Vacuum $\langle T_\mu{}^\nu \rangle$ in Schwarzschild spacetime

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The vacuum expectation value of the stress-energy tensor for the Hartle-Hawking vacuum in Schwarzschild spacetime has been calculated by means of the method of covariant point separation. It is found that $\langle T_\mu{}^\nu \rangle$ separates naturally into the sum of two terms. The first coincides with an approximate expression suggested by Page on the basis of a Gaussian approximation to the proper-time Green's function. The second term is a "remainder" which comprises sums over mode functions that may be evaluated numerically. It is found that the total expression is in good qualitative agreement with Page's approximation. These results are at variance with earlier numerical results given by Fawcett which purported to show that the true value of $\langle T_\mu{}^\nu \rangle$ differed in important respects from Page's approximation. The error in Fawcett's calculation is explained.

I. INTRODUCTION

In this paper we calculate the vacuum expectation value of the stress-energy tensor for the conformally invariant scalar field in the region exterior to the horizon of a Schwarzschild black hole. The vacuum state under consideration is the Hartle-Hawking[1] vacuum, which represents a black hole of mass M in unstable thermal equilibrium with a bath of blackbody radiation[2] of local temperature

$$T_{\rm loc}=\frac{1}{8\pi M}\left[1-\frac{2M}{r}\right]^{-1/2} . \tag{1.1}$$

Preliminary results of this calculation have been presented in Ref. 3.

We find that the expression for $\langle T_\mu{}^\nu \rangle_{\rm ren}$ splits naturally into two parts:

$$\langle T_\mu{}^\nu\rangle_{\rm ren}=\frac{\pi^2}{90(8\pi M)^4}\left\{\left[1-\frac{2M}{r}\right]^{-2}\left[1-\left[\frac{2M}{r}\right]^6\left[4-\frac{6M}{r}\right]^2\right]\begin{bmatrix}-3&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix}\right.$$
$$\left.+24\left[\frac{2M}{r}\right]^6\begin{bmatrix}3&0&0&0\\0&1&0&0\\0&0&0&0\\0&0&0&0\end{bmatrix}+1920\Delta_\mu{}^\nu\right\} . \tag{1.2}$$

The first part, consisting of the first two terms, is identical to the expression obtained by Page[4] by means of a Gaussian[5] approximation to the proper-time propagator. As described by Page, this part has the trace and asymptotic behavior expected from the stress tensor.

The second part contains $\Delta_\mu{}^\nu$ which is composed of a traceless combination of several types of mode sums. Our numerical evaluation of $\Delta_\mu{}^\nu$ shows that Page's approximate expression dominates $\langle T_\mu{}^\nu \rangle_{\rm ren}$; $\Delta_\mu{}^\nu$ does not significantly affect its character.

These results clearly differ from earlier numerical work published by Fawcett,[6] which purported to show that the true value of $\langle T_\mu{}^\nu \rangle_{\rm ren}$ had significant differences from Page's approximation. We show that this discrepancy arises from an oversight in Fawcett's analysis prior to his numerical calculation.

II. RENORMALIZATION OF $\langle T_\mu{}^\nu \rangle$

We shall calculate the vacuum expectation value of the stress-energy tensor using the operator expression

$$\langle T_\mu{}^\nu\rangle=\langle(\tfrac{2}{3}\phi_{;\mu}\phi^{;\nu}-\tfrac{1}{6}g_\mu{}^\nu\phi_{;\alpha}\phi^{;\alpha}-\tfrac{1}{3}\phi\phi_{;\mu}{}^\nu)\rangle . \tag{2.1}$$

Using the geodesic point-separation scheme of DeWitt[7] and Christensen,[8] the renormalized value of $\langle T_\mu{}^\nu \rangle$ is given in terms of the Hartle-Hawking propagator G and the bivectors of parallel transport $g^{\alpha'}{}_\mu$ by the expression

$$\langle T_\mu{}^\nu\rangle_{\rm ren}=\lim_{x\to x'}\{-i[\tfrac{1}{3}(G_{;\mu\alpha'}g^{\alpha'\nu}+G^{;\nu}{}_{\alpha'}g^{\alpha'}{}_\mu)-\tfrac{1}{6}G_{;\alpha\beta'}g^{\alpha\beta'}g_\mu{}^\nu-\tfrac{1}{6}(G_{;\mu}{}^\nu+G_{;\alpha'\beta'}g^{\alpha'}{}_\mu g^{\beta'\nu})]-\langle T_\mu{}^\nu\rangle_{\rm subtract}\}\ , \tag{2.2}$$

where $\langle T_\mu{}^\nu\rangle_{\rm subtract}$ are the subtraction terms of Christensen.[8] These subtraction terms are

$$\begin{aligned}-8\pi^2\langle T_{\mu\nu}\rangle_{\rm subtract}=&\frac{-4}{\sigma^2}\left[g_{\mu\nu}-\frac{4\sigma_\mu\sigma_\nu}{\sigma}\right]+\tfrac{1}{360}R^{\alpha\beta\gamma\delta}R_{\alpha\beta\gamma\delta}\frac{\sigma_\mu\sigma_\nu}{\sigma}+\tfrac{1}{45}(R_\mu{}^\rho{}_\nu{}^\tau R_{\rho\alpha\tau\beta}+R_{\mu\tau\rho\alpha}R_\nu{}^{\rho\tau}{}_\beta+R_{\mu\rho\tau\alpha}R_\nu{}^{\rho\tau}{}_\beta)\frac{\sigma^\alpha\sigma^\beta}{\sigma}\\&-\tfrac{1}{45}R^{\rho\tau\kappa}{}_\alpha R_{\rho\tau\kappa(\mu}\frac{\sigma_{\nu)}\sigma^\alpha}{\sigma}-\tfrac{1}{180}R_{\rho\tau\kappa\alpha}R^{\rho\tau\kappa}{}_\beta\frac{\sigma^\alpha\sigma^\beta}{\sigma}\left[g_{\mu\nu}-\frac{2\sigma_\mu\sigma_\nu}{\sigma}\right]\\&-\tfrac{4}{45}R_{\rho\beta\tau\gamma}R^\tau{}_\alpha{}^\rho{}_{(\mu}\frac{\sigma_{\nu)}\sigma^\alpha\sigma^\beta\sigma^\gamma}{\sigma^2}-\tfrac{1}{90}R^\rho{}_\alpha{}^\tau{}_\beta R_{\rho\gamma\tau\gamma}\frac{\sigma^\alpha\sigma^\beta\sigma^\gamma\sigma^\delta}{\sigma^2}\left[g_{\mu\nu}-\frac{4\sigma_\mu\sigma_\nu}{\sigma}\right]\\&+(\tfrac{1}{90}R_{\mu\alpha\nu\beta;\gamma\delta}-\tfrac{1}{45}R^\rho{}_{\alpha\mu\beta}R_{\rho\gamma\nu\delta}+\tfrac{1}{36}R^\rho{}_\alpha{}^\tau{}_\beta R_{\rho\gamma\tau\delta}g_{\mu\nu})\frac{\sigma^\alpha\sigma^\beta\sigma^\gamma\sigma^\delta}{\sigma^2}\end{aligned} \tag{2.3}$$

(σ and σ^μ are defined in Appendix B).

In principle, the knowledge of a single component of $\langle T_\mu{}^\nu\rangle$ in Schwarzschild spacetime is sufficient to determine all of its components. Conventionally, $\langle T_\theta{}^\theta\rangle$ is selected as the component to be evaluated. We will continue this tradition and demonstrate only the calculation of $\langle T_{\theta\theta}\rangle_{\rm ren}$.

