the masses becomes very heavy. We can exhibit the nature of these extra terms by performing a canonical transformation on Eq. (49) for the case $\mathscr{A}=-\alpha/r$. The transformation[16]

$$\vec{r}\to\vec{r}\,'=\vec{r}-\frac{\alpha}{2(m_1+m_2)}\frac{\vec{r}}{r},$$
$$\vec{p}\to\vec{p}\,'=\vec{p}+\frac{\alpha}{2(m_1+m_2)}\left[\frac{\vec{p}}{r}-\vec{r}\,\vec{r}\cdot\frac{\vec{p}}{r^3}\right]=\vec{p}-\frac{\alpha}{2(m_1+m_2)}\frac{1}{r^3}\vec{r}\times\vec{L} \tag{50}$$

is canonical through first order in α. When this transformation is applied to (49) we obtain (for $\mathscr{A}=-\alpha/r$)

$$w=m_1+m_2+\frac{\vec{p}^{\,2}}{2\mu}-\frac{\alpha}{r}-\frac{(\vec{p}^{\,2})^2}{8}\left[\frac{1}{m_1{}^3}+\frac{1}{m_2{}^3}\right]-\frac{\alpha}{2m_1m_2}\vec{p}\cdot\left[\frac{1}{r}+\frac{\vec{r}\,\vec{r}}{r^3}\right]\cdot\vec{p}. \tag{51}$$

The last term in this equation is that first derived from classical field theory by Darwin;[17] its quantization (with proper ordering) is the Breit interaction for spinless particles. For particles of comparable mass, this Breit term produces corrections of the same order of importance as relativistic kinematical effects. Thus, at the level of relativistic classical mechanics, the constraint approach has built into it the recoil and "retardation" effects of a field theory. The naive classical kinetic plus potential energy forms of (44) and (41) lack the Darwin-Breit term necessary for correct classical and quantum semirelativistic dynamics.

III. QUANTUM CONSTRAINT MECHANICS

In order to quantize the constraint formalism we construct quantum versions of the $\mathscr{H}_i$'s for cases I and II in which the gauge structure is maintained for Hermitian π_i^μ. To make the construction as transparent as possible, we employ the quantum analogs of the collective variables p^μ and P^μ defined in Eqs. (6) and (7). The variables p^μ and $\hat{P}^\mu$ are well defined if we restrict our space so that P^2 has only timelike eigenvalues. Such general results as the compatibility of quantum constraints in no way depend on this restriction and may be verified using the original constituent variables. Our classical expressions (23) for the π_i^μ suggest the following Hermitian quantum forms:

$$\pi_1^\mu=G\left[\hat{P}^\mu(\epsilon_1-\mathscr{A}_1)+p^\mu+\frac{1}{2i}\nabla^\mu\ln G\right],$$
$$\pi_2^\mu=G\left[\hat{P}^\mu(\epsilon_2-\mathscr{A}_2)-p^\mu-\frac{1}{2i}\nabla^\mu\ln G\right]. \tag{52}$$

