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# Stochastic mechanics of spin-$\frac{1}{2}$ particles

Francesco Guerra
*Department of Mathematics, Istituto "G. Castelnuovo," Università di Roma, I-00185, Roma, Italy*

Rossana Marra
*Istituto di Fisica dell'Università, Salerno, Italy*
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Following the strategy of stochastic quantization, based on variational principles for processes taking values on a discrete configuration space, we give a detailed and self-contained description of the stochastic processes associated to a two-level quantum system, of interest for the physical description of spin-$\frac{1}{2}$ particles and Fermi fields. In particular we investigate the class of processes related to unperturbated quantum evolution and those simulating quantum-measurement procedures leading to mixture formation.

## I. INTRODUCTION

In a previous work[1] we considered stochastic variational principles for processes taking values on a discrete configuration space (finite or denumerable). For an appropriate choice of the stochastic action, suggested by the discrete form of the Hamilton-Jacobi-Madelung equation for an associated quantum system, we found a class of processes directly related to quantum states, in the same way as Nelson's stochastic mechanics[2] associates diffusion processes to quantum states in the case of a continuous configuration-space manifold.

For the same stochastic action there exists a different class of critical processes, whose behavior simulates the behavior of quantum systems undergoing a measurement procedure.

The purpose of this paper is to give a detailed and self-contained description of the whole proposed scheme in the simplest nontrivial case of a two-level quantum system, of interest for the physical description of spin-$\frac{1}{2}$ particles and Fermi fields (considered as assemblies of Fermi oscillators after a Jordan-Wigner transformation). In fact, the analysis presented here goes well beyond the general frame outlined in Ref. 1, because in this simple case all elements of this procedure acquire a very simple, direct, and explicit formulation.

The material presented in this paper is organized as follows. In Sec. II we recall the general structure of a two-level quantum system in various representations. The representation most suitable for a stochastic description is based on the splitting of the complex Schrödinger equation into two real equations, interpreted as a continuity equation (conservation of probability) and the Hamilton-Jacobi-Madelung equation in the discrete form. In Sec. III we collect all general properties of controlled Markov random processes on a two-site set, necessary for the following considerations. In Sec. IV we derive a candidate for the stochastic action from the Hamilton-Jacobi-Madelung equation and introduce a basic additional property on the controlled process specifying the behavior of the osmotic part of the transition probability per unit time. In Sec. V we take as the basic assumption of the theory the stochastic action considered in Sec. IV as a candidate, and we show, by direct application of the stochastic variational principle, that the Hamilton-Jacobi-Madelung equation can be derived as a programming equation for the controlled problem. Then we analyze the properties of the associated processes. Section VI deals with the additional class of critical processes arising from the proposed stochastic action. We clarify the analogy between the behavior of these processes and the behavior of quantum systems under measurement procedures. It is rather surprising that a unique stochastic variational principle gives rise to two completely different classes of processes, associated with bifurcations of orbits in phase space. Section VII contains some concluding remarks and outlook for further developments.

## II. THE QUANTUM CANONICAL STRUCTURE

For this very simple quantum system wave functions and norms are given by

$$(-1,1)=Z_2\ni x\to\psi(x)\in C\ ,\qquad \psi\in L^2(Z_2),\quad \|\psi\|^2=\sum_x|\psi(x)|^2\ . \tag{1}$$

From a physical point we understand $x$ as twice the component of the spin along the axis $x_1$. We consider the Hamiltonian defined by

$$(H\psi)(x)=\tfrac{1}{2}[\psi(x)-\psi(-x)] \tag{2}$$

with eigenstates

$$\phi_0(x)=1/2^{1/2},\quad \phi_1(x)=x/2^{1/2}\ ,\qquad H\phi_0=0,\quad H\phi_1=\phi_1\ . \tag{3}$$