It is first necessary to evaluate $\langle T_{\theta\theta}\rangle_{\rm subtract}$. In Schwarzschild spacetime the Riemann tensor may be written in the form

$$R_{abcd}=\frac{2M}{r^3}(g_{ac}g_{bd}-g_{ad}g_{cb}),\quad R_{aibj}=\frac{-M}{r^3}g_{ab}g_{ij},\quad R_{ijkl}=\frac{2M}{r^3}(g_{ik}g_{jl}-g_{il}g_{jk})\ , \tag{2.4}$$

where $abcd$ run over r and t and $ijkl$ run over θ and ϕ. An elementary calculation reveals

$$\langle T_{\theta\theta}\rangle_{\rm subtract}=\frac{r^2}{2\pi^2\sigma^2}+\frac{M^2}{360\pi^2r^4}\ . \tag{2.5}$$

The form above is valid in Schwarzschild spacetime whenever σ^μ has zero angular components.

From Eq. (2.2) we see that the expression for $\langle T_{\theta\theta}\rangle_{\rm ren}$ takes the form

$$\begin{aligned}\langle T_{\theta\theta}\rangle_{\rm ren}&=\lim_{x\to x'}\{-i[\tfrac{2}{3}G_{;\theta\theta'}g^{\theta'}{}_\theta-\tfrac{1}{6}G_{;\alpha\beta'}g^{\alpha\beta'}g_{\theta\theta}-\tfrac{1}{6}(G_{;\theta\theta}+G_{;\theta'\theta'}g^{\theta'}{}_\theta g^{\theta'}{}_\theta)]-\langle T_{\theta\theta}\rangle_{\rm subtract}\}\\&=\lim_{x\to x'}\left\{-i\left[\tfrac{2}{3}G_{,\theta\theta'}\frac{r'}{r}-\tfrac{1}{6}r^2\left[g^{tt'}G_{,tt'}+g^{rr'}G_{,rr'}+g^{rt'}G_{,rt'}+g^{tr'}G_{,tr'}+\frac{1}{rr'}G_{,\theta\theta'}+G_{,\phi\phi'}\frac{1}{rr'\sin\theta\sin\theta'}\right]\right.\right.\\&\qquad\left.\left.-\tfrac{1}{6}\left[G_{,\theta\theta}-\Gamma^r{}_{\theta\theta}G_{,r}+G_{,\theta'\theta'}\frac{r^2}{r'^2}-\Gamma^{r'}{}_{\theta'\theta'}G_{,r'}\frac{r^2}{r'^2}\right]\right]-\frac{r^2}{2\pi^2\sigma^2}-\frac{M^2}{360\pi^2r^4}\right\}\ ,\end{aligned} \tag{2.6}$$

where in the last equation we have explicitly shown the forms of $g^{\theta'}{}_\theta$, $g^{\phi'}{}_\phi$, and $g_{\theta\theta}$. The covariant derivatives have been written as partial derivatives plus Christoffel symbols. The only Christoffel symbol which appears in Eq. (2.6) is

$$\Gamma^r{}_{\theta\theta}=-(r-2M)\ . \tag{2.7}$$

It is shown in Ref. 9 that, for imaginary values of the Schwarzschild "time" coordinate $t=-i\tau$, this propagator may be expressed, when $\tau'\neq\tau$, in the form

$$G_H(-i\tau,r,\theta,\phi;-i\tau',r',\theta',\phi')=\frac{i}{32\pi^2M^2}\sum_{n=1}^{\infty}\cos n\kappa(\tau-\tau')\sum_{l=0}^{\infty}(2l+1)P_l(\cos\gamma)[p_l^n(\xi_<)q_l^n(\xi_>)/n-2P_l(\xi_<)Q_l(\xi_>)]\ , \tag{2.8}$$

where

$$\cos\gamma=\cos\theta\cos\theta'+\sin\theta\sin\theta'\cos(\phi-\phi'),\quad \xi=\frac{r}{M}-1,\quad \kappa=\frac{1}{4M}\ ,$$

and $\xi_<$ and $\xi_>$ denote the smaller and the greater of ξ and ξ'. P_l and Q_l are Legendre functions and p_l^n and q_l^n are solutions of the radial equation

$$\left[\frac{d}{d\xi}(\xi^2-1)\frac{d}{d\xi}-l(l+1)-\frac{n^2}{16}\frac{(1+\xi)^4}{(\xi^2-1)}\right]R(\xi)=0 \tag{2.9}$$

specified by the requirements that, for $n>0$, $p_l^n(\xi)$ is the solution that remains bounded as $\xi\to 1$ and $q_l^n(\xi)$ is the solution that tends to zero as $\xi\to\infty$. These solutions are normalized by requiring

$$P_l^n(\xi)\sim(\xi-1)^{n/2},\quad q_l^n(\xi)\sim(\xi-1)^{-n/2}\quad\text{as } \xi\to 1\ . \tag{2.10}$$

As discussed in Ref. 9, this form for the propagator is well suited to the type of problem at hand. The expression (2.8) for the propagator will not contain divergent l sums as partial coincidence limits are taken.

Use of the expression (2.8) for the Hartle-Hawking propagator brings $\langle T_\theta{}^\theta\rangle_{\rm ren}$ into the form

$$\begin{aligned}
\langle T_\theta{}^\theta\rangle_{\rm ren}=\frac{1}{192M^4\pi^2}\lim_{x\to x'}\Bigg[&\frac{g^{tt'}}{16}\sum_{n=1}^{\infty}n^2\cos n\kappa\epsilon\sum_{l=0}^{\infty}(2l+1)P_l(\cos\gamma)\left[\frac{p_l^n(\xi)q_l^n(\xi')}{n}-2P_l(\xi)Q_l(\xi')\right]\\
&-\frac{g^{rr'}}{2}\sum_{n=1}^{\infty}\cos n\kappa\epsilon\sum_{l=0}^{\infty}(2l+1)P_l(\cos\gamma)\left[\frac{1}{n}\frac{\partial p_l^n(\xi)}{\partial\xi}\frac{\partial q_l^n(\xi')}{\partial\xi'}-2\frac{\partial P_l(\xi)}{\partial\xi}\frac{\partial Q_l(\xi')}{\partial\xi'}\right]\\
&-\frac{1}{2}\left[\frac{(\xi-1)}{(\xi+1)^2}\frac{\partial}{\partial\xi}+\frac{(\xi'-1)}{(\xi'+1)^2}\frac{\partial}{\partial\xi'}\right]\\
&\times\sum_{n=1}^{\infty}\cos n\kappa\epsilon\sum_{l=0}^{\infty}(2l+1)P_l(\cos\gamma)\left[\frac{1}{n}p_l^n(\xi)q_l^n(\xi')-2P_l(\xi)Q_l(\xi')\right]\\
&+\frac{1}{4(\xi+1)^2}\left[1+\frac{\xi+1}{\xi'+1}\right]^2\sum_{n=1}^{\infty}\cos n\kappa\epsilon\sum_{l=0}^{\infty}(2l+1)l(l+1)P_l(\cos\gamma)\\
&\qquad\times\left[\frac{1}{n}p_l^n(\xi)q_l^n(\xi')-2P_l(\xi)Q_l(\xi')\right]\\
&+\frac{i}{8}\left[g^{rt'}\frac{\partial}{\partial\xi}-g^{tr'}\frac{\partial}{\partial\xi'}\right]\sum_{n=1}^{\infty}n\sin n\kappa\epsilon\sum_{l=0}^{\infty}(2l+1)P_l(\cos\gamma)\left[\frac{1}{n}p_l^n(\xi)q_l^n(\xi')-2P_l(\xi)Q_l(\xi')\right]\\
&-\frac{384(\xi+1)^2M^4}{(\xi-1)^2\epsilon^4}+\frac{64M^2}{\epsilon^2(\xi^2-1)^2}-\frac{64(\xi+1)^2}{(\xi-1)^2}\left[\frac{1}{5(\xi+1)^7}-\frac{37}{120(\xi+1)^8}\right]-\frac{16}{15(\xi+1)^6}\Bigg]\,,
\end{aligned} \tag{2.11}$$

where $\epsilon=\tau-\tau'$. In the partial coincidence limit $\xi=\xi'$, $\theta=\theta'$, and $\phi=\phi'$, Eq. (2.11) becomes