We must use the quantum brackets

$$[x_1^\mu,p^\nu]=i(g^{\mu\nu}+\hat{P}^\mu\hat{P}^\nu) \tag{53}$$

to verify that the commutator $[\mathscr{H}_1,\mathscr{H}_2]$ vanishes. The Hermitian ordering produces extra terms in the compatibility calculation. But since the commutator $[\mathscr{H}_1,\mathscr{H}_2]$ can be reduced to $[\mathscr{H}_1-\mathscr{H}_2,\Phi]$ and since the difference of the quantum constraints still has the form $\mathscr{H}_1-\mathscr{H}_2=2P\cdot p$, the necessary condition for quantum compatibility remains $P\cdot(\partial\Phi/\partial x)=0$, which is still satisfied by having the invariant scalar α_i's and β_i's depend on x only through $x_\perp$ just as in the classical case.[18] Since $[\mathscr{H}_1,\mathscr{H}_2]$ is strongly zero, $(\mathscr{H}_1+N_1)\psi=0$ and $(\mathscr{H}_2+N_2)\psi=0$ are compatible conditions on the wave function when N_1 and N_2 are neutral elements, that is, objects that commute with the $\mathscr{H}_1$, $\mathscr{H}_2$ algebra and with each other. Excluding redefinitions of Φ, such elements may be constants, functions of P^2, or noncanonical operators whose classical limit vanishes so that we recover the original classical constraints. Since the N_1 and N_2 are independent of $x_\perp$, their sole role is to alter the constraints for distant particles from their naive Klein-Gordon forms. If we demand that our quantum equations degenerate for vanishing interaction to two uncoupled Klein-Gordon equations for particles whose quantum masses equal their classical masses, we are forced to set $N_1=N_2=0$. The quantum versions of the classical $\mathscr{H}_i\approx0$ then become sharp conditions on the wave function:[19]

$$\mathscr{H}_1\psi=(\pi_1{}^2+m_1{}^2)\psi=0, \tag{54}$$

$$\mathscr{H}_2\psi=(\pi_2{}^2+m_2{}^2)\psi=0 \tag{55}$$

with

$$[\mathscr{H}_1,\mathscr{H}_2]=0 \tag{56}$$

enforcing their compatibility.

Since our two-body system is an isolated one, P^μ is a constant of the motion. We restrict our attention to the case in which the total wave function is an eigenstate of P^μ with positive total energy (eigenvalue). We separate the c.m. and internal parts of the wave function by introducing the "center of mass" position variable[20]

$$X^\mu=[(\epsilon_1/w)x_1^\mu+(\epsilon_2/w)x_2^\mu+(m_1{}^2-m_2{}^2)^2x\cdot PP^\mu/(P^2)^2]/2+\text{H.c.}\ , \tag{57}$$

which is canonically conjugate to P^μ and which completes a set of canonical variables with x^μ and p^μ. That is,

$$[X^\mu,X^\nu]=0=[X^\mu,x^\nu]=[X^\mu,p^\nu] \tag{58}$$

as well as

$$[X^\mu,P^\nu]=ig^{\mu\nu}\ . \tag{59}$$

Then P^μ can be represented by $-i\partial/\partial X'_\mu$ (the prime denotes c number) whose eigenfunctions are plane waves of the form[21] $e^{iP'X'}$ where $P'^2\equiv -w^2$, $P'^0_{\text{c.m.}}=w>0$. Thus

$$\psi(x_1',x_2')=e^{iP'X'}\psi_{P'}(x')\ . \tag{60}$$

Below we shall see that $\psi_{P'}(x')$ has a finite norm for bound states. In what follows, we will drop the prime superscript so that P^μ becomes a c number. The condition

$$P\cdot p\psi=0 \tag{61}$$

then becomes a differential equation in the relative variable x_μ since in the coordinate representation, the relative momentum p^μ can be represented by $-i\partial/\partial x_\mu$. Once the c.m. part has been factored off, any solution to (61) must have the general form[22,1]

$$\psi_P(x)=\psi_P(x_\perp)\left[=(2\pi)^{-3}\int \chi(p,P)\delta(p\cdot\hat{P})e^{ipx}d^4p\right]. \tag{62}$$

One possible norm for $\psi_P(x_\perp)$ is[1]

$$\int \psi_P^*(x_\perp)\psi_P(x_\perp)d^3x_\perp\ , \tag{63}$$

where $d^3x_\perp$ is the covariant measure $\delta(\hat{P}\cdot x)d^4x$. We thus have a Hilbert space for bound-state wave functions (corresponding to discrete eigenvalues of $w=\sqrt{-P^2}$) satisfying (61). The quantum counterpart to the remaining independent constraint $\mathscr{H}\approx 0$ is a condition on the wave function that serves as our relativistic Schrödinger equation for the effective particle of relative motion:

$$\mathscr{H}\psi_P=\left[\frac{\epsilon_2}{w}\mathscr{H}_1+\frac{\epsilon_1}{w}\mathscr{H}_2\right]\psi_P=0\ . \tag{64}$$

