Quantum states and the Hadamard form. II. Energy minimization for spin-$\frac{1}{2}$ fields

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We investigate properties of the states of a Hermitian spin-$\frac{1}{2}$ field in a Robertson-Walker space-time constructed by an energy-minimization requirement. It is shown that the singularity structure of the commutator function $\langle[\hat{\Psi}(x),\hat{\bar{\Psi}}(x')]\rangle$ for these states is such that current renormalization theory yields an infinite value for the renormalized expectation value of the trace of the stress tensor.

I. INTRODUCTION

The problem of constructing the physical Hilbert space of states for a free quantum field propagating on a nonstatic curved background space-time has been the subject of much research.[1] In the first paper in this series[2] we examined one such construction for a scalar field theory and showed that it was unacceptable in that it gave rise to infinite values for $\langle\phi^2\rangle_{\rm ren}$ and $\langle T^{\mu\nu}\rangle_{\rm ren}$. This construction was based on the principle that the physical state space on a Cauchy surface S should be a Fock space whose vacuum state was such as to minimize the total energy on S.[3]

To investigate the generality of the above result we have been led to consider the construction of the physical space of states for a Hermitian spin-$\frac{1}{2}$ field. The construction of the space of states by an energy-minimization condition has recently been detailed by one of us,[4] and it is the states obtained through this construction that we shall investigate in this paper. Our space-time conventions will be those of Misner, Thorne, and Wheeler,[5] while our field-theory conventions will be those of DeWitt.[6]

II. SPINORS IN CURVED SPACE-TIME

In this section we shall briefly review the topic of spinor fields in a general curved space-time before restricting ourselves later to the case of Robertson-Walker universes. We assume that the space-time is globally hyperbolic and has vanishing second Stiefel-Whitney class so that there is no obstruction to the construction of a global spin structure.[7]

First we introduce a vierbein field $L^p(x)$ ($p=0,1,2,3$) whose components $L^p{}_\mu$ are related to the space-time metric by the equation

$$g_{\mu\nu}=L^p{}_\mu L^q{}_\nu \eta_{pq} , \tag{2.1}$$

where $\eta_{pq}=\mathrm{diag}(-1,1,1,1)_{pq}$ is the Minkowski metric. Vierbein indices, which will be denoted by latin letters, are raised and lowered by means of the Minkowski metric.

If $\{\gamma^p\}$ denotes a set of flat-space Dirac matrices satisfying the anticommutation relations $\{\gamma^p,\gamma^q\}=2\eta^{pq}I$ then their curved-space analogs are defined by the equations

$$\gamma^\mu=L_p{}^\mu\gamma^p \tag{2.2}$$

and satisfy the anticommutation relations

$$\{\gamma^\mu,\gamma^\nu\}=2g^{\mu\nu}I . \tag{2.3}$$

With our metric signature we may choose a representation of these anticommutation relations in which the γ^μ are real[6] (we will use such a representation below). There then exists a real antisymmetric matrix γ such that

$$\gamma^{\mu\sim}=-\gamma\gamma^\mu\gamma^{-1} . \tag{2.4}$$

Under a proper Lorentz transformation $\gamma'^r=L^r{}_p\gamma^p$ the anticommutation relations (2.3) remain invariant. It follows that there exists a real matrix $S(L)$ of unit determinant such that

$$\gamma'^r=S^{-1}\gamma^r S . \tag{2.5}$$

The spinor field Ψ is defined to provide a spin representation of the vierbein group according to the transformation law

$$\Psi'=S\Psi . \tag{2.6}$$

The contragradient representation is obtained by taking $\bar{\Psi}=\Psi^\sim\gamma$ which transforms according to the law

$$\bar{\Psi}'=\bar{\Psi}S^{-1} . \tag{2.7}$$

The covariant derivative of a spinor field Ψ is given by

$$\Psi_{;\mu}=\Psi_{,\mu}+\Gamma_\mu\Psi , \tag{2.8}$$

where

$$\Gamma_\mu=\tfrac{1}{2}\Sigma^{pq}L_p{}^\nu L_{q\nu,\mu} \tag{2.9}$$

with $\Sigma^{pq}\equiv\frac{1}{4}[\gamma^p,\gamma^q]$, while the covariant derivative of the adjoint spinor $\bar{\Psi}$ is given by

$$\bar{\Psi}_{;\mu}=\Psi^\sim{}_{,\mu}\gamma-\Psi^\sim\gamma\Gamma_\mu . \tag{2.10}$$

The action functional for the free Hermitian spin-$\frac{1}{2}$ field is

$$S[\Psi]=\tfrac{1}{2}i\int g^{1/2}d^4x\,\overline{\Psi}(\gamma^\mu\Psi_{;\mu}+m\Psi)\ . \tag{2.11}$$