$$\begin{aligned}
\langle T_\theta{}^\theta\rangle_{\rm ren}=\lim_{x\to x'}\frac{1}{192\pi^2M^4}\Bigg\{&\frac{g^{tt'}}{16}\sum_{n=1}^{\infty}n^2\cos(n\kappa\epsilon)\sum_{l=0}^{\infty}\left[\frac{(2l+1)}{n}p_l^n(\xi)q_l^n(\xi)-\frac{2}{(\xi^2-1)^{1/2}}\right]\\
&+\frac{i}{8}g^{rt'}\frac{d}{d\xi}\sum_{n=1}^{\infty}n\sin(n\kappa\epsilon)\sum_{l=0}^{\infty}\left[\frac{(2l+1)}{n}p_l^n(\xi)q_l^n(\xi)-\frac{2}{(\xi^2-1)^{1/2}}\right]\\
&-g^{rr'}\sum_{n=1}^{\infty}\cos(n\kappa\epsilon)\sum_{l=0}^{\infty}\Bigg[\frac{2l+1}{n}\frac{dp_l^n(\xi)}{d\xi}\frac{dq_l^n(\xi)}{d\xi}+\frac{2l(l+1)}{(\xi^2-1)^{3/2}}\\
&\qquad+\frac{n^2(\xi+1)^4}{16(\xi^2-1)^{5/2}}-\frac{3}{4(\xi^2-1)^{5/2}}\Bigg]\\
&+\frac{2}{(\xi+1)^2}\sum_{n=1}^{\infty}\cos(n\kappa\epsilon)\sum_{l=0}^{\infty}\Bigg[\frac{2l+1}{n}l(l+1)p_l^n(\xi)q_l^n(\xi)
\end{aligned}$$

$$-\frac{2l(l+1)}{(\xi^2-1)^{1/2}}+\frac{n^2(\xi+1)^4}{16(\xi^2-1)^{3/2}}-\frac{1}{4(\xi^2-1)^{3/2}}\Bigg]$$

$$-\frac{\xi-1}{(\xi+1)^2}\frac{d}{d\xi}\sum_{n=1}^{\infty}\cos(n\kappa\epsilon)\sum_{l=0}^{\infty}\left[\frac{2l+1}{n}p_l^n(\xi)q_l^n(\xi)-\frac{2}{(\xi^2-1)^{1/2}}\right]$$

$$-\frac{1}{16}\left[\frac{\xi+1}{\xi-1}\right]^2\sum_{n=1}^{\infty}n^3\cos(n\kappa\epsilon)+\frac{1}{(\xi^2-1)^2}\sum_{n=1}^{\infty}n\cos(n\kappa\epsilon)$$

$$+\frac{1}{1920}\left[1+10\left[\frac{2}{\xi+1}\right]^4-48\left[\frac{2}{\xi+1}\right]^7+37\left[\frac{2}{\xi+1}\right]^8\right]\left[\frac{\xi+1}{\xi-1}\right]^2\Bigg\}\,, \tag{2.12}$$

where terms involving Legendre functions have been replaced using identities proved in Appendix A. We have freely added terms to Eq. (2.12) which are proportional to

$$\sum_{n=1}^{\infty}n^2\cos(n\kappa\epsilon)=0\,. \tag{2.13}$$

These terms have been inserted to make explicit the convergence of the l sums. For large l the WKB approximants indicate that the l summands in Eq. (2.12) are all $O(l^{-2})$. The inverse powers of ϵ in (2.11) have been replaced with n sums shown in Appendix B to be equivalent up to terms of order ϵ^2 which will not survive the limiting process.

Unfortunately, while the expression on the left-hand side of (2.12) is finite and unambiguous, it is expressed in terms of n sums which are not separately finite. We may remedy this by distributing the $O(n^3)$ and $O(n)$ terms, which originated as subtraction terms, among the other sums. In this effort we are guided by the WKB form of the product $p_l^n(\xi)q_l^n(\xi)$. This procedure provides

$$\langle T_\theta{}^\theta\rangle_{\rm ren}=\frac{1}{192M^4\pi^2}\lim_{\epsilon\to0}\Bigg\{\frac{g^{tt'}}{16}\sum_{n=1}^{\infty}n^2\cos(n\kappa\epsilon)\left[\sum_{l=0}^{\infty}\left[\frac{2l+1}{n}p_l^n(\xi)q_l^n(\xi)-\frac{2}{(\xi^2-1)^{1/2}}\right]+\frac{n}{2}\left[\frac{\xi+1}{\xi-1}\right]\right]$$

$$-g^{rr'}\sum_{n=1}^{\infty}\cos(n\kappa\epsilon)\Bigg[\sum_{l=0}^{\infty}\Bigg[\frac{2l+1}{n}\frac{dp_l^n(\xi)}{d\xi}\frac{dq_l^n(\xi)}{d\xi}+\frac{2l(l+1)}{(\xi^2-1)^{3/2}}$$

$$+\frac{n^2(\xi+1)^4}{16(\xi^2-1)^{5/2}}-\frac{3}{4(\xi^2-1)^{5/2}}\Bigg]-\frac{n^3(\xi+1)^3}{96(\xi-1)^3}+\frac{n}{3(\xi-1)^3}\Bigg]$$

$$+\frac{2}{(\xi+1)^2}\sum_{n=1}^{\infty}\cos(n\kappa\epsilon)\Bigg[\sum_{l=0}^{\infty}\Bigg[\frac{2l+1}{n}l(l+1)p_l^n(\xi)q_l^n(\xi)$$

$$-\frac{2l(l+1)}{(\xi^2-1)^{1/2}}+\frac{n^2(\xi+1)^4}{16(\xi^2-1)^{3/2}}-\frac{1}{4(\xi^2-1)^{3/2}}\Bigg]$$

$$-\frac{n^3(\xi+1)^4}{48(\xi-1)^2}-\frac{n(\xi-2)}{3(\xi-1)^2}\Bigg]$$

$$-\frac{\xi-1}{(\xi+1)^2}\frac{d}{d\xi}\sum_{n=1}^{\infty}\cos(n\kappa\epsilon)\left[\sum_{l=0}^{\infty}\left[\frac{2l+1}{n}p_l^n(\xi)q_l^n(\xi)-\frac{2}{(\xi^2-1)^{1/2}}\right]+\frac{n(\xi+1)}{2(\xi-1)}\right]$$

$$+\frac{\pi^2}{90(8\pi M)^4}\left[\frac{\xi+1}{\xi-1}\right]^2\left[1-\left[\frac{2}{\xi+1}\right]^6\left[4-\frac{6}{\xi+1}\right]^2\right]\Bigg\}\,. \tag{2.14}$$

Each of the n sums is now individually finite, and we may take the indicated limit.

The final term in (2.14) contains the finite contributions from Christensen's subtraction terms, as well as contributions from products of bivectors of parallel transport with inverse powers of ϵ. We will return to discuss this term after examining the remainder of Eq. (2.14).

When the limit is taken we find

$$\langle T_\theta{}^\theta\rangle_{\text{ren}}=\frac{\pi^2}{90(8\pi M)^4}\left\{\left[\frac{\xi+1}{\xi-1}\right]^2\left[1-\left[\frac{2}{\xi+1}\right]^6\left[4-\frac{6}{\xi+1}\right]^2\right]+1920\Delta_\theta{}^\theta\right\}, \tag{2.15}$$

where

$$\Delta_\theta{}^\theta=-S_1-S_2+2S_3-(\xi-1)S_4-2S_5\;. \tag{2.16}$$

The forms of the S_i are given in Table I.