Note that the $\mathscr{H}_i$'s used in this equation are functions of the c number P^μ. Since $\mathscr{H}_1$, $\mathscr{H}_2$, and $\mathscr{H}$ differ only by terms proportional to $P\cdot p$, they are equivalent conditions on ψ's that satisfy (61). Thus all three lead in the c.m. rest frame ($P^0=w,\vec{P}=0$) to the wave equation

$$\mathscr{H}_1\psi=\mathscr{H}_2\psi=\mathscr{H}\psi=G^2\left[\vec{p}^{\,2}-(\epsilon_w-\mathscr{A})^2+m_w{}^2+\frac{2}{i}\vec{\nabla}\ln G\cdot\vec{p}-\tfrac{1}{2}\vec{\nabla}^2\ln G-\tfrac{3}{4}(\vec{\nabla}\ln G)^2\right]\psi=0\ . \tag{65}$$

Equation (61) implies the equivalence of the three forms in the first line of (65) as well as the relation $p^2\psi=\vec{p}^{\,2}\psi$. The general form (65) includes both case I (for $\mathscr{V}=\mathscr{A}$ and $G=1$) and case II. If we let $\Psi=G^{1/2}\psi$, then our eigenvalue equation can be reduced to

$$[\vec{p}^{\,2}-(\epsilon_w-\mathscr{A})^2+m_w{}^2-i\vec{\nabla}\ln G\cdot\vec{p}]\Psi=0\ . \tag{66a}$$

If we let $\phi=G^{1/2}\Psi$, then we obtain the Hermitian form

$$[\vec{p}^{\,2}-(\epsilon_w-\mathscr{A})^2+m_w{}^2+\tfrac{1}{2}\vec{\nabla}^2\ln G+\tfrac{1}{4}(\vec{\nabla}\ln G)^2]\phi=0\ . \tag{66b}$$

These transformations do nothing in case I where $G=1$ but amount to scale transformations in case II.[23] The Hermitian form of the wave equations is preferred in applications where numerical techniques are needed since its symmetrical form provides for more reliable and rapid convergence of eigenvalue calculations. The quantum equations for case I (where $G=1$) and case II [where $G=(1-2\mathscr{A}/w)^{-1/2}$] are inequivalent, in sharp contrast to the equivalence of the corresponding classical forms (28) and (38). We explicitly display their different spectral predictions for $\mathscr{A}=-\alpha/r$ where $\alpha=e_1e_2/\hbar c$. The bound-state equation in case I is

$$\left[\vec{p}^{\,2}-\left[\epsilon_w+\frac{\alpha}{r}\right]^2+m_w{}^2\right]\Psi=0\ , \tag{67}$$

yielding the relativistic Balmer formula

$$w^2(n,l)=m_1{}^2+m_2{}^2+2m_1m_2\left[1+\frac{\alpha^2}{n'^2}\right]^{1/2}\ , \tag{68}$$

where

$$n'=n_r+\lambda+1 \tag{69}$$

in which n_r is the number of intermediate radial nodes and

$$\lambda=+[(l+\tfrac{1}{2})^2-\alpha^2]^{1/2}-\tfrac{1}{2}\ . \tag{70}$$

Thus $w(n,l)$ up through terms of order α^4 is

$$w(n,l)=m_1+m_2-\frac{\mu\alpha^2}{2n^2}-\frac{\mu\alpha^4}{n^3}\left[\frac{1}{2l+1}-\frac{3}{8n}+\frac{\mu}{8(m_1+m_2)n}\right]. \quad (71)$$

As Todorov points out, since this spectrum does not include all recoil effects it does not agree with the results obtained from the conventional Breit equation.[24] This discrepancy should have been anticipated since the vector potential of case I has no transverse part.