Demanding that this action be stationary with respect to arbitrary variations $\delta\overline{\Psi}(x)$ gives rise to the Dirac equation

$$\gamma^\mu\Psi_{;\mu}+m\Psi=0\ . \tag{2.12}$$

The stress tensor for the spin-$\frac{1}{2}$ field is defined by

$$T_{\mu\nu}=g^{-1/2}L_{p\nu}\frac{\delta S}{\delta L_p{}^\mu}$$
$$=\tfrac{1}{4}i(\overline{\Psi}_{;(\mu}\gamma_{\nu)}\Psi-\overline{\Psi}\gamma_{(\mu}\Psi_{;\nu)})\ , \tag{2.13}$$

where we have used the Dirac equation in writing $T_{\mu\nu}$ in this form.

To conclude this brief section we define the intrinsic derivative D_ν of a spinor restricted to a spacelike hypersurface S:[8]

$$D_\nu\Psi\,|_S=h_\nu{}^\mu\Psi_{;\mu}+\tfrac{1}{2}\chi_{\mu\nu}n_\rho\gamma^\mu\gamma^\rho\Psi\ , \tag{2.14}$$

where n^μ is the unit future-pointing normal to the hypersurface, $h_{\mu\nu}\equiv g_{\mu\nu}+n_\mu n_\nu$ is its intrinsic metric and $\chi_{\mu\nu}$ is the second fundamental form on S. The intrinsic Dirac operator on S is then given by

$$\not{D}\Psi\,|_S=\gamma^\mu h_\mu{}^\nu D_\nu\Psi=\gamma^\mu h_\mu{}^\nu\Psi_{;\nu}+\tfrac{1}{2}\chi n_\nu\gamma^\nu\Psi\ . \tag{2.15}$$

III. QUANTIZATION AND ENERGY MINIMIZATION

The first step in the passage from the classical to the quantum theory of the spin-$\frac{1}{2}$ field is the construction of the operator algebra. This is achieved by replacing $\Psi(x)$ by an operator-valued distribution $\hat{\Psi}(x)$ which satisfies the Dirac equation (2.12) and imposing on it the covariant anticommutation relation[6]

$$\{\hat{\Psi}(x),\hat{\Psi}^{\sim}(x')\}=i\widetilde{G}(x,x')\hat{1}\ , \tag{3.1}$$

where $\widetilde{G}$ is the difference of the advanced and the retarded Green's function of the classical theory.

The remaining step in the passage to the quantum theory is the construction of the Hilbert space of states. We shall limit ourselves to cases in which this space is an antisymmetric Fock space. To construct such a Fock space one must first identify "positive- and negative-frequency" solutions to the Dirac equation. The one-particle Hilbert space H^1 of the theory is then taken to be the direct sum $H^+\oplus\overline{H}^-$ where H^+ and H^- are the Hilbert spaces of positive- and negative-frequency solutions, respectively, and $\overline{H}^-$ is the dual space to H^-. The full Hilbert space of states is then taken to be the antisymmetric Fock space constructed from H^1:[9]

$$\mathcal{F}=C\oplus H^1\oplus(H^1\otimes H^1)_a\oplus\cdots\ ,$$

where the subscript a denotes that the antisymmetric tensor product is to be taken.

In static space-times the choice of positive- and negative-frequency solutions can be made by Fourier transforming solutions with respect to the preferred time coordinate. In more general space-times it is necessary to give a more general prescription for determining the space of positive-frequency solutions which defines the physical Hilbert space of states. Indeed in nonstatic space-times one expects on general grounds that there should be a production of physical particles by the time-varying gravitational field. This leads one to associate a Hilbert space with each instant, that is, with each Cauchy surface S. The prescription for the construction of the space of states that we shall be investigating in this paper is that of energy minimization. We review this construction briefly below; for a fuller description the reader is referred to Ref. 4.

Let $\{\psi_k(x),\psi_k^*(x)\}$ be a complete set of solutions to the Dirac equation satisfying the orthonormality conditions

$$\int_s\overline{\psi}{}_k^*\gamma^\mu\psi_{k'}d\Sigma_\mu=\delta_{kk'}\ , \tag{3.2}$$

$$\int_s\overline{\psi}_k\gamma^\mu\psi_{k'}d\Sigma_\mu=0\ . \tag{3.3}$$

These equations are independent of the choice of Cauchy surface S by virtue of the Dirac equation. Here ψ_k and ψ_k^* are to be identified with positive- and negative-frequency solutions, respectively. The generalized index k must include at least one discrete index taking two distinct values.