To show the convergence of the n sums which appear in the S_i we may follow the procedure of Ref. 9. A short examination reveals

$$S_1=\frac{\xi+1}{16(\xi-1)}\sum_{n=1}^{\infty}n^2[\mathcal{T}_n(\xi)+\mathcal{J}_n(\xi)]\;, \tag{2.17}$$

$$S_4=\frac{1}{(\xi+1)^2}\frac{d}{d\xi}\sum_{n=1}^{\infty}[\mathcal{T}_n(\xi)+\mathcal{J}_n(\xi)]\;, \tag{2.18}$$

$$S_5=\frac{1}{4(\xi+1)^2}\sum_{n=1}^{\infty}[\mathcal{T}_n(\xi)+\mathcal{J}_n(\xi)]\;, \tag{2.19}$$

with

$$\mathcal{T}_n(\xi)=\int_{-1/2}^{\infty}dl\,(2l+1)\left[\frac{1}{n}p_l^n(\xi)q_l^n(\xi)-W_l^{(1)n}(\xi)-W_l^{(2)n}(\xi)-W_l^{(3)n}(\xi)\right] \tag{2.20}$$

and

$$\mathcal{J}_n(\xi)=4\mathcal{P}\int_0^{\infty}\frac{d\lambda\lambda}{e^{2\pi\lambda}+1}\left[\frac{1}{n}p^n_{-1/2+i\lambda}(\xi)q^n_{-1/2+i\lambda}(\xi)-W^{(1)n}_{-1/2}(\xi)-W^{(2)n}_{-1/2}(\xi)+\frac{\lambda^2}{2}\frac{\partial^2}{\partial\lambda^2}W^{(1)n}_{-1/2}(\xi)\right]. \tag{2.21}$$

The $W_l^{(1)n}(\xi)$ are given in Table II. $\mathcal{T}_n(\xi)$ and $\mathcal{J}_n(\xi)$ were shown in Ref. 9 to be $O(n^{-5})$. Only S_2 and S_3 require further study. We will show that the summands of S_2 and S_3 are both $O(n^{-3})$ for large n.

TABLE I. A table of the sums that appear in $\Delta_\mu{}^\nu$.

$$S_1=\frac{1}{16}\left[\frac{\xi+1}{\xi-1}\right]\sum_{n=1}^{\infty}n^2\left[\sum_{l=0}^{\infty}\left[\frac{2l+1}{n}p_l^n(\xi)q_l^n(\xi)-\frac{2}{(\xi^2-1)^{1/2}}\right]+\frac{n}{2}\left[\frac{\xi+1}{\xi-1}\right]\right],$$

$$S_2=\left[\frac{\xi-1}{\xi+1}\right]\sum_{n=1}^{\infty}\left[\sum_{l=0}^{\infty}\left[\frac{2l+1}{n}\frac{dp_l^n(\xi)}{d\xi}\frac{dq_l^n(\xi)}{d\xi}+\frac{2l(l+1)}{(\xi^2-1)^{3/2}}+\frac{n^2(\xi+1)^4}{16(\xi^2-1)^{5/2}}-\frac{3}{4(\xi^2-1)^{5/2}}\right]-\frac{n^3(\xi+1)^3}{96(\xi-1)^3}+\frac{n}{3(\xi-1)^3}\right],$$

$$S_3=\frac{1}{(\xi+1)^2}\sum_{n=1}^{\infty}\left[\sum_{l=0}^{\infty}\left[\frac{2l+1}{n}(l+\tfrac{1}{2})^2p_l^n(\xi)q_l^n(\xi)-\frac{2(l+\frac{1}{2})^2}{(\xi^2-1)^{1/2}}+\frac{n^2(1+\xi)^4}{16(\xi^2-1)^{3/2}}-\frac{1}{4(\xi^2-1)^{3/2}}\right]\right.$$
$$\left.-\frac{n^3(\xi+1)^4}{48(\xi-1)^2}+\frac{n}{24(\xi-1)^2}(3\xi^2-8\xi+13)\right],$$

$$S_4=\frac{1}{(\xi+1)^2}\frac{\partial}{\partial\xi}\sum_{n=1}^{\infty}\left[\sum_{l=0}^{\infty}\left[\frac{2l+1}{n}p_l^n(\xi)q_l^n(\xi)-\frac{2}{(\xi^2-1)^{1/2}}\right]+\frac{n(\xi+1)}{2(\xi-1)}\right],$$

$$S_5=\frac{1}{4(\xi+1)^2}\sum_{n=1}^{\infty}\left[\sum_{l=0}^{\infty}\left[\frac{2l+1}{n}p_l^n(\xi)q_l^n(\xi)-\frac{2}{(\xi^2-1)^{1/2}}\right]+\frac{n}{2}\left[\frac{\xi+1}{\xi-1}\right]\right].$$

TABLE II. The WKB approximants.

$$\chi^2=(l+\tfrac{1}{2})^2(\xi^2-1)+\frac{n^2}{16}(1+\xi)^4,\quad \chi_0=\frac{n}{4}(1+\xi)^2\,,$$

$$W^{(1)}=\frac{1}{\chi}\,,$$

$$W^{(2)}=\frac{1}{8\chi^3}-\frac{\chi_0^2}{4\chi^5}(2\xi^2-6\xi+7)+\frac{5\chi_0^4}{8\chi^7}(\xi-2)^2\,,$$

$$W^{(3)}=\frac{16\xi^2+11}{128\chi^5}-\frac{\chi_0^2}{32\chi^7}(16\xi^4-60\xi^3+88\xi^2-70\xi+171)+\frac{7\chi_0^4}{64\chi^9}(56\xi^4-320\xi^3+773\xi^2-1020\xi+666)$$
$$-\frac{231\chi_0^6}{32\chi^{11}}(2\xi^2-6\xi+7)(\xi-2)^2+\frac{1155\chi_0^8}{128\chi^{13}}(\xi-2)^4\,,$$

$$W^{(4)}=\frac{1}{1024\chi^7}(128\xi^4+824\xi^2+173)-\frac{\chi_0^2}{512\chi^9}(256\xi^6-1008\xi^5+1344\xi^4-776\xi^3+2058\xi^2-12\,510\xi+11\,901)$$
$$+\frac{3\chi_0^4}{1024\chi^{11}}(10\,112\xi^6-66\,976\xi^5+195\,648\xi^4-337\,216\xi^3+392\,289\xi^2-382\,324\xi+333\,472)$$
$$-\frac{11\chi_0^6}{256\chi^{13}}(6304\xi^6-53\,172\xi^5+198\,348\xi^4-430\,770\xi^3+599\,925\xi^2-536\,904\xi+248\,042)\,,$$
$$+\frac{429\chi_0^8}{1024\chi^{15}}(1968\xi^4-11\,488\xi^3+28\,963\xi^2-37\,948\xi+23\,448)(\xi-2)^2$$
$$-\frac{255\,255\chi_0^{10}}{512\chi^{17}}(\xi-2)^4(2\xi^2-6\xi+7)+\frac{425\,425\chi_0^{12}}{1024\chi^{19}}(\xi-2)^6\,.$$

We will begin by examining the l sum in S_3. As was shown in Ref. 9, it may be replaced by a contour integral

$$\sum_{l=0}^{\infty}\left[\frac{2l+1}{n}(l+\tfrac{1}{2})^2p_l^n(\xi)q_l^n(\xi)-\frac{2(l+\frac{1}{2})^2}{(\xi^2-1)^{1/2}}+\frac{n^2(\xi+1)^4}{16(\xi^2-1)^{3/2}}-\frac{1}{4(\xi^2-1)^{3/2}}\right]$$
$$=-\mathrm{Re}\left[\frac{1}{\pi i}\int_\gamma dl\,\pi\cot\pi l\left[\frac{2l+1}{n}(l+\tfrac{1}{2})^2p_l^n(\xi)q_l^n(\xi)-\frac{2(l+\frac{1}{2})^2}{(\xi^2-1)^{1/2}}+\frac{n^2(\xi+1)^4}{16(\xi^2-1)^{3/2}}-\frac{1}{4(\xi^2-1)^{3/2}}\right]\right] \tag{2.22}$$