As we shall now demonstrate, case II with the invariant $\mathscr{A}=-\alpha/r$ gives a two-body Klein-Gordon equation with the following properties: (i) The recoil contributions of the $\ln G$ terms of case II (missing in case I) help reproduce the correct spectrum through order α^4. (ii) In the weak-potential limit, the case II wave equation reproduces Todorov's homogeneous quasipotential equation for two spinless particles under mutual electromagnetic interaction. This provides the most direct perturbative justification of (i). (iii) A semirelativistic approximation (i.e., a slow motion as well as a weak-potential expansion) of the constraint approach produces the Fermi-Breit approximation to the Bethe-Salpeter equation.

We examine the approximations (ii) and (iii) first before discussing the exact equation. In the weak-potential limit where we treat $\mathscr{A}$ as small compared to w we have

$$\ln G \to \frac{\mathscr{A}}{w}. \quad (72)$$

Then Eq. (66a) reduces to

$$\left[\vec{p}^{\,2}-(\epsilon_w-\mathscr{A})^2+m_w^{\,2}-i\vec{\nabla}\ln\left[\frac{\mathscr{A}}{w}\right]\cdot\vec{p}\right]\Psi=0 \quad (73a)$$

while (66b) becomes

$$\left[\vec{p}^{\,2}-(\epsilon_w-\mathscr{A})^2+m_w^{\,2}+\frac{1}{2w}\vec{\nabla}^2\mathscr{A}+\frac{1}{4w^2}(\vec{\nabla}\mathscr{A})^2\right]\phi=0. \quad (73b)$$

When $\mathscr{A}=-\alpha/r$, (73b) exactly reproduces Todorov's quasipotential equation.[25,26] As he points out, the term $(\vec{\nabla}\mathscr{A}/2w)^2[=\alpha^2/(4w^2r^4)]$ does not contribute until order α^6 while $\vec{\nabla}^2\mathscr{A}/2w[=2\pi\alpha\delta(\vec{r})/w]$ contributes the needed extra α^4 recoil term. The complete spectrum through order α^4 (agreeing with the Breit result) is then

$$w(n,l)=m_1+m_2-\frac{\mu\alpha^2}{2n^2}-\frac{\mu\alpha^4}{n^3}\left[\frac{1}{2l+1}-\frac{3}{8n}+\frac{\mu}{8(m_1+m_2)n}\right]+\frac{\mu^2}{(m_1+m_2)}\frac{\alpha^4}{n^3}\delta_{l0}. \quad (74)$$

The same recoil contribution can be obtained from the term $-i(\vec{\nabla}\mathscr{A}/w)\cdot\vec{p}[=-i\alpha/(r^3w)\vec{r}\cdot\vec{p}]$ in (73a) treated as a perturbation.

We emphasize that the constraint equation (66a) or (66b) and the quasipotential equation (73a) or (73b) are bona-fide relativistic wave equations, not $O(1/c^2)$ approximations. Hence their simple momentum dependence holds up to all orders of v/c in contrast with the Fermi-Breit $O(1/c^2)$ truncation of the Bethe-Salpeter equation seen below

$$\left[\frac{\vec{p}^{\,2}}{2\mu}-\frac{\alpha}{r}-\frac{1}{8}\left[\frac{1}{m_1^{\,3}}+\frac{1}{m_2^{\,3}}\right](\vec{p}^{\,2})^2-\frac{\alpha}{2m_1m_2}\vec{p}\cdot\left[\frac{1}{r}+\frac{\vec{r}\vec{r}}{r^3}\right]\cdot\vec{p}\right]\psi=(w-m_1-m_2)\psi \quad (75)$$

whose kinetic and dynamical recoil corrections get successively more complicated as one goes from one order of v^2/c^2 to the next. Thus the computation of the perturbative fine structure is technically more attractive in the constraint or quasipotential approach. In fact, as we shall see below, one of the standard methods used to compute the expectation value of the Fermi-Breit Hamiltonian involves reducing it to a form identical to the semirelativistic expansion of the constraint equation.