The field $\hat{\Psi}(x)$ can now be expanded in terms of these solutions:

$$\hat{\Psi}(x)=\sum_k[\hat{a}_k\psi_k(x)+\hat{a}{}_k^*\psi_k^*(x)]\ . \tag{3.4}$$

According to Eqs. (3.1), (3.2), and (3.3) the annihilation and creation operators $\hat{a}_k$ and $\hat{a}{}_k^*$ must satisfy the anticommutation relations

$$\{\hat{a}_k,\hat{a}{}_{k'}^*\}=\delta_{kk'}\hat{1}\ , \tag{3.5}$$

$$\{\hat{a}_k,\hat{a}_{k'}\}=\{\hat{a}{}_k^*,\hat{a}{}_{k'}^*\}=\hat{0}\ . \tag{3.6}$$

To determine the physical Fock space associated with the Cauchy surface S we consider the total energy of each possible vacuum state $|\,\hat{a};\mathrm{vac}\rangle$. To be specific, consider the formal quantity

$$E[\,|\,\hat{a};\mathrm{vac}\rangle]=\int_s dS\,n^\mu n^\nu\langle\hat{a};\mathrm{vac}\,|\,T_{\mu\nu}[\hat{\Psi}]\,|\,\hat{a};\mathrm{vac}\rangle\ , \tag{3.7}$$

where $T_{\mu\nu}[\hat{\Psi}]$ denotes the operator obtained on replacing $\Psi(x)$ by $\hat{\Psi}(x)$ in Eq. (2.13). Since at this stage we are only interested in comparing $E[\,|\,\hat{a};\mathrm{vac}\rangle]$ for different choices of $|\,\hat{a};\mathrm{vac}\rangle$ we need not worry about problems of renormalization. Inserting the expansion (3.4) into Eq. (3.7) we obtain

$$E[\,|\,\hat{a};\mathrm{vac}\rangle]=-\tfrac{1}{2}i\int_s dS\sum_k\psi_k(\not{D}+m)\psi_k^*\ . \tag{3.8}$$

The energy-minimization ansatz identifies the physical choice of positive-frequency modes ψ_k with that which minimizes expression (3.8) subject to the constraints (3.2) and (3.3).

The solution to this variational problem has recently been found by one of us.[4] The physical positive-frequency spinors should be negative-eigenvalue eigenvectors of the operator $\vec{\mathrm{O}}$ defined on S by

$$\vec{\mathrm{O}}\equiv in_\mu\gamma^\mu(\not{D}+m)\ . \tag{3.9}$$

For later convenience we record here that

$$\vec{O}^2=-\not{D}^2+m^2 . \tag{3.10}$$

Furthermore, this expression can be related to the Laplace-Beltrami operator ${}^3\Delta$ on the hypersurface by the Weitzenbock formula[8]

$$\not{D}^2={}^3\Delta+\tfrac{1}{4}{}^3R , \tag{3.11}$$

where ${}^3R$ is the Ricci scalar of the hypersurface.

IV. ENERGY MINIMIZATION IN COSMOLOGY

In this section we apply the energy-minimization ansatz of the previous section to the construction of the Hilbert space of states in a cosmological space-time.

For simplicity we shall deal with a spatially flat Robertson-Walker universe. It is convenient to write the metric in the manifestly conformally flat form

$$ds^2=a^2(t)[-dt^2+(dx^1)^2+(dx^2)^2+(dx^3)^2] , \tag{4.1}$$

where $t\in(0,\infty)$, $x^i\in(-\infty,\infty)$. Since the second Stiefel-Whitney class of this space-time vanishes there is no obstruction to the construction of a spinor structure on it. The nonzero Ricci tensor components are given by

$$R_t{}^t=-\frac{3}{a^4}(\dot a^2-a\ddot a),\quad R_i{}^j=\frac{1}{a^4}(\dot a^2+a\ddot a)\delta_i{}^j \tag{4.2}$$

and the Ricci scalar by

$$R=6\frac{\ddot a}{a^3} . \tag{4.3}$$

The nonzero components of the second fundamental form of a hypersurface $t=$constant with future-pointing normal n^μ are given by

$$\chi_i{}^j=\tfrac{1}{3}\chi\delta_i{}^j=\frac{\dot a}{a^2}\delta_i{}^j . \tag{4.4}$$

We can choose a vierbein adapted to the above coordinate system by defining

$$L^0=adt,\quad L^i=a\,dx^i . \tag{4.5}$$

The spin-connection coefficients can be calculated from Eq. (2.9) as

$$\Gamma_t=0,\quad \Gamma_{x^i}=-\frac{\dot a}{2a}\gamma^i\gamma^0 , \tag{4.6}$$

where γ^0 and γ^i are flat-space Dirac matrices.