The Hamiltonian corresponds to a magnetic field directed along the axis $x_3$. The Schrödinger equation (for $\hbar=1$) is

$$i(\partial_t\psi)(x,t)=(H\psi)(x,t)=\tfrac{1}{2}[\psi(x,t)-\psi(-x,t)]\ . \tag{4}$$

In a canonical setting (see, for example, Ref. 3), for functions $F,G$ of the basic variables $(\psi(\cdot),\psi^*(\cdot))$ we can introduce the Poisson brackets

$$\{F,G\}=-i\sum_x\{[\partial F/\partial\psi(x)][\partial G/\partial\psi^*(x)]-[\partial G/\partial\psi(x)][\partial F/\partial\psi^*(x)]\}\ , \tag{5}$$

$$\{\psi(x),\psi^*(x')\}=-i\delta_{xx'}=-i(1+xx')/2\ . \tag{6}$$

If we define the canonical Hamiltonian $\mathscr{H}$ on the symplectic phase space described by $(\psi(\cdot),\psi^*(\cdot))$,

$$\mathscr{H}(\psi,\psi^*)\equiv\langle\psi,H\psi\rangle=\sum_x\psi^*(x)\tfrac{1}{2}[\psi(x)-\psi(-x)]\ , \tag{7}$$

then the Schrödinger equation (4) can be understood as a canonical Hamilton equation

$$\partial_t\psi=\{\psi,\mathscr{H}\}\ . \tag{8}$$

In analogy with the continuous case[4] it is convenient to work in a different representation, where the density is one of the basic variables. We perform the canonical transformation

$$\psi(x)=\rho^{1/2}\exp[iS(x)]\ . \tag{9}$$

Then the relation for the one-forms

$$\sum_x\rho(x)\delta S(x)=i\sum_x\tfrac{1}{2}[\psi(x)\delta\psi^*(x)-\psi^*(x)\delta\psi(x)] \tag{10}$$

implies that Poisson brackets (5) can be written also as

$$\{F,G\}=\sum_x\{[\partial F/\partial\rho(x)][\partial G/\partial S(x)]-[\partial G/\partial S(x)][\partial F/\partial\rho(x)]\}\ . \tag{11}$$

Therefore, Eq. (7) becomes

$$\mathscr{H}=\tfrac{1}{2}[\rho(1)+\rho(-1)]-[\rho(1)\rho(-1)]^{1/2}\cos[S(1)-S(-1)] \tag{12}$$

and the canonical equation (8) [equivalent to (4)] splits in the continuity equation

$$\dot\rho(x)=\{\rho(x),\mathscr{H}\}=\partial\mathscr{H}/\partial S(x)=x\{[\rho(1)\rho(-1)]^{1/2}\sin[S(1)-S(-1)]\} \tag{13}$$

and the Hamilton-Jacobi equation

$$\dot S(x)=\{S(x),\mathscr{H}\}=-\partial\mathscr{H}/\partial\rho(x)=-H(x)\ ,$$
$$H(x)=\{1-[\rho(-x)/\rho(x)]^{1/2}\cos[S(1)-S(-1)]\}/2\ , \tag{14}$$

so that

$$\mathscr{H}=\sum_x H(x)\rho(x)\ . \tag{15}$$

Notice that $\rho$ and $S$ in (13) and (14) depend also on time $t$ (notationally suppressed). As a consequence of the phase invariance

$$\mathscr{H}(\rho,S)=\mathscr{H}(\rho,S+\chi)\ ,$$
$$(S+\chi)(x)=S(x)+\chi,\ \ \chi\in u(1)\ ,$$

we have the conservation of probability

$$\dot\rho(1)+\dot\rho(-1)=0\ ,$$
$$\rho(1)+\rho(-1)=1\ \text{(at all times)}\ . \tag{16}$$