The Breit equation (75) and the weak-potential form (73b) [or (73a)] of the covariant constraint equation are related through the quantum version of the classical semirelativistic approximation of Sec. II. Equation (73b) is of the form

$$[\vec{p}^{\,2}+\Phi(r,w)]\psi=b^2(w)\psi, \quad (76)$$

where

$$\Phi(r,w)=\frac{-2\epsilon_w\alpha}{r}+\frac{\alpha^2}{r^2}+\frac{2\pi\alpha\delta(\vec{r})}{w}. \quad (77)$$

Using this information about Φ the quantum version of (49) becomes

$$\left[\frac{\vec{p}^{\,2}}{2\mu}-\frac{\alpha}{r}-\frac{1}{8}\left[\frac{1}{m_1^{\,3}}+\frac{1}{m_1^{\,3}}\right](\vec{p}^{\,2})^2-\frac{1}{2m_1m_2}\{\vec{p}^{\,2},\alpha/r\}+\frac{1}{2(m_1+m_2)}\frac{\alpha^2}{r^2}+\frac{\pi\alpha}{m_1m_2}\delta(\vec{r})\right]\psi=(w-m_1-m_2)\psi. \quad (78)$$

By using the quantum version of the inverse of the canonical transformation given in Eq. (50), Schwinger[7] has shown that the expectation value of the Breit interaction

$$-\alpha\vec{p}\cdot(1/r+\vec{r}\,\vec{r}/r^3)\cdot\vec{p}/2m_1m_2$$

can be transformed into that of the last three terms of the left-hand side of (78).

Let us now return to the exact forms (66a) and (66b) to see that they actually reproduce the perturbative results of the weak-potential form of the equations. As we shall see, extra care is required in the handling of the perturbation, at least in the case of (66b). For $\mathscr{A}=-\alpha/r$, Eq. (66a) becomes

$$\left[\vec{p}^{\,2}-2\frac{\epsilon_w\alpha}{r}-\frac{\alpha^2}{r^2}-\frac{i\alpha}{wr^2(rw+2\alpha)}\vec{r}\cdot\vec{p}\right]\Psi=b^2\Psi\ . \tag{79a}$$

Here we will restrict our discussion to the ground state, the S state most sensitive to the recoil term. When the unperturbed eigenfunction Ψ_0 is taken as a solution of $(\vec{p}^{\,2}-2\epsilon_w\alpha/r)\Psi_0=b^2\Psi_0$ the expectation value of the perturbation

$$-\alpha^2/r^2-i\alpha/[r^2(rw+2\alpha)]\vec{r}\cdot\vec{p}$$

yields a power-series expansion for Δb^2 in α^2 beginning with $-3\epsilon_w^{\,2}\alpha^2/2$. Specializing to the case of equal masses leads to

$$w=m\left[2-\frac{\alpha^2}{4}-\frac{13\alpha^4}{64}\right] \tag{80}$$

agreeing with (74) to the appropriate order. In the other exact equation (66b), the logarithmic terms produce denominators that soften the singular character of the operators. When $\mathscr{A}=-\alpha/r$, the delta-function term is forced to vanish by its denominator; the other terms combine to give

$$\left[\vec{p}^{\,2}-2\frac{\epsilon_w\alpha}{r}-\frac{\alpha^2}{r^2}+\frac{5}{4}\frac{\alpha^2}{r^2}\frac{1}{(rw+2\alpha)^2}\right]\varphi=b^2\varphi\ . \tag{79b}$$

The last term on the left-hand side has the limiting form $5\alpha^2/(4w^2r^4)$ for large r while for small r it becomes $5/(16r^2)$. But, is this term capable of reproducing the corrections that the delta-function term gives in the weak coupling case of (73b)? If one naively treats both of the last two terms on the left-hand side of (79b) as a perturbation, one produces an incorrect result for the $w(n,l)$ through order α^4.[27] The source of this error is not too hard to find. As it turns out, the last term cannot be considered a weak perturbation in the small-r region. When we make explicit the α dependence by using dimensionless Coulomb variables

$$\vec{r}=\vec{x}/\epsilon_w\alpha,\quad \lambda=\frac{b^2}{\epsilon_w^{\,2}\alpha^2}\ , \tag{81}$$