We choose the following representation for the Dirac matrices:

$$\gamma^0=\begin{bmatrix}-i\sigma_2 & 0\\ 0 & -i\sigma_2\end{bmatrix},\quad \gamma^1=\begin{bmatrix}0 & \sigma_1\\ \sigma_1 & 0\end{bmatrix},$$
$$\gamma^2=\begin{bmatrix}\sigma_1 & 0\\ 0 & -\sigma_1\end{bmatrix},\quad \gamma^3=\begin{bmatrix}\sigma_3 & 0\\ 0 & \sigma_3\end{bmatrix}, \tag{4.7}$$

where

$$\sigma_1=\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix},\quad \sigma_2=\begin{bmatrix}0 & -i\\ i & 0\end{bmatrix},$$

and

$$\sigma_3=\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}$$

are the Pauli spin matrices. We note that all these γ matrices are real, giving us a Majorana representation. The matrix γ defined by Eq. (2.4) can and will be chosen equal to $-\gamma^0$.

Taking advantage of the spatial homogeneity of the Robertson-Walker universe we seek solutions to the Dirac equation of the form

$$\Psi(t,\vec{x})=[2\pi a(t)]^{-3/2}\psi(t)e^{i\vec{k}\cdot\vec{x}} . \tag{4.8}$$

Acting on such a spinor the Dirac equation becomes

$$\gamma^0\psi_{,t}+ik_i\gamma^i\psi+ma\psi=0 . \tag{4.9}$$

If we act on this equation with the operator $(\gamma^0\partial_t+ik_i\gamma^i-ma)$ we find that ψ must satisfy

$$\psi_{,tt}+(k^2+m^2a^2)\psi-m\dot a\gamma^0\psi=0 . \tag{4.10}$$

Writing

$$\psi=\begin{bmatrix}u\\ v\end{bmatrix},$$

Eq. (4.10) requires that u obey the equation

$$u_{,tt}+(k^2+m^2a^2)u+im\dot a\sigma_2u=0 . \tag{4.11}$$

This equation is easily solved by defining $U=Su$, where $S\equiv\frac{1}{2}(I-i\sigma_1+i\sigma_2+\sigma_3)$ so $S\sigma_2S^{-1}=\sigma_3$. If

$$U=\begin{bmatrix}U_1\\ U_2\end{bmatrix},$$

then Eq. (4.11) yields the equations

$$U_{1,tt}+(k^2+m^2a^2)U_1+im\dot aU_1=0 , \tag{4.12a}$$

$$U_{2,tt}+(k^2+m^2a^2)U_2-im\dot aU_2=0 . \tag{4.12b}$$

Finally, using Eq. (4.9) to relate v to u, we find that the general solution to the Dirac equation can be written as

$$\psi=\begin{bmatrix}k_1u\\ i\sigma_3u_{,t}-k_2u+ik_3\sigma_2u+ima\sigma_1u\end{bmatrix}, \tag{4.13}$$

where

$$u=\begin{bmatrix}U_1-U_2\\ i(U_1+U_2)\end{bmatrix}$$

with U_1 and U_2 satisfying Eqs. (4.12a) and (4.12b), respectively.

We now turn to the determination of the orthonormal positive-frequency solutions satisfying the energy-minimization condition on a hypersurface $t=t_0$. According to Sec. III we need to solve the eigenvalue problem

$\vec{O}\Psi=\lambda\Psi$. Using Eqs. (3.10) and (3.11) it can be shown that $\vec{O}^2\Psi=\lambda^2\Psi=a_0^{-2}(k^2{}_0+m^2a_0^2)\Psi$, where $a_0\equiv a(t_0)$. It follows that for positive-frequency solutions we must solve the equation

$$\vec{O}\psi\equiv\frac{1}{a_0}\gamma^0(k_i\gamma^i-ima_0)\psi$$
$$=-\frac{1}{a_0}(k^2+m^2a^2)^{1/2}\psi\ , \qquad (4.14)$$

subject to the constraints (3.2) and (3.3). Denoting the two independent solutions by $\psi_{1\vec{k}}$ and $\psi_{2\vec{k}}$ the constraints take on the simple form

$$\psi^{\sim *}_{r\vec{k}}\psi_{s\vec{k}}=\delta_{rs} \qquad (4.15)$$

and

$$\psi^{\sim}_{r\vec{k}}\psi_{s(-\vec{k})}=0\ . \qquad (4.16)$$

After much tedious algebra it can be shown that these solutions can be expressed through the boundary conditions

$$\psi_{r\vec{k}}(t_0)=A_r\begin{bmatrix}k_1(k_3+ima_0)\\ -k_1[\eta+(-1)^r\kappa]\\ -[k_2+(-1)^r\kappa](k_3+ima_0)\\ -[k_2+(-1)^r\kappa][\eta+(-1)^r\kappa]\end{bmatrix} \qquad (4.17)$$

with

$$4\kappa\eta[k_2+(-1)^r\kappa][\eta+(-1)^r\kappa](-1)^r|A_r|^2=1\ , \qquad (4.18)$$

where

$$\kappa\equiv(k_1^2+k_2^2)^{1/2} \text{ and } \eta\equiv(k^2+m^2a_0^2)^{1/2}\ .$$