Therefore, we can reduce the number of variables by defining

$$\alpha=S(1)-S(-1)\in U(1)\ ,$$
$$M=\sum_x x\rho(x)=\rho(1)-\rho(-1)\ , \tag{17a}$$

$$\rho(x)=(1+Mx)/2,\ \ -1\le M\le 1\ . \tag{17b}$$

Then (12), (13), and (14) become

$$\mathscr{H}=\tfrac{1}{2}[1-(1-M^2)^{1/2}\cos\alpha]\ \ (0\le\mathscr{H}\le 1)\ , \tag{18}$$

$$\dot M=\partial(2\mathscr{H})/\partial\alpha=(1-M^2)^{1/2}\sin\alpha\ , \tag{19a}$$

$$\dot\alpha=-\partial(2\mathscr{H})/\partial M=-M(1-M^2)^{1/2}\cos\alpha\ . \tag{19b}$$

The factor 2 in (19) appears because, according to (11), the normalized canonical variables are $(M,\alpha/2)$. It is very well known that for this reduced system of variables the canonical phase space (equivalent to the quantum state space[3]) is $S_2$, the two-dimensional spherical surface. In fact, let us define $n_1=M$ and consider the plane $x_1=n_1$ in $\mathbb{R}^3$, intersecting the unit sphere $S_2$ along a circle of radius $(1-M^2)^{1/2}$. Define $n_2=(1-M^2)^{1/2}\sin\alpha$ and $n_3=(1-M^2)^{1/2}\cos\alpha$ so that $n\equiv(n_1,n_2,n_3)$ is on $S_2$. The natural symplectic structure on $S_2$ is given by the Poisson brackets

$$\{n_1,n_2\}=n_3,\ \text{etc., cyclically}\ . \tag{20}$$

Then one can immediately verify that this natural symplectic structure agrees with (11) reduced on $S_2$, while $\mathscr{H}=\frac{1}{2}(1-n_3)$. Any quantum-mechanical observable $A$ is associated to a phase-space observable $\mathscr{A}=\vec{a}\cdot\vec{n}$, according to the general analysis of Ref. 3, so that

$$\mathscr{A}=\langle\psi,A\psi\rangle,\ \ \{\mathscr{A},\mathscr{B}\}=-i\langle\psi,[A,B]\psi\rangle\ , \tag{21}$$

while the time evolution can be represented in the equivalent unitary and canonical forms

$$\dot A=i[H,A],\ \ \dot{\mathscr{A}}=\{\mathscr{A},\mathscr{H}\}\ . \tag{22}$$

Notice that (21) and (22) represent all essential quantum physical content of the theory of this model.

Since the canonical phase space is compact we can introduce a Lagrangian theory only locally. In fact, take $M$ as a configuration variable, $-1\le M\le 1$, and introduce $\alpha=\alpha(M,\dot M)$ implicitly defined by (19a). Then the Lagrangian is

$$\mathscr{L}(M,\dot M)=(\alpha\sin\alpha+\cos\alpha)(1-M^2)^{1/2} \tag{23}$$

and the Euler-Lagrange equation is

$$\ddot M+M=0\ , \tag{24}$$

which could have also been found as an easy consequence of (19). Therefore, for this system $M(t)$ oscillates periodically in time while $\alpha(t)$ moves accordingly to (19b). The point $n$ on the sphere moves at uniform speed along parallels $n_3=1-2\mathscr{H}$, $0\leq\mathscr{H}\leq 1$, $\mathscr{H}=$constant, as is also clear from the canonical equations arising from (20),

$$\dot{n}(t)=\{n(t),\mathscr{H}\},\quad \mathscr{H}=\tfrac{1}{2}(1-n_3)\ . \tag{25}$$

Our objective is to introduce stochastic variational principles for random processes taking values on $Z_2$ such that (19) are the corresponding continuity and programming equations.