with γ the contour in Fig. 1. Using

$$\cot\pi l=-i+\frac{2i}{1-e^{-2\pi il}} \tag{2.23}$$

and rotating the contour of the integral of the second term to γ' depicted in Fig. 2 [the rotation of the contour is allowable in view of the analytic properties of $p_l^n(\xi)$ and $q_l^n(\xi)$ (Ref. 10)], we have

$$\sum_{l=0}^{\infty}\left[\frac{2l+1}{n}(l+\tfrac{1}{2})^2p_l^n(\xi)q_l^n(\xi)-\frac{2(l+\frac{1}{2})^2}{(\xi^2-1)^{1/2}}+\frac{n^2(\xi+1)^4}{16(\xi^2-1)^{3/2}}-\frac{1}{4(\xi^2-1)^{3/2}}\right]={}_3\mathscr{I}_n(\xi)+{}_3\mathscr{J}_n(\xi)\,, \tag{2.24}$$

where

$${}_3\mathscr{I}_n(\xi)=\int_{-1/2}^{\infty}dl\left[\frac{2l+1}{n}(l+\tfrac{1}{2})^2p_l^n(\xi)q_l^n(\xi)-\frac{2(l+\frac{1}{2})^2}{(\xi^2-1)^{1/2}}+\frac{n^2(\xi+1)^4}{16(\xi^2-1)^{3/2}}-\frac{1}{4(\xi^2-1)^{3/2}}\right] \tag{2.25}$$

and

$${}_3\mathscr{J}_n(\xi)=-4\mathscr{P}\int_0^{\infty}\frac{d\lambda\,\lambda^3}{(e^{2\pi\lambda}+1)}\frac{1}{n}p^n_{-1/2+i\lambda}(\xi)q^n_{-1/2+i\lambda}(\xi)\,. \tag{2.26}$$

To ${}_3\mathscr{I}_n(\xi)$ we may subtract and add again the third-order WKB approximant to $p_l^n(\xi)q_l^n(\xi)/n$:

$$ {}_3\mathscr{F}_n(\xi)=\int_{-1/2}^{\infty}dl\,(2l+1)(l+\tfrac{1}{2})^2\left[\frac{1}{n}p_l^n(\xi)q_l^n(\xi)-W_l^{(1)n}(\xi)-W_l^{(2)n}(\xi)-W_l^{(3)n}(\xi)\right] $$
$$ +\int_{-1/2}^{\infty}dl\left[(2l+1)(l+\tfrac{1}{2})^2[W_l^{(1)n}(\xi)+W_l^{(2)n}(\xi)+W_l^{(3)n}(\xi)]\right. $$
$$ \left.-\frac{2(l+\frac{1}{2})^2}{(\xi^2-1)^{1/2}}+\frac{n^2(\xi+1)^4}{16(\xi^2-1)^{3/2}}-\frac{1}{4(\xi^2-1)^{3/2}}\right]. \tag{2.27} $$

The second integral in (2.27) may be evaluated explicitly. We obtain

$$ {}_3\mathscr{F}_n(\xi)=\int_{-1/2}^{\infty}dl\,(2l+1)(l+\tfrac{1}{2})^2\left[\frac{1}{n}p_l^n(\xi)q_l^n(\xi)-W_l^{(1)n}(\xi)-W_l^{(2)n}(\xi)-W_l^{(3)n}(\xi)\right] $$
$$ +\frac{n^3(\xi+1)^4}{48(\xi-1)^2}-\frac{n}{24(\xi-1)^2}(3\xi^2-8\xi+13)+\frac{7}{120n(\xi+1)^2}. \tag{2.28} $$

We observe that the terms cubic and linear in n exactly cancel the last two terms in large square brackets in S_3.

We may isolate the $O(n^{-1})$ contribution to ${}_3\mathscr{F}_n(\xi)$ by subtracting and adding again the first WKB approximant evaluated at $\lambda=0$:

$$ {}_3\mathscr{F}_n(\xi)=-4\mathscr{P}\int_0^{\infty}\frac{d\lambda\lambda^3}{e^{2\pi\lambda}+1}\left[\frac{1}{n}p^n_{-1/2+i\lambda}(\xi)q^n_{-1/2+i\lambda}(\xi)-W^{(1)n}_{-1/2}(\xi)\right]-4\int_0^{\infty}\frac{d\lambda\lambda^3}{e^{2\pi\lambda}+1}W^{(1)n}_{-1/2}(\xi). \tag{2.29} $$

Evaluating the second integral, we find

$$ {}_3\mathscr{F}_n(\xi)=-4\mathscr{P}\int_0^{\infty}\frac{d\lambda\lambda^3}{e^{2\pi\lambda}+1}\left[\frac{1}{n}p^n_{-1/2+i\lambda}(\xi)q^n_{-1/2+i\lambda}(\xi)-W^{(1)n}_{-1/2}(\xi)\right]-\frac{7}{120n\,(\xi+1)^2}. \tag{2.30} $$

Note that the final term in (2.30) exactly cancels the $O(n^{-1})$ behavior of ${}_3\mathscr{F}_n(\xi)$. Assembling these results we have

$$ S_3=\frac{1}{(\xi+1)^2}\sum_{n=1}^{\infty}[{}_3\bar{\mathscr{F}}_n(\xi)+{}_3\bar{\bar{\mathscr{F}}}_n(\xi)], \tag{2.31} $$

where

$$ {}_3\bar{\mathscr{F}}_n(\xi)=\int_{-1/2}^{\infty}dl\,(2l+1)l(l+\tfrac{1}{2})^2\left[\frac{1}{n}p_l^n(\xi)q_l^n(\xi)-W_l^{(1)n}(\xi)-W_l^{(2)n}(\xi)-W_l^{(3)n}(\xi)\right] \tag{2.32} $$

and

$$ {}_3\bar{\bar{\mathscr{F}}}_n(\xi)=-4\mathscr{P}\int_0^{\infty}\frac{d\lambda\lambda^3}{e^{2\pi\lambda}+1}\left[\frac{1}{n}p^n_{-1/2+i\lambda}(\xi)q^n_{-1/2+i\lambda}(\xi)-W^{(1)n}_{-1/2}(\xi)\right]. \tag{2.33} $$

We have shown that ${}_3\bar{\mathscr{F}}_n(\xi)$ and ${}_3\bar{\bar{\mathscr{F}}}_n(\xi)$ are $O(n^{-3})$ for large n.

A similar procedure shows that the n summand of S_2 is $O(n^{-3})$ for large n. We will not show the steps involved, only note the necessary intermediate results for the interested reader. We have

$$ \int_{-1/2}^{\infty}dl\,(2l+1)\{-(\xi^2-1)^{-2}[WS_l^{(1)n}(\xi)+WS_l^{(2)n}(\xi)+WS_l^{(3)n}(\xi)]+WP_l^{(1)n}(\xi)+WP_l^{(2)n}(\xi)\} $$
$$ =\frac{n^3(\xi+1)^3}{96(\xi-1)^3}-\frac{n(\xi+1)^2}{48(\xi^2-1)^3}(\xi^2-16\xi+17)-\frac{97\xi-143}{n\,240(\xi+1)^4(\xi-1)} \tag{2.34} $$

and

$$ 4\int_0^{\infty}\frac{d\lambda\lambda}{e^{2\pi\lambda}+1}\left[(\xi^2-1)^{-2}\left[WS^{(1)n}_{-1/2}(\xi)+WS^{(2)n}_{-1/2}(\xi)-\frac{\lambda^2}{2}\frac{d^2}{d\lambda^2}WS^{(1)n}_{-1/2}(\xi)\right]-WP^{(1)n}_{-1/2}(\xi)\right] $$
$$ =\frac{n}{48(\xi-1)^2}+\frac{97\xi-143}{n\,240(\xi+1)^4(\xi-1)}, \tag{2.35} $$

where the $WS_l^n(\xi)$ and $WP_l^n(\xi)$ are defined in Appendix C.