It is interesting to note that these boundary conditions take a remarkably simple form when expressed in terms of the auxiliary functions U_1 and U_2 of Eq. (4.13). The boundary conditions on these functions are simply

$$\dot{U}_i(t_0)=-i\eta U_i(t_0),\ \ i=1,2 \qquad (4.19)$$

together with appropriate normalization conditions. Equation (4.19) is also of interest in that it provides a link between the energy-minimization formalism for spinor fields and that for scalar fields.[3]

V. ENERGY MINIMIZATION AND RENORMALIZATION

To study the physical properties of the states constructed in the previous section we now examine the vacuum commutator function $G(x,x')\equiv\langle[\hat{\Psi}(x),\hat{\bar{\Psi}}(x')]\rangle$. In terms of the mode decomposition (3.4) we can write

$$G(t,\vec{x};t',\vec{x}\,')=\frac{1}{8\pi^3[a(t)a(t')]^{3/2}}\int d^3\vec{k}e^{i\vec{k}\cdot(\vec{x}-\vec{x}\,')}\sum_{r=1}^{2}[\psi_{r\vec{k}}(t)\bar{\psi}^*_{r\vec{k}}(t')-\psi^*_{r(-\vec{k})}(t)\bar{\psi}_{r(-\vec{k})}(t')]\ . \qquad (5.1)$$

If we restrict x and x' to the initial hypersurface $t=t_0$ then we can express the energy-minimization commutator function $G^{\mathrm{EM}}(x,x')$ in closed form by inserting Eqs. (4.17) and (4.18) into Eq. (5.1). This yields

$$G^{\mathrm{EM}}{}_{\alpha}{}^{\beta'}(t_0,\vec{x};t_0,\vec{x}\,')=\frac{1}{(2\pi a_0)^3}\int\frac{d^3\vec{k}}{(k^2+m^2a_0^2)^{1/2}}e^{i\vec{k}\cdot(\vec{x}-\vec{x}\,')}[-(k^2+m^2a_0^2)^{1/2}\gamma^0+k_i\gamma^i+ima_0I]_{\alpha}{}^{\beta'}\ , \qquad (5.2)$$

where $a_0\equiv a(t_0)$, and for clarity we have displayed the spinor indices explicitly.

We wish to compare the singularity structure of the above commutator function with that assumed by standard renormalization theory. The first step in that theory is to introduce an auxiliary propagator $\mathcal{G}(x,x')$ which is defined to satisfy the equation[6]

$$G(x,x')=-i(\gamma^\mu\nabla_\mu-m)\mathcal{G}(x,x')\ . \qquad (5.3)$$

It then follows from the Dirac equation that $\mathcal{G}(x,x')$ satisfies the second-order differential equation

$$(\Box-\tfrac{1}{4}R-m^2)\mathcal{G}=0\ . \qquad (5.4)$$

Standard renormalization theory assumes that the singularity structure of $\mathcal{G}$ is of the form first discussed by Hadamard.[10] The Hadamard ansatz postulates that, at least for x sufficiently close to x', $\mathcal{G}(x,x')$ can be written in the form

$$\mathcal{G}^H(x,x')=\frac{1}{4\pi^2}\left[\frac{u}{\sigma}+v\ln|\sigma|+w\right]\ , \qquad (5.5)$$

where $\sigma(x,x')$ is one half the square of the geodesic distance between x and x'. In Eq. (5.5) u, v, and w are assumed to be smooth bispinor functions of x and x' with v and w possessing expansions of the form

$$v_\alpha{}^{\beta'}(x,x')=\sum_{n=0}^{\infty}v_{n\alpha}{}^{\beta'}(x,x')\sigma^n\ , \qquad (5.6a)$$

$$w_\alpha{}^{\beta'}(x,x')=\sum_{n=0}^{\infty}w_{n\alpha}{}^{\beta'}(x,x')\sigma^n\ . \qquad (5.6b)$$

Imposing Eq. (5.4) determines $u_\alpha{}^{\beta'}=\Delta^{1/2}\mathcal{I}_\alpha{}^{\beta'}$, where $\Delta(x,x')\equiv g^{-1/2}(x)\det(\sigma_{;\mu\nu})g^{-1/2}(x')$ is the biscalar form of the VanVleck-Morette determinant, and $\mathcal{I}_\alpha{}^{\beta'}(x,x')$ is the bispinor of parallel transport which is defined by the equation

$$\sigma^{;\mu}\mathcal{I}_\alpha{}^{\beta'}{}_{;\mu}=0$$

together with the boundary condition $\mathcal{I}_\alpha{}^{\beta}(x,x)=I_\alpha{}^{\beta}$.[6]