the radial equation corresponding to the ground state becomes

$$\left[-\frac{d^2}{dx^2}-\frac{2}{x}-\frac{\alpha^2}{x^2}+\frac{5}{16}\frac{\alpha^4}{x^2(2x+\alpha^2)^2}\right]u=\lambda u\ , \tag{82}$$

where we have used $w/\epsilon_w\approx4$ in the perturbation. Without the two α^2-dependent perturbations the ground-state eigenfunction and eigenvalue are $u_0=xe^{-x}$ and $\lambda_0=-1$, respectively. If one naively neglects the α^2 in the denominator of the last term on the left-hand side, then that perturbation would appear to be ignorable since its form $5\alpha^4/64x^4$ is two orders higher than the preceding term. However, since its small-x behavior $5/16x^2$ is nonperturbative, we can not treat this term as a perturbation in the standard way. Instead, we must fold its influence into the unperturbed wave function. We do this in a way which retains the unperturbed eigenvalue at $\lambda_0=-1$ but changes the unperturbed part of the potential to accommodate the new unperturbed wave function. This new unperturbed wave function is obtained from $\phi_0=G^{1/2}\Psi_0$, where the radial part of Ψ_0 is the u_0 given above. Since to lowest order in α, G is $1/(1+\alpha^2/2x)^{1/2}$, our choice for the modified unperturbed ground state is

$$u_0=\frac{cx^{5/4}e^{-x}}{(2x+\alpha^2)^{1/4}}\ . \tag{83}$$

That this wave function corresponds to $\lambda_0=-1$ can be seen from its large-x behavior (the same as that of u_0). For comparison, we write (82) as

$$\left[-\frac{d^2}{dx^2}+V\right]u=\lambda u\ . \tag{84}$$

We define $\widetilde{V}_0$ by

$$\left[-\frac{d^2}{dx^2}-\lambda_0\right]u_0=-\widetilde{V}_0u_0\ . \tag{85}$$

This leads to

$$\widetilde{V}_0=-\frac{4x+\frac{5}{2}\alpha^2}{x(2x+\alpha^2)}+\frac{5\alpha^4}{16x^2(2x+\alpha^2)^2}\ . \tag{86}$$

Now we solve (84) with $\widetilde{V}_0$ as our unperturbed potential instead of $-2/x$. Our perturbation is thus

$$V-\widetilde{V}_0=-\frac{2}{x}-\frac{\alpha^2}{x^2}+\frac{2}{x}\frac{(2x+\frac{5}{4}\alpha^2)}{(2x+\alpha^2)}\simeq-\frac{3}{4}\frac{\alpha^2}{x^2}\ . \tag{87}$$

Its expectation value with respect to u_0 yields a power series for $\Delta\lambda(=\lambda-\lambda_0)$ beginning with $-3\alpha^2/2$. This leads back to (80) agreeing with the weak-potential result (74) for the equal-mass ground state. The various perturbative arguments given in Eqs. (72)–(87) lead to the standard spectral results and thereby reassure us of the validity of the perturbative structure of our equations.

But our new strong-potential equation (79b) has more than perturbative content. Its extra structure gives it a quantum mechanically well-defined short-distance behavior [as opposed to the delta-function behavior in (73b)] that allows a nonperturbative (i.e., numerical) treatment of the whole equation. Using an iterative finite difference method, we solve for the ground state in the case of equal masses for $\alpha=0.2,0.3$. The numerical solutions to (79b) (with uncertainties $<0.01\alpha^6$) are (for $m=1$)

$$w=1.989\,664\,5\cdots\ ,\qquad w=1.975\,742\,1\cdots\ , \tag{88}$$

which differ from the perturbative results given in (74)

$$w = 1.989\,675\,0 \cdots ,$$
$$w = 1.975\,854\,7 \cdots \qquad (89)$$

by roughly $0.16\alpha^6$. There is no particular significance to the values of α we have chosen for out test other than numerical convenience. Our results indicate that the general solution to (79b) agrees with the perturbative solution (74) through order α^4.