## III. MARKOV PROCESSES ON $Z_2$

The occupation probability $\rho(x,t)$ and transition probabilities $p(x,t;x',t')$, $t\geq t'$, of a Markov process $q(t)$ on $Z_2$, satisfy the equations

$$\sum_x \rho(x,t)=1,\quad \rho(x,t)=\sum_{x'} p(x,t;x',t')\rho(x',t')\ ,\quad t\geq t' \tag{26}$$

$$\partial_t p(x,t;x',t')=\sum_{x''} a^+_{x,x''}(t)p(x'',t;x',t')\ , \tag{27a}$$

$$\partial_t \rho(x,t)=\sum_{x'} a^+_{x,x'}(t)\rho(x',t)\ , \tag{27b}$$

$$a^+_{x,x'}(t)=\lim_{\Delta t\to 0^+}(\Delta t)^{-1}[p(x,t+\Delta t;x',t)-\delta_{xx'}]\ , \tag{28}$$

$$\sum_x a^+_{x,x'}(t)=0\ ,\quad a^+_{x,x'}(t)\geq 0,\quad x\neq x'\ . \tag{29}$$

In our strategy we assume a given initial $\rho(\cdot,t_0)$ and let the process evolve, according to (27), under the control of some given arbitrary $a^+(t)$ in a time interval $t_0\leq t\leq t_1$. According to (29) we can introduce functions $a(t)$, $b(t)$ such that

$$\begin{gathered} 0\leq |a(t)|\leq b(t)\ ,\\ a^+_{x,x'}(t)=\tfrac{1}{2}x[a(t)-b(t)x']\ . \end{gathered} \tag{30}$$

By exploiting (17b), formula (27b) reduces to

$$\dot{M}(t)=a(t)-b(t)M(t)\ . \tag{31}$$

We assume $a(t)$, $b(t)$ as basic controlling parameters; then (27) can be explicitly solved in the form

$$\begin{gathered} p(x,t;x_0,t_0)=\tfrac{1}{2}[1+xM(x_0,t)]\ ,\\ \rho(x,t)=\tfrac{1}{2}[1+xM(t)]\ , \end{gathered} \tag{32}$$

$$M(t)=\left[M(t_0)+\int_{t_0}^{t}dt'a(t')\exp\left[\int_{t_0}^{t'}b(t'')dt''\right]\right]\times\exp\left[-\int_{t_0}^{t}b(t')dt'\right]\ , \tag{33}$$

where $M(x_0,t)$ has the same expression as $M(t)$ in (33), with the substitution of $M(t_0)$ with $x_0$ in (33). We define

$$(D_{(\pm)}F)(x,t)=\pm\lim_{t\to 0^+}E(F(q(t\pm\Delta t),t\pm\Delta t)-F(q(t),t)\,|\,q(t)=x) \tag{34}$$

and find, in analogy with the result in Refs. 1–3,

$$(D_{(\pm)}F)(x,t)=(\partial_t F)(x,t)+[F(-x,t)-F(x,t)]a^{\pm}_{-x,x}\ , \tag{35}$$

$$\begin{aligned} a^+_{-x,x}&=\tfrac{1}{2}(b-ax)\ ,\\ a^-_{-x,x}&=-a^+_{x,-x}\rho(-x)/\rho(x)=\tfrac{1}{2}\{2aM-b(1+M^2)+[2bM-a(1+M^2)]\}/(1-M^2)\ . \end{aligned} \tag{36}$$

Let us also introduce

$$a_{x,x'}=\tfrac{1}{2}(a^+_{x,x'}+a^-_{x,x'})\ , \tag{37}$$

$$a^0_{x,x'}=\tfrac{1}{2}(a^+_{x,x'}-a^-_{x,x'}) \tag{38}$$

and notice

$$\partial_t\rho(x,t)=2a_{x,-x}\rho(-x,t)\ , \tag{39}$$

$$a_{-x,x}=(a-bM)(M-x)/2(1-M^2)\ , \tag{40}$$

$$a^0_{-x,x}=(b-aM)(1-Mx)/2(1-M^2)\ , \tag{41}$$

$$\begin{aligned} a^+_{x,-x}a^+_{-x,x}&=a^-_{x,-x}a^-_{-x,x}\\ &=a_{x,-x}a_{-x,x}+a^0_{x,-x}a^0_{-x,x}\\ &=\tfrac{1}{4}(b^2-a^2)\ . \end{aligned} \tag{42}$$

## IV. A CANDIDATE FOR THE STOCHASTIC ACTION

In order to build up the machinery of stochastic variational principles in this particular case we need a candidate for the stochastic action, as starting point for the strategy presented in Refs. 5 and 1.