Imposing Eq. (5.4) and equating equal powers of σ to zero yields the differential recursion relations

$$(n+1)(n+2)v_{n+1}+(n+1)v_{n+1;\mu}\sigma^{;\mu} -(n+1)v_{n+1}\Delta^{-1/2}\Delta^{1/2}{}_{;\mu}\sigma^{;\mu}+\tfrac{1}{2}(\Box-\tfrac{1}{4}R-m^2)v_n =0\,, \quad (5.7)$$

$$(n+1)(n+2)w_{n+1}+(n+1)w_{n+1;\mu}\sigma^{;\mu} -(n+1)w_{n+1}\Delta^{-1/2}\Delta^{1/2}{}_{;\mu}\sigma^{;\mu}+\tfrac{1}{2}(\Box-\tfrac{1}{4}R-m^2)w_n +(2n+3)v_{n+1}+v_{n+1;\mu}\sigma^{;\mu}-\Delta^{-1/2}\Delta^{1/2}{}_{;\mu}\sigma^{;\mu}v_{n+1} =0\,, \quad (5.8)$$

together with the boundary condition

$$v_{0;\mu}\sigma^{;\mu}+(1-\Delta^{-1/2}\Delta^{1/2}{}_{;\mu}\sigma^{;\mu})v_0 +\tfrac{1}{2}(\Box-\tfrac{1}{4}R-m^2)(\Delta^{1/2}\mathscr{I})=0\,. \quad (5.9)$$

These recursion relations are identical to the equivalent scalar relations with $\xi=\frac{1}{4}$, except that here of course the covariant derivatives are understood to be acting on spinors. As in the scalar case the singular part of $\mathscr{G}^H$ is completely determined by these recursion relations. On the other hand, w_0 is undetermined corresponding to the freedom to add any nonsingular solution to the homogeneous equation.

To obtain the small-distance behavior of $\mathscr{G}^H(x,x')$ we may use Eqs. (5.7) and (5.9) to obtain a Taylor series expansion for $\mathscr{I}_{\beta'}{}^{\alpha}v_{0_\alpha}{}^{\gamma'}$ and $\mathscr{I}_{\beta'}{}^{\alpha}v_{1_\alpha}{}^{\gamma'}$. These expansions are displayed in the Appendix together with other useful bispinor expansions.

To compare the singularity structure of $G^{\rm EM}(x,x')$ with that arising from $\mathscr{G}^H(x,x')$ it is convenient to examine the divergences in the trace of the stress tensor $\langle T_\mu{}^\mu[\hat{\Psi}]\rangle$. From Eq. (2.13) it follows that classically

$$T_\mu{}^\mu[\Psi]=\tfrac{1}{2}im\overline{\Psi}\Psi\,. \quad (5.10)$$

Formally we can write the expectation value of $T_\mu{}^\mu[\hat{\Psi}]$ in the energy-minimization vacuum state as

$$T_\mu{}^\mu(\mathrm{EM})=-\tfrac{1}{4}im\lim_{x\to x'}\mathscr{I}_{\beta'}{}^{\alpha}G^{\rm EM}{}_{\alpha}{}^{\beta'}(x,x') =-\tfrac{1}{4}im\lim_{x\to x'}\mathrm{Tr}(\mathscr{I}G^{\rm EM})\,. \quad (5.11)$$

The corresponding expression for the expectation value in a state with auxiliary propagator $\mathscr{G}^H$ is given by Eqs. (5.11) and (5.3) as

$$T_\mu{}^\mu(H)=-\tfrac{1}{4}m\lim_{x\to x'}\mathrm{Tr}[\mathscr{I}(\gamma^\mu\nabla_\mu-m)\mathscr{G}^H]\,. \quad (5.12)$$

To compare (5.11) with (5.12) it is necessary to calculate $\mathscr{I}(x,x')$ for x close to x'. First we note that[2]

$$\sigma(t,\vec{x};t',\vec{x}\,')=-\tfrac{1}{2}a^2\tau^2+\tfrac{1}{2}a^2r^2-\tfrac{1}{2}a\dot{a}\tau^3+\tfrac{1}{2}a\dot{a}\tau r^2 +\tfrac{1}{24}[-(3\dot{a}^2+4a\ddot{a})\tau^4 +2(\dot{a}^2+2a\ddot{a})\tau^2r^2+\dot{a}^2r^4]+\cdots\,, \quad (5.13)$$