We may write

$$S_2=\frac{\xi+1}{\xi-1}\sum_{n=1}^{\infty}[{}_2\mathscr{T}_n(\xi)+{}_2\mathscr{J}_n(\xi)]\;, \tag{2.36}$$

where

$$\begin{aligned}{}_2\mathscr{T}_n(\xi)= \int_{-1/2}^{\infty}dl\,(2l+1)\Bigg[&\frac{1}{n}\frac{dp_l^n(\xi)}{d\xi}\frac{dq_l^n(\xi)}{d\xi}-WP_l^{(1)n}(\xi)-WP_l^{(2)n}(\xi)\\&+(\xi^2-1)^{-2}[WS_l^{(1)n}(\xi)+WS_l^{(2)n}(\xi)+WS_l^{(3)n}(\xi)]\Bigg]\end{aligned} \tag{2.37}$$

and

$$\begin{aligned}{}_2\mathscr{J}_n(\xi)=4\mathscr{P}\int_0^\infty\frac{d\lambda\lambda}{e^{2\pi\lambda}+1}\Bigg[&\frac{1}{n}\frac{dp^n_{-1/2+i\lambda}(\xi)}{d\xi}\frac{dq^n_{-1/2+i\lambda}(\xi)}{d\xi}\\&+(\xi^2-1)^{-2}\left[WS^{(1)n}_{-1/2}(\xi)+WS^{(2)n}_{-1/2}(\xi)-\frac{\lambda^2}{2}\frac{d^2}{d\lambda^2}WS^{(1)n}_{-1/2}(\xi)\right]-WP^{(1)n}_{-1/2}(\xi)\Bigg]\;.\end{aligned} \tag{2.38}$$

${}_2\mathscr{T}_n(\xi)$ and ${}_2\mathscr{J}_n(\xi)$ are each $O(n^{-3})$ for large n. These results are exact.

The other components of the stress-energy tensor are also amenable to this process. We find

$$\langle T_\mu{}^\nu\rangle_{\text{ren}}=\frac{\pi^2}{90(8\pi M)^4}\left\{\left[\left(\frac{\xi+1}{\xi-1}\right)^2\left[1-\left(\frac{2}{\xi+1}\right)^6\left(4-\frac{6}{\xi+1}\right)^2\right]\right]\begin{bmatrix}-3&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix}+24\left(\frac{2}{\xi+1}\right)^6\begin{bmatrix}3&0&0&0\\0&1&0&0\\0&0&0&0\\0&0&0&0\end{bmatrix}+1920\Delta_\mu{}^\nu\right\} \tag{2.39}$$

with

$$\Delta_t{}^t=5S_1-S_2-S_3-S_4+S_5\;, \tag{2.40}$$

$$\Delta_r{}^r=-3S_1+3S_2-3S_3+(2\xi-1)S_4+3S_5\;. \tag{2.41}$$

The first two terms in (2.39) are exactly the expression that Page obtained. We will show presently that the contribution of $\Delta_\mu{}^\nu$ does not significantly alter the character of $\langle T_\mu{}^\nu\rangle_{\text{ren}}$ from that expected on the basis of Page's approximation.

At this point we may approximate the S_i's by replacing the integrands of the $\mathscr{T}_n(\xi)$'s and $\mathscr{J}_n(\xi)$'s by WKB terms of the next higher order. For ${}_2\mathscr{T}_n(\xi)$, ${}_3\mathscr{T}_n(\xi)$, ${}_2\mathscr{J}_n(\xi)$, and ${}_3\mathscr{J}_n(\xi)$ this amounts to using the $O(n^{-3})$ terms. This procedure yields the following asymptotic forms, valid for large r:

$$S_1\sim{}_aS_1=\tfrac{9}{448}(\xi-1)^2\left[\frac{2}{\xi+1}\right]^{10}\zeta(3)\;, \tag{2.42}$$

$$S_2\sim{}_aS_2=\tfrac{-1}{430080}\left[\frac{2}{\xi+1}\right]^{10}\times\zeta(3)(931\xi^4-2058\xi^3-5256\xi^2+6298\xi+1653)\;, \tag{2.43}$$

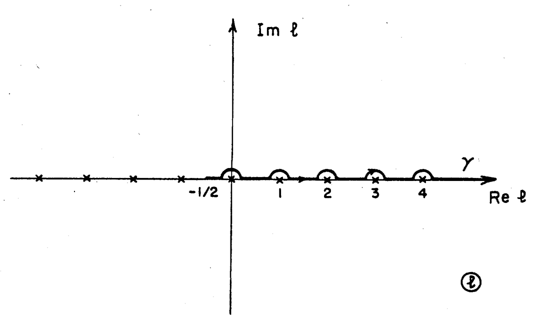

FIG. 1. The curve γ used in converting the l sum to a contour integral.

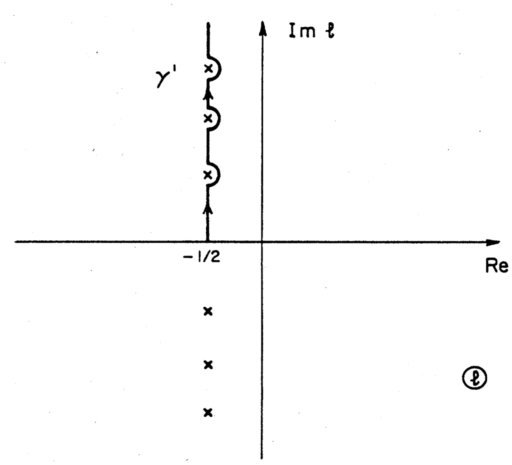

FIG. 2. The curve γ' used to evaluate ${}_3\mathscr{J}_n(\xi)$. The crosses correspond to the poles of the function $q_l^n(\xi)$.

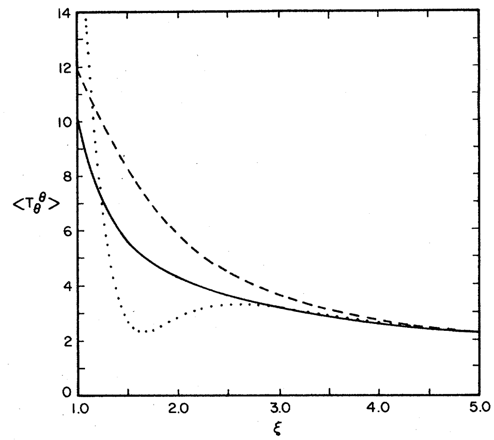

FIG. 3. $[90(8\pi M)^4/\pi^2]\langle T_\theta{}^\theta\rangle$ as a function of ξ. The dashed line represents Page's approximation.

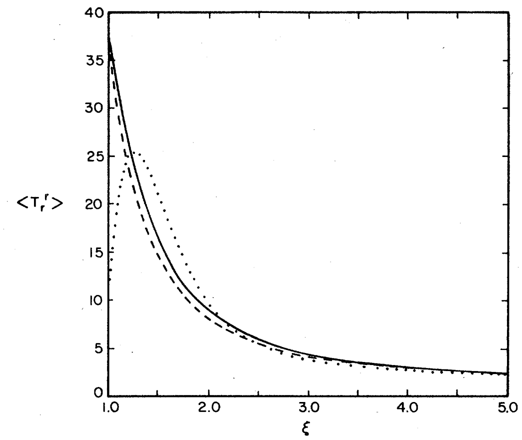

FIG. 5. $[90(8\pi M)^4/\pi^2]\langle T_r{}^r\rangle$ as a function of ξ. The dashed line represents Page's approximation.

$$S_3 \sim {}_aS_3 = \frac{-(\xi-1)}{322\,560}\left[\frac{2}{\xi+1}\right]^{10} \times \zeta(3)(155\xi^3-465\xi^2+7311\xi-7931)\,, \quad (2.44)$$

$$S_4 \sim {}_aS_4 = \tfrac{-9}{224}(\xi-1)^2\left[\frac{2}{\xi+1}\right]^{14}\zeta(5)(4\xi-7)\,, \quad (2.45)$$

$$S_5 \sim {}_aS_5 = \tfrac{9}{448}(\xi-1)^3\left[\frac{2}{\xi+1}\right]^{13}\zeta(5)\,. \quad (2.46)$$

The approximate values for $\langle T_\mu{}^\nu\rangle_{\rm ren}$ obtained by the use of the ${}_aS_i$ are presented in Figs. 3–5 as dotted curves.