In the continuous case we have strong hints coming from the semiclassical limit.[5,6] Here we are obliged to work along different lines, by exploiting the method followed in Ref. 1.

Therefore, let us consider the forward derivative $D_{(+)}S_x$ of the phase function defined in (9). By exploiting (35), (14), (17), and (30) we find

$$\mathscr{L}^{(+)}_x=D_{(+)}S_x=\tfrac{1}{2}\alpha(a-xb)+\tfrac{1}{2}(1-Mx)(1-M^2)^{-1/2}\cos\alpha-\tfrac{1}{2}\ . \tag{43}$$

If $\mathscr{L}^{(+)}$ must be assumed as a stochastic Lagrangian, then it must be only a function of $a^+$ or equivalently of $a$ and $b$. So we must try to express $\alpha$ and $M$ in (43) as functions of $a$ and $b$. This will need some basic assumption about the osmotic part $a^0$, as in Refs. 1 and 7.

In fact if we compare (39) with (13) we find

$$a_{x,-x}=\tfrac{1}{2}x[\rho(x)/\rho(-x)]^{1/2}\sin\alpha\ . \tag{44}$$

On the other hand, on the basis of (29), we must have

$$a^+_{x,-x}=a_{x,-x}+a^0_{x,-x}\geq 0\ . \tag{45}$$

In the continuous case the osmotic part depends only on the density,[2,5] therefore a natural assumption is[1,7]

$$a^0_{x,-x}=\tfrac{1}{2}[\rho(x)/\rho(-x)]^{1/2}\ . \tag{46}$$

Straightforward consequences of (44), (46), (40), and (41) are the following:

$$b^2-a^2=\cos^2\alpha \ , \tag{47}$$

$$M=(a-b\sin\alpha)/(b-a\sin\alpha) \ . \tag{48}$$

Then we have accomplished the objective of expressing $\mathscr{L}^+$ as a function of $a$ and $b$ only. In fact, Eq. (47) can be assumed as defining $\alpha$ as a function of $a$, $b$, while (48) allows us the rewrite (43) in the form

$$\mathscr{L}_x^{(+)}=-\tfrac{1}{2}+(b-ax)\tfrac{1}{2}[(\cos\alpha)^{-1}-x(\alpha-\tan\alpha)] \ . \tag{49}$$

Let us notice explicitly, that, even though (47) does not define $\alpha$ uniquely, nevertheless (49) and (47) are sufficient as a starting point for a variational principle. In fact, $\alpha$ can be considered as an additional controlling parameter, on which $\mathscr{L}^{(+)}$ depends, such that the variations $\delta a$, $\delta b$, $\delta\alpha$ are constrained by (47) in the form

$$a\delta a-b\delta b=\sin\alpha\cos\alpha\delta\alpha \ . \tag{50}$$

Notice that the action corresponding to (49) is

$$A=\int_{t_0}^{t_1}E(\mathscr{L}^{(+)})dt=-\tfrac{1}{2}(t_1-t_0)+\tfrac{1}{2}\int_{t_0}^{t_1}[(b-aM)(\cos\alpha)^{-1}+(a-bM)(\alpha-\tan\alpha)]dt \ . \tag{51}$$

## V. THE STOCHASTIC VARIATIONAL PRINCIPLE AND THE ASSOCIATED CRITICAL PROCESSES

Let us now explicitly state the basic assumptions. We consider Markov processes on $Z_2$ with kinematical properties described in Sec. III, in terms of the controlling parameters $a^+$, or $a$ and $b$. For these processes we introduce the stochastic Lagrangian defined by (49), (47), and the associated action (51). Notice that the discussion in Sec. IV must be considered only as a motivation for the choices (47), (49), and (51). As a matter of fact, these three equations are the basic assumptions of the stochastic theory. We will show that they allow us to derive all peculiar properties of the associated quantum system, giving also a stochastic model for the quantum measurement process.