where $\tau\equiv(t-t')$, $r\equiv|\vec{x}-\vec{x}\,'|$ and a, $\dot{a}$, $\ddot{a}$ are all evaluated at t'. Writing the equation $\sigma^{;\mu}\mathscr{I}_{;\mu}=0$ as a partial differential equation for the components of $\mathscr{I}$ in the vierbein (4.5), we find

$$\mathscr{I}_{\beta'}{}^{\alpha}=I_{\beta'}{}^{\alpha}+\frac{\dot{a}}{2a}r_i(\gamma^i\gamma^0)_{\beta'}{}^{\alpha} +\tfrac{1}{4}\left[\frac{\ddot{a}}{a}-\frac{\dot{a}^2}{a^2}\right]\tau r_i(\gamma^i\gamma^0)_{\beta'}{}^{\alpha} +\frac{\dot{a}^2}{8a^2}r^2I_{\beta'}{}^{\alpha}+\cdots\,, \quad (5.14)$$

where $r_i\equiv(\vec{x}-\vec{x}\,')_i$ and a, $\dot{a}$, $\ddot{a}$ are again evaluated at t'.

We can now evaluate $\mathrm{Tr}(\mathscr{I}G^{\rm EM})$ and $\mathrm{Tr}[\mathscr{I}(\gamma^\mu\nabla_\mu-m)\mathscr{G}^H]$ for two points x and x' which lie close to each other on the initial hypersurface, so that $\tau=0$ in Eq (5.14). From Eq. (5.2), for $x\neq x'$, we have

$$G^{\rm EM}{}_{\alpha}{}^{\beta'}|_{t_0}=\frac{1}{(2\pi a)^3}(imaGI_\alpha{}^{\beta'}-i\partial_iG\gamma^i{}_\alpha{}^{\beta'})\,, \quad (5.15)$$

where

$$G\equiv\int\frac{d^3\vec{k}}{(k^2+m^2a^2)^{1/2}}e^{i\vec{k}\cdot(\vec{x}-\vec{x}\,')-\epsilon k} =\frac{4\pi}{r^2}+\pi m^2a^2\sum_{k=0}^{\infty}\frac{1}{k!(k+1)!}\left[\frac{mar}{2}\right]^{2k} \times\left[2\ln\left[\frac{mar}{2}\right]-\frac{1}{k+1}-2\psi(k+1)\right] \quad (5.16)$$

with $\psi(1)\equiv-\gamma$ and

$$\psi(k+1)\equiv\sum_{m=1}^{k}\frac{1}{m}-\gamma\quad(k\geq1)\,,$$

γ being Euler's constant. It follows that

$$G^{\rm EM}|_{t_0}=\frac{\pi i}{(2\pi a)^3}\left[\frac{4ma}{r^2}I+\frac{8}{r^4}r_i\gamma^i-2m^2a^2\frac{1}{r^2}r_i\gamma^i +4\frac{\dot{a}}{a}\frac{1}{r^2}\gamma^0-2m\dot{a}\frac{1}{r^2}r_i(\gamma^i\gamma^0) +\frac{\dot{a}^2}{a^2}\frac{1}{r^2}r_i\gamma^i+m^3a^3\ln r^2I+\cdots\right], \quad (5.17)$$

where we have dropped the spinor indices for clarity. Combining Eq. (5.17) with Eq. (5.14) we find

$$\mathrm{Tr}(\mathscr{I}G^{\rm EM})|_{t_0}=\frac{im}{2\pi^2}\left[\frac{4}{a^2r^2}+m^2\ln r^2+\cdots\right], \quad (5.18)$$

where the remaining terms are nonsingular in the limit $x\to x'$.

On the other hand, using the expansions of $\mathscr{I}v_0$ and $\mathscr{I}v_1$ given in the Appendix, it can readily be shown that $\mathrm{Tr}(\mathscr{I}\gamma^\mu\nabla_\mu\mathscr{G}^H)$ is nonsingular in the limit $x\to x'$ while

$$\mathrm{Tr}(\mathscr{I}\mathscr{G}^H)=\frac{1}{4\pi^2}\left[\frac{4}{\sigma}+2(m^2+\tfrac{1}{12}R)\ln\sigma+\cdots\right]. \quad (5.19)$$

Restricting x and x' to the initial hypersurface, it follows that

$$(-i)\mathrm{Tr}[\mathscr{I}(\gamma^\mu\nabla_\mu - m)\mathscr{G}^H]\,|_{t_0} = \frac{im}{2\pi^2}\left[\frac{4}{a^2r^2} + (m^2 + \tfrac{1}{12}R)\ln r^2 + \cdots\right]. \quad (5.20)$$

As a check on our calculation we note that Eq. (5.20) can also be obtained from the results of Christensen.[11]

Comparison of Eq. (5.18) with Eq. (5.20) reveals that the divergences in $T_\mu{}^\mu(\mathrm{EM})$ are not of the same form as those in $T_\mu{}^\mu(H)$. It follows that standard renormalization theory cannot yield a finite value for $T_\mu{}^\mu(\mathrm{EM})_{\mathrm{ren}}$.