One might expect these asymptotic forms to also apply as $r\to 2M$ because the criterion for the validity of the WKB approximation holds both as $r\to\infty$ and as $r\to 2M$. This cannot be the case as evidenced by the fact that the ${}_aS_i$ predict $\langle T_t{}^t\rangle_{\rm ren}\neq\langle T_r{}^r\rangle_{\rm ren}$ at $r=2M$, whereas these two components must be equal in order for $\langle T_\mu{}^\nu\rangle_{\rm ren}$ to be regular in a freely falling frame on the future horizon. As discussed in Ref. 9, this breakdown can be traced to the fact that the radial function $q_l^n(\xi)$ contains a logarithm which is not present in its WKB approximation.

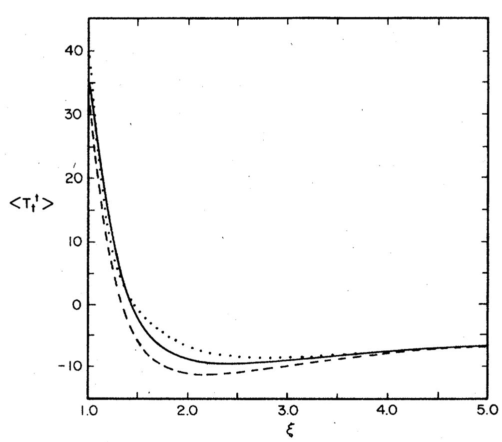

FIG. 4. $[90(8\pi M)^4/\pi^2]\langle T_t{}^t\rangle$ as a function of ξ. The dashed line represents Page's approximation.

III. NUMERICAL EVALUATION OF $\langle T_\mu{}^\nu\rangle$

The expressions for the S_i in terms of integrals are not amenable to numerical evaluation. We will begin with the forms of the S_i given in Table I and obtain more suitable expressions.

First we must consider the evaluation of the l sums which appear in the S_i. Each l sum is of the form

$$\sum_{l=0}^{\infty}\left[\frac{2l+1}{n}F(l,p_l^n(\xi),q_l^n(\xi))-G(l,\xi)\right],$$

where F is a functional which may contain derivatives. For each of these sums, we will subtract and then add again the WKB approximant to $F(l,p_l^n(\xi),q_l^n(\xi))$ through terms of order l^{-5}. Symbolically, this procedure yields

$$\sum_{l=0}^{\infty}\frac{2l+1}{n}[F(l,p_l^n(\xi),q_l^n(\xi))-F_{\rm WKB}(l,n,\xi)] + \sum_{l=0}^{\infty}\left[\frac{2l+1}{n}F_{\rm WKB}(l,n,\xi)-G(l,\xi)\right].$$

All of the sums of the first type will converge rapidly. An explicit expression exists for the $F_{\rm WKB}(l,n,\xi)$, so sums of the second type are easily evaluated. For each n and ξ, the first 28 $F_{\rm WKB}(l,n,\xi)$ are obtained as double precision numbers. A 15-term Richardson extrapolation (see Appendix D) then returns a value for the sum which is accurate to ten significant figures.

Values for sums of the first type must be obtained in a more tedious manner. In general, these sums of the difference between a functional of $p_l^n(\xi)$ and $q_l^n(\xi)$ and a WKB approximant of it, will converge more slowly than the similar sum which appeared in $\Delta(r)$, so higher l terms will provide a larger contribution. To achieve dependable values, the functions $p_l^n(\xi)$ and $q_l^n(\xi)$ and their derivatives must be evaluated very accurately to prevent the greater cancellation, which occurs for large l, from contributing significant errors.

The function $q_l^n(\xi)$ is defined in terms of $p_l^n(\xi)$ by

$$q_l^n(\xi)=2np_l^n(\xi)\int_\xi^\infty \frac{dx}{(x^2-1)[p_l^n(x)]^2} \,. \tag{3.1}$$

Its derivative gives

$$\frac{dq_l^n(\xi)}{d\xi}=2n\frac{dp_l^n(\xi)}{d\xi}\int_\xi^\infty \frac{dx}{(x^2-1)[p_l^n(x)]^2} - \frac{2n}{(\xi^2-1)p_l^n(\xi)} \,. \tag{3.2}$$

For values of their arguments less than 3.0, $p_l^n(\xi)$ and its derivative may be evaluated as double precision sums and for values of their arguments greater than 3.0 a sixth-order predictor-corrector method applies. Transformation of the integral to a finite range and an extended ten-point integration routine achieves an accuracy of 12 significant figures in $q_l^n(\xi)$.

The use of double precision in the evaluation of the l summands allows the use of an iterated Shanks transformation scheme to be used to evaluate the l sums. The desired accuracy is reached with the first 30 terms.

The evaluation of the integral which appears in (3.1) and (3.2) is by far the most time-consuming portion of the numerical program. Each of the S_i requires this integral, so the most efficient use of computer time is achieved if all five S_i are evaluated simultaneously. Given the S_i, the form (2.39) allows all the components of $\langle T_\mu{}^\nu \rangle_{\text{ren}}$ to be evaluated. No further numerical work is necessary, as would be were we extracting $\langle T_t{}^t \rangle_{\text{ren}}$ and $\langle T_r{}^r \rangle_{\text{ren}}$ from $\langle T_\theta{}^\theta \rangle_{\text{ren}}$.

The rate of convergence of the n sums in the S_i, established in the previous section, allows us to obtain acceptable accuracy in the S_i by evaluating only the $n=1$ to $n=4$ contributions and approximating the remaining terms by their contribution to ${}_aS_i$.

The above method allows $\langle T_\mu{}^\nu \rangle$ to be calculated to three-figure accuracy in about 12 minutes of C.P.U. time. Our results are presented in Table III and depicted in Figs. 3—5.

TABLE III. Values of $\langle T_\mu{}^\nu \rangle_{\text{ren}}$.

ξ	$\frac{90(8\pi M)^4}{\pi^2}\langle T_\theta{}^\theta \rangle_{\text{ren}}$	$\frac{90(8\pi M)^4}{\pi^2}\langle T_r{}^r \rangle_{\text{ren}}$	$\frac{90(8\pi M)^4}{\pi^2}\langle T_t{}^t \rangle_{\text{ren}}$
1.0	10.29	37.728	37.728
1.1	8.793	31.490	24.827
1.2	7.585	26.233	14.472
1.3	6.624	21.892	6.381
1.4	6.034	18.709	1.402
1.5	5.565	16.132	−2.096
1.6	5.205	14.059	−4.579
1.7	4.922	12.344	−6.334
1.8	4.683	10.942	−7.560
1.9	4.473	9.771	−8.400
2.0	4.296	8.794	−8.963
2.1	4.136	7.970	−9.321
2.2	3.991	7.269	−9.530
2.3	3.859	6.672	−9.631
2.4	3.735	6.158	−9.651
2.5	3.621	5.714	−9.615
2.6	3.514	5.328	−9.536
2.7	3.414	4.993	−9.428
2.8	3.321	4.697	−9.299
2.9	3.233	4.435	−9.156
3.0	3.151	4.203	−9.005
3.2	2.999	3.810	−8.693
3.4	2.865	3.498	−8.381
3.6	2.743	3.243	−8.078
3.8	2.634	3.029	−7.795
4.0	2.535	2.852	−7.529
4.2	2.446	2.701	−7.283
4.4	2.366	2.572	−7.055
4.6	2.292	2.460	−6.845
4.8	2.226	2.363	−6.652
5.0	2.165	2.278	−6.475
5.2	2.113	2.202	−6.310
5.4	2.057	2.134	−6.159
5.6	2.009	2.074	−6.019
5.8	1.966	2.020	−5.889