In fact, by following the same methods as in Refs. 5 and 1, we can state our first result. If the controlled process is such that the action (51) is stationary under small variations of the control $a^+$, subject to the condition that the occupation probability is kept fixed at times $t_0, t_1$, then there must necessarily be a function $S(x,t)$ such that

$$(D_+S)_x=\mathscr{L}_x^{(+)} \ , \tag{52}$$

and moreover,

$$E(\delta\mathscr{L}^{(+)})+\tfrac{1}{2}(M\delta b-\delta a)[S(1,t)-S(-1,t)]=0 \tag{53}$$

for any $\delta a$, $\delta b$. The proof is straightforward and employs the same technique as in Refs. 1 and 5.

Now we calculate $\delta\mathscr{L}^{(+)}$. Starting from (49) and taking into account (50), we find

$$\delta\mathscr{L}_x^{(+)}=(\delta b-x\delta a)[(\cos\alpha)^{-1}-x(\alpha-\tan\alpha)]/2+(b-ax)(1+x\sin\alpha)(a\delta a-b\delta b)/2(\cos\alpha)^3 \ . \tag{54}$$

Now we take the average and consider that (53) must hold for any $\delta a$, $\delta b$, then we get the two equations

$$\begin{aligned}&(\cos\alpha)^{-1}+M[S(1)-S(-1)-\alpha+\tan\alpha]-bc=0 \ ,\\ &\alpha-\tan\alpha-S(1)+S(-1)-M(\cos\alpha)^{-1}+ac=0 \ ,\end{aligned} \tag{55}$$

where

$$c=[(b-a\sin\alpha)+M(b\sin\alpha-a)](\cos\alpha)^{-3} \ . \tag{56}$$

It is immediately found that Eqs. (55) have two solutions. The first, which we call standard, is given by

$$M=(a-b\sin\alpha)/(b-a\sin\alpha) \ , \tag{57a}$$

$$\alpha=S(1)-S(-1) \ . \tag{57b}$$

The second, called nonstandard, is

$$M=a/b \ , \tag{58a}$$

$$\alpha-\tan\alpha=S(1)-S(-1) \ . \tag{58b}$$

The quite unexpected fact that the stochastic action (51) has two families of critical processes, ruled by (57) and (58), respectively, is a general feature of this general scheme and is confirmed by the analysis made in Ref. 1.

The rest of this section is devoted to the analysis of standard solutions (57). The others, given by (58), are analyzed in Sec. VI.

If we take (57a) and substitute in (40), we find, *as a consequence of the variational principle,* the expressions (44) and (46) for $a_{x,-x}$ and $a^0_{x,-x}$. Moreover, substitution of (57b) in the transport equation (52) gives immediately, after a simple calculation, the Hamilton-Jacobi equation (14). On the other hand, (13) also is satisfied. Therefore, we can conclude that the stochastic variational principle, based on (51), reproduces completely the quantum-mechanical content given by (13) and (14). Moreover, the stochastic process associated to each state is completely specified by (44) and (46).