Finally, the log derivative forms in our two-body Klein-Gordon equations give them an advantage over the quasipotential and Breit equations in nonperturbative phenomenological applications. The terms in (66b) that correspond to the singular terms in (73b), (75), or (78) provide a natural smoothing mechanism that eliminates the need for introducing extra parameters or finite particle sizes to avoid unacceptable short-range behavior in the effective potential.[28]

IV. CONCLUSION

In this paper we have shown that vector interactions can be introduced into relativistic constraint mechanics through minimal substitutions on the constituent momenta. Among the resulting vector interactions are ones that are purely timelike (case I) and ones that are electromagneticlike (case II). The corresponding quantum wave equations have simple local momentum and potential structures regardless of the strength of the potential, are fully relativistic, and reproduce the standard field-theoretic perturbative spectrum through order α^4.

In quantum electrodynamics, the conventional computation of relativistic corrections can be handled adequately by a perturbative treatment of the weak-potential approximation of the chosen operator appearing in the wave equation. This approximation cannot be justified in cases (such as with phenomenological potentials inspired by QCD), where those corrections may not be small. Our work in this paper shows that the constraint equation in its general form (with no weak-potential restriction) can be used not only in circumstances for which a perturbative treatment is inadequate (e.g., when relativistic effects of the $\mathscr{H}$ operator have significant effects on the wave function) but also in situations where the perturbation is in fact weak. In the latter case, one may use such equations to perform spectral calculations both by perturbative or numerical (nonperturbative) means. Other approaches have not demonstrated this degree of flexibility. When relativistic effects are small, the Breit approximation is good enough for perturbative calculations in QED and quark models with phenomenological potentials.[29] When relativistic effects are large, one might restore the unexpanded kinetic terms to the Breit equation or pass on to a phenomenological modification of the Bethe-Salpeter equation. But the resulting equations will be nonlocal due to complicated momentum dependences of kinetic and interaction terms and may still be defined only as perturbative forms. The main advantage of the constraint approach is that it leads to well-defined local wave equations whose structural complexity is independent of the velocity of constituents or the strength of interaction.

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[8] Our notation is for the most part that used by Todorov in Refs. 1 and 5.

[9] This covariant point function structure for the A_i's is similar to the structure which would appear in a phase-space version of the electrodynamics of Wheeler and Feynman; however, there the roles of the α's and the β's are played by sums of multiple proper-time integrals over products of half-advanced and half-retarded Green's functions.

[10] Since the effective constituent vector potentials depend on the positions only through the relative coordinate (in $\mathscr{A}$ or $\mathscr{V}$) any gauge condition "inherited" from the structure of a related field theory will end up being expressed in terms of the relative coordinate.

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[13]The equivalence of the weak-potential form of our two-body Klein-Gordon equation with Todorovs' homogeneous quasipotential equation [see Eq. (73b)] has the direct benefit of allowing one to work backward from the corresponding inhomogeneous quasipotential equation and infer perturbative field-theoretic information about the form of $\mathscr{A}$ [see also the Introduction and the footnote after Eq. (73b)].

[14]An appropriate fieldlike theory for this matching might be the electrodynamics of Wheeler and Feynman which, like the constraint approach, lacks radiation loss terms. [E. Kerner has developed a systematic way of obtaining a Hamiltonian formulation expanded in a semirelativistic manner in J. Math. Phys. **3**, 35 (1962).]

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[18]The extra terms in Φ produced by the Hermitian ordering are proportional to $p\cdot x_\perp$ and have commutators with $P\cdot p$ proportional to $P\cdot p_\perp(\equiv 0)$.

[19]An alternative quantization that makes use of many proper times, in which the neutral elements are the operators $i\partial/\partial\tau_i$, has been applied by L. P. Horwitz and F. Rohrlich in Phys. Rev. D **24**, 1528 (1981) to the scattering problem. Instead our quantization is based upon the sharp wave-function conditions $\mathscr{H}_i\psi=0$ (which from their point of view are "null eigenvalue" equations) which lead to two free Klein-Gordon equations when the potential is turned off. While their quantization makes use of many proper times, ours is based on the single system time τ that is implicit in the connection between Todorov's quasipotential formalism and the two-body constraint approach.