VI. CONCLUSION

In this paper we have extended the results of the first paper in this series[2] to show that for spin-$\frac{1}{2}$ field theories, as for scalar field theories, the infinities arising in physical expectation values in states constructed by an energy-minimization requirement cannot be dealt with by standard renormalization theory. The consequences of this circumstance were discussed in Ref. 2 and we will not repeat the discussion here.

So far in this series of papers we have set ourselves the task of answering the question of whether particular schemes for the construction of the space of states give rise to two-point functions with the Hadamard form. If one accepts the renormalization prescription as it is understood today (which perhaps one should not) then it seems that a minimal requirement for physical states is that their corresponding two-point function (the anticommutator function for scalar fields, the auxiliary propagator for spin-$\frac{1}{2}$ fields) should have the Hadamard form. In the next paper in this series we shall examine the constraints on the construction of the physical space of states imposed by this requirement. That these constraints must be nontrivial has been demonstrated above.

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APPENDIX

We list here the Taylor series expansions for various bispinors which were used in the text; other useful expansions can be found in Ref. 11. We shall use the notation $\sigma^\mu \equiv \sigma^{;\mu}$ and $\Sigma^{\mu\nu} \equiv \frac{1}{4}[\gamma^\mu, \gamma^\nu]$.

$$\mathscr{I}v_0 = \tfrac{1}{2}(m^2 + \tfrac{1}{12}R)I - \tfrac{1}{48}R_{;\mu}\sigma^\mu I - \tfrac{1}{12}R_\mu{}^{\rho;\tau}\sigma^\mu\Sigma_{\rho\tau} + \tfrac{1}{24}[m^2R_{\mu\nu} + \tfrac{1}{12}RR_{\mu\nu} + \tfrac{1}{6}R_{;\mu\nu} + \tfrac{1}{20}(C^\kappa{}_{\mu\nu}{}^\lambda R_{\kappa\lambda} + 2C^\kappa{}_{\mu\nu}{}^\lambda{}_{;\kappa\lambda}) - \tfrac{1}{60}g_{\mu\nu}(R_{\rho\tau\kappa\lambda}R^{\rho\tau\kappa\lambda} - R_{\kappa\lambda}R^{\kappa\lambda} + \Box R)]\sigma^\mu\sigma^\nu I + \tfrac{1}{24}R_\mu{}^{\rho;\tau}{}_{;\nu}\sigma^\mu\sigma^\nu\Sigma_{\rho\tau} - \tfrac{1}{96}R^{\rho\tau}{}_{\lambda\mu}R^{\xi\eta\lambda}{}_\nu\sigma^\mu\sigma^\nu\Sigma_{\rho\tau}\Sigma_{\xi\eta} + \cdots, \quad (A1)$$

$$\mathscr{I}v_1 = \tfrac{1}{8}(m^2 + \tfrac{1}{6}Rm^2 + \tfrac{1}{144}R^2 - \tfrac{1}{60}\Box R + \tfrac{1}{90}R_{\rho\tau\kappa\lambda}R^{\rho\tau\kappa\lambda} - \tfrac{1}{90}R_{\kappa\lambda}R^{\kappa\lambda})I + \tfrac{1}{192}R^{\rho\tau}{}_{\kappa\lambda}R^{\xi\eta\kappa\lambda}\Sigma_{\rho\tau}\Sigma_{\xi\eta} + \cdots, \quad (A2)$$

$$\Delta^{1/2;\mu}\mathscr{I}\mathscr{I}_{;\mu} = -\tfrac{1}{24}R^\lambda{}_\mu R^{\rho\tau}{}_{\lambda\nu}\sigma^\mu\sigma^\nu\Sigma_{\rho\tau} + \cdots, \quad (A3)$$

$$\Delta^{1/2}\mathscr{I}\mathscr{I}_{;\mu}{}^{;\mu} = \tfrac{1}{3}R_\mu{}^{\rho;\tau}\sigma^\mu\Sigma_{\rho\tau} + \tfrac{1}{12}(R^{\rho\tau}{}_{\kappa\mu}R^\kappa{}_\nu - R_\mu{}^{\rho;\tau}{}_{;\nu})\sigma^\mu\sigma^\nu\Sigma_{\rho\tau} + \tfrac{1}{16}R^{\rho\tau}{}_{\kappa\mu}R^{\xi\eta\kappa}{}_\nu\sigma^\mu\sigma^\nu\Sigma_{\rho\tau}\Sigma_{\xi\eta} + \cdots. \quad (A4)$$

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