For this simple system the time dependence of all variables can easily be expressed in explicit form. In fact, from (24) we have

$$\begin{aligned}&M(t)=M_0\cos(t-t_0), \quad -1\le M_0\le 1 \ ,\\ &\dot M(t)=-M_0\sin(t-t_0) \ .\end{aligned} \tag{59}$$

Therefore, (19a) gives

$$\sin\alpha(t)=-[M_0\sin(t-t_0)]/[1-M_0^2\cos^2(t-t_0)]^{1/2} \ . \tag{60}$$

On the other hand, we also have

$$\begin{aligned}&a(t)=(M+\sin\alpha)/(1-M^2)^{1/2} \ ,\\ &b(t)=(1+M\sin\alpha)/(1-M^2)^{1/2} \ ,\end{aligned} \tag{61}$$

so that all the properties of the process are explicitly known. It is amusing to see what happens in the extreme case of equatorial states. Take, for example, $t_0=0$, $M_0=1$, so that $M(t)=\cos t$, $\sin\alpha(t)=-\text{sign}(\sin t)$. Clearly this is a case where the chart on $S_2$ given by $(M,\alpha)$ becomes singular (recall the meaning of $M,\alpha$ given in Sec. II). For the transition probabilities per unit time we have

$$\begin{aligned} &a^+_{1,-1}=0 \quad \text{for } \sin t>0 , \\ &a^+_{1,-1}=(1+\cos t)/|\sin t| \quad \text{for } \sin t<0 , \\ &a^+_{-1,1}=(1-\cos t)/|\sin t| \quad \text{for } \sin t>0 , \\ &a^+_{-1,1}=0 \quad \text{for } \sin t<0 . \end{aligned} \tag{62}$$

Therefore, the process is completely concentrated at $+1$ at time $t=0$. During the time interval $0\le t\le \pi$, there is a continuous transition toward $-1$, while no transition back to $+1$ is possible. The transition probability per unit time $a^+_{-1,1}$ from $+1$ to $-1$ becomes infinite as $t\to\pi$ ($|\sin t|\to 0$), so that the site $+1$ is completely void at time $t=\pi$ and the process is completely concentrated at the site $-1$. Then, as time progresses, the same situation is met with $+1,-1$ exchanged. On the other hand, for the south-pole process (ground-state process), we have $M_0=0$, $\sin\alpha=0$, $\alpha=\pi$, therefore $M(t)=0$, while $a^+_{x,-x}=\frac{1}{2}$. Therefore, on the average the two sites $+1,-1$ have the same occupation probability $\frac{1}{2}$ and the transition probability per unit time is constant. Since now $a(t)=0$, $b(t)=1$, we can derive from (32), and (33) the explicit expression of the transition probability

$$p(x,t;x_0,t_0)=\tfrac{1}{2}\{1+xx_0\exp[-(t-t_0)]\} , \tag{63}$$

which corresponds to a time-homogeneous Markov process on $Z_2$.

For a general state we have a situation intermediate between the equatorial and polar ones. Also here the transition probability can be easily found starting from (32), (33), and the explicit expressions (59), (60), and (61). In the general case the process is not time homogeneous, because it must cope with the variability of $M(t)$ given by (59). Equatorial states, as we have shown before, correspond to alternate drastic depletions of the two sites $+1,-1$.

It must also be observed that two quantum states, obtained by reflection with respect to the equatorial plane, give rise to the same process (but $\alpha$ is different in general). This degeneracy can be easily understood. In fact, in our model we have taken a fixed magnetic field along the third axis, while we have taken a representation where the spin component along the first axis has been diagonalized (compare with the discussion in Sec. II). Clearly the degeneracy is removed if we consider the process corresponding to a representation where the component of the spin along a generic axis in space has been diagonalized. Alternatively, we can consider magnetic fields in generic directions by changing the Hamiltonian of Sec. II. Then the general method, explained in Ref. 1, allows us to build the complete stochastic frame also for these more general cases.

On the other hand, the degeneracy is also removed by looking at the sign of $\cos\alpha$. This makes it clear that a deeper understanding of the physical meaning of the parameter $\alpha$ of (47) is really necessary.