[20]I. T. Todorov, Lectures presented at the Summer School in Mathematical Physics, Bogazici University, Bebek, Istanbul, Turkey, 1979 (unpublished); *Developements in the Theory of Fundamental Interactions,* Proceedings of the XVIIth Winter School of Theoretical Physics, Karpacz, Poland, edited by L. Turko and A. Pekalski (Harwood Academic, New York, 1981), pp. 543–81.

[21]Normalizable wave packets can be constructed which are eigenfunctions of P^2 and have positive energy, e.g., $\psi(X')=\int e^{iP'\cdot X'}\delta(P'^2+w^2)\theta(P'^0)f(P')d^4P'$ satisfying $(P^\mu P_\mu+w^2)\psi(X')=0$. With an appropriate choice of $f(P')$, $f_{\underline{P}}(P')$, one can construct a $\psi_{\underline{P}}(X')$ that is arbitrarily close to being an eigenfunction of P^μ (a "pseudo-eigenfunction" of P^μ) that has a unit norm. See Appendix 2 of T. Biswas and F. Rohrlich in Ref. 12. Since the system Klein-Gordon equation implies that the current $\psi^*(X')i\overleftrightarrow{\partial}\psi(X')/\partial X^\mu$ is conserved, the Poincaré invariant finite norm $i\int\psi^*(X')\overleftrightarrow{\partial}_\mu\psi(X')\underline{P}'^\mu d^3\sigma(X')$ is suitable for a system centered about $\underline{P}$.

[22]I. T. Todorov, in *Relativistic Action at a Distance: Classical and Quantum Aspects,* edited by J. Llosa (Springer Berlin, 1982).

[23]The two cases I (purely timelike) and II (which as it turns out is electromagneticlike) are not the only possibilities. In general the constraint approach allows an arbitrary combination of the two. There is (conflicting) phenomenological evidence that the effective QCD potential has both of these types of vector interaction. See E. Eichten and F. Feinberg, Phys. Rev. Lett. **43**, 1205 (1979) and H. Crater and P. Van Alstine (Ref. 12).

[24]As we shall establish in a later paper, however, the spectrum in (68) results from an exact solution for the two-body Dirac equation derived from the constraint approach when applied to singlet positronium (for $\mathscr{A}=-\alpha/r$).

[25]We point out a misprint in Eq. (4.21) of the second reference of 5. π^2 should be replaced by $\frac{1}{4}$.

[26]Alternatively, one can derive from this relativistic Schrödinger-type equation a relativistic Lippmann-Schwinger equation equivalent to Todorov's inhomogeneous quasipotential equation by methods analogous to those used in nonrelativistic quantum mechanics. By postulating that the scattering matrix that appears in this equation is the (off-shell) S matrix of the appropriate quantum field theory, it is possible to relate, perturbatively, the vector potentials of our constraint formalism to field theory. It is noteworthy that with just the barest (lowest-order) input from field theory (the nonrelativistic Coulomb potential term), the gauge structure of our constraint formalism leads to spectral results that include the recoil effects of the field-theoretic Darwin interaction. This is in fact an improvement over the results of Todorov in which all aspects of the field-theoretic Born diagram are needed to identify the various parts of the quasipotential. We are able to determine the relativistic $\mathscr{A}$ (and thus the A_i^μ's) from an analysis of merely the nonrelativistic limit of the Born diagram.

[27]Note that the reason this problem does not occur for (79a) is that there the small-r behavior is blunted in the perturbation expectation value by the extra factor of r in the numerator.

[28]This was discovered in Ref. 6 in the constraint approach for spinning particles under mutual scalar interactions.

[29]For a treatment of quark phenomenology using the full Breit equation for two spin-$\frac{1}{2}$ particles see R. W. Childers, Phys. Lett. **121B**, 485 (1983).