## VI. CRITICAL PROCESSES LEADING TO MIXTURES

Let us now consider the process associated to the nonstandard solution (58). First of all, let us notice that (58a), together with the purely kinematical (31), implies $M(t)=M_0=\text{constant}$. Therefore, the occupation probabilities do not change and the phase-space representative moves along circles perpendicular to the $x_1$ axis (recall Sec. II). Let us now exploit (52), together with (58) and (49). A simple computation leads to

$$\dot S(x)=\tfrac{1}{2}[\eta(1-Mx)(1-M^2)^{-1/2}-1] , \tag{64}$$

where $\eta$ is the sign of $\cos\alpha$. Therefore, we have

$$S(1)-S(-1)=-\eta M/(1-M^2)^{1/2} . \tag{65}$$

Consider time intervals where $\alpha$ does not cross $\pm\pi/2$, where $\cos\alpha$ changes sign, so that $\eta$ stays constant. Then (65) is easily integrated and (58b) gives

$$\begin{aligned} \alpha(t)-\tan\alpha(t) = &-\eta M(1-M^2)^{-1/2}(t-t_0) \\ &+\alpha_0-\tan\alpha_0 , \end{aligned} \tag{66}$$

where we assume $|M|<1$. A simple look at the graph of $\alpha$-$\tan\alpha$ shows that $\alpha(t)$ evolves in time, according to (66), so that the previous points are never crossed, but are asymptotic limits as $t\to\pm\infty$ provided $M\neq 0$. Therefore as $t\to\infty$, for $M\neq 0$ we have $\alpha(t)\to\pm\pi/2$ and $\cos\alpha\to 0$. Therefore $a(t)\to 0$, $b(t)\to 0$, while the ratio $M=a(t)/b(t)$ stays constant, as (47) and (58a) show. Therefore, in the asymptotic time limit the diffusion disappears and the process becomes a mixture of sticky processes at $+1$ or $-1$, with the given occupation probabilities $(1\pm M)/2$.

This behavior strongly resembles the behavior of a quantum system subject to a measurement. Here the measured quantity is the spin component along the $x_1$ axis. In the time-asymptotic region the process, evolving along the nonstandard solution, reduces to a mixture corresponding to the two pure states with spin along the two directions of $x_1$. Notice that the relaxation given by (66) has a speed ruled by $M$. In the case $M=0$ corresponding to the maximum uncertainty, no relaxation appears and $\alpha(t)$ stays constant. If $|M|$ increases, then the speed increases, and in fact becomes infinite as $|M|\to 1$. The points $M=\pm 1$ are to be considered singular points, in fact $\alpha$ is not defined there. In the generic case $\alpha(t)$ moves to reach the points $\pm\pi/2$ as $t\to\pm\infty$. Therefore the orbits on $S_2$ are portions of circles lying on the plane $x_1=M$. It is immediately seen that standard and nonstandard orbits have a locus of bifurcation along the maximum circle on the plane $x_2=0$, which contains the $x_1$ and $x_3$ axis.

It is surely remarkable that a single variational principle gives rise to two completely different qualitative behaviors, in particular to a bifurcation for the orbits. On the other hand, this mechanism can be proposed as a model for quantum measurement only after some additional investigation. In fact, here the relaxation to mixture formation is a time-asymptotic phenomenon, while in the standard quantum-measurement theory it is assumed as instantaneous (even though opinions differ on this point).

## VII. CONCLUSIONS AND OUTLOOK

For the simple case of a two-level quantum system, adequate for the description of the rotational degrees of freedom of a spin-$\frac{1}{2}$ particle in a magnetic field, we have shown that a suitable form of stochastic variational principle allows us to derive all properties of stochastic processes, associated to quantum states, along the general strategy of stochastic mechanics. A by-product of this construction is the emergence of bifurcations leading to random processes, which simulate some typical behavior of quantum-measurement theory. We have explicitly described all essential features of the two classes of processes.

The analysis is in agreement with the general one, about discrete quantum systems, presented elsewhere.[1]

A natural frame of application of the ideas presented here could be provided by the study of quantum thermal mixture, exploiting methods of stochastic mechanics, according to the general frame presented, for example, in Ref. 8.

Other possible applications refer to the study of Fermi quantum fields, for which a satisfactory stochastic theory is still lacking.

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