Our results are seen to alter slightly the values of $\langle T_\mu{}^\nu\rangle_{\rm ren}$ from those predicted by Page's approximation, without changing the overall character of the curves. This is in marked disagreement with the previous numerical work of Fawcett.[6]

We feel that Fawcett's work is in error for the following reason. Prior to his numerical analysis he effectively writes[11]

$$\langle T_{\mu\nu}\rangle_{\rm ren}=\langle(\phi_{;\mu}\phi_{;\nu}+\tfrac{1}{4}g_{\mu\nu}\phi\Box\phi)\rangle+\frac{1}{6\pi^2}g_{\mu\nu}a_2 -\tfrac{1}{12}g_{\mu\nu}\Box\langle\phi^2\rangle-\tfrac{1}{6}\langle\phi^2\rangle_{;\mu\nu}\,. \tag{3.3}$$

He employs (3.3) to evaluate $\langle T_{\theta\theta}\rangle_{\rm ren}$ from which he obtains the other components of $\langle T_{\mu\nu}\rangle_{\rm ren}$. In his calculation, he assumes that the last term

$$-\tfrac{1}{6}\langle\phi^2\rangle_{;\theta\theta}$$

vanishes on the grounds that $\langle\phi^2\rangle$ is a function of r only. However,

$$\langle\phi^2\rangle_{;\mu\nu}=\langle\phi^2\rangle_{,\mu\nu}-\Gamma^\lambda_{\mu\nu}\langle\phi^2\rangle_{,\lambda} \tag{3.4}$$

and therefore

$$\langle\phi^2\rangle_{;\theta\theta}=-\Gamma^r_{\theta\theta}\langle\phi^2\rangle_{,r}\ , \tag{3.5}$$

which is nonzero. In Fig. 6 we show our results, Fawcett's published results, and his results plus the term (3.5). It is apparent that this term brings his results into agreement with ours.

In view of the success of Page's expression for $\langle T_\mu{}^\nu\rangle_{\rm ren}$, work is currently in progress to solve the back-reaction problem for a static (nonevaporating) black hole. Using his expression on the right-hand side of the semi-classical Einstein equation

$$G_\mu{}^\nu=8\pi\langle T_\mu{}^\nu\rangle_{\rm ren}\ , \tag{3.6}$$

an approximation for the one-loop quantum correction to the metric may be obtained. The expression thereby obtained, while not exact, should contain the important characteristics of the true one-loop solution.

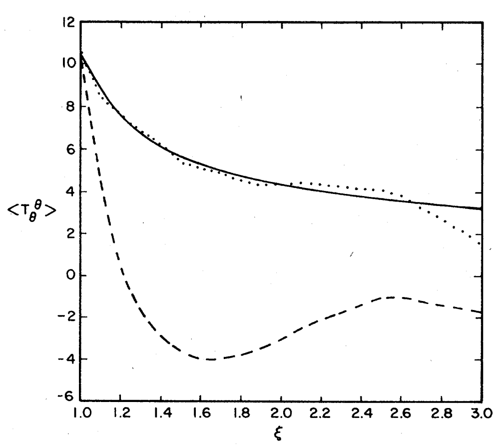

FIG. 6. $[90(8\pi M)^4/\pi^2]\langle T_\theta{}^\theta\rangle$ as a function of ξ. The solid line is our result, the dashed line represents the published values of Fawcett, and the dotted line is obtained by adding the term (3.5) to his results.

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APPENDIX A: IDENTITIES INVOLVING SUMS OF LEGENDRE FUNCTIONS

In this appendix we will prove a series of identities of the form

$$\sum_{l=0}^{\infty}\left[(2+1)F\left[P_l(x),Q_l(x),\frac{\partial P_l(x)}{\partial x},\frac{\partial P_l(x)}{\partial x}\right]+G(l,x)\right], \tag{A1}$$

where $P_l(x)$ and $Q_l(x)$ are Legendre functions satisfying

$$\left[\frac{\partial}{\partial x}(x^2-1)\frac{\partial}{\partial x}-l(l+1)\right]P_l(x)=0\ . \tag{A2}$$

We will use the identities

$$\sum_{l=0}^{\infty}P_l(\cos\gamma)=\tfrac{1}{2}(1-\cos\gamma)^{-1/2} \underset{\gamma\to0}{\sim}1/\gamma+O(\gamma)\ , \tag{A3}$$

$$\sum_{l=0}^{\infty}l(l+1)P_l(\cos\gamma)=\tfrac{1}{2}(\cos\gamma-3)[2(1-\cos\gamma)]^{-3/2} \underset{\gamma\to0}{\sim}-\gamma^{-3}-\tfrac{3}{8}\gamma+O(\gamma)\ , \tag{A4}$$

where (A4) follows from (A3) by the use of (A2). We also use the identity

$$\sum_{l=0}^{\infty}(2l+1)P_l(x)Q_l(x')P_l(\cos\gamma) =(x^2+x'^2-2xx'\cos\gamma-\sin^2\gamma)^{-1/2}\ , \tag{A5}$$

which is proved in the following way.

We find the zero-frequency term of the Fourier series expansion for the standard identity (12) (assuming $x'>x$),

$$Q_l(xx'-(x^2-1)^{1/2}(x'^2-1)^{1/2}\cos\psi) =P_l(x)Q_l(x')+2\sum_{n=1}^{\infty}(-1)^n\cos(n\psi)P_l^{-n}(x)Q_l^n(x') \tag{A6}$$

to be

$$P_l(x)Q_l(x') = \frac{1}{2\pi}\int_0^{2\pi} d\psi\, Q_l(xx'-(x^2-1)^{1/2}(x'^2-1)^{1/2}\cos\psi) \;. \tag{A7}$$

This relation and Heine's formula,

$$\sum_{l=0}^{\infty}(2l+1)P_l(y)Q_l(z)=\frac{1}{z-y}\;, \tag{A8}$$

allow us to show that

$$\sum_{l=0}^{\infty}(2l+1)P_l(x)Q_l(x')P_l(\cos\gamma) = \frac{1}{2\pi}\int_0^{2\pi} d\psi[xx'-\cos\gamma - (x^2-1)^{1/2}(x'^2-1)^{1/2}\cos\psi]^{-1} = (x^2+x'^2-2xx'\cos\gamma-\sin^2\gamma)^{-1/2}\;, \tag{A9}$$

which proves (A5).

The first identity of the form (A1) is

$$\sum_{l=0}^{\infty}[(2l+1)P_l(x)Q_l(x)-2(x^2-1)^{1/2}]=0\;. \tag{A10}$$

Examining (A5) for $x=x'$ and $\gamma\sim 0$ we find

$$\sum_{l=0}^{\infty}(2l+1)P_l(x)Q_l(x)P_l(\cos\gamma)\sim\frac{1}{\gamma(x^2-1)^{1/2}}+O(\gamma)\;. \tag{A11}$$

Dividing (A3) by $(x^2-1)^{1/2}$, subtracting it from (A11), and letting $\gamma\to 0$, we see

$$\sum_{l=0}^{\infty}[(2l+1)P_l(x)Q_l(x)-(x^2-1)^{-1/2}]=0\;. \tag{A12}$$

A single derivative with respect to x on (A5) gives

$$\sum_{l=0}^{\infty}(2l+1)\frac{\partial P_l(x)}{\partial x}Q_l(x')P_l(\cos\gamma) = \frac{x'\cos\gamma-x}{(x^2=x'^2-2xx'\cos\gamma-\sin^2\gamma)^{3/2}}\;, \tag{A13}$$

which, for $x=x'$ and γ near 0 behaves as

$$\frac{-x}{2(x^2-1)^{3/2}\gamma}+O(\gamma)\;.$$

From this we may deduce that

$$\sum_{l=0}^{\infty}\left[(2l+1)\frac{\partial P_l(x)}{\partial x}Q_l(x)+\frac{x}{2(x^2-1)^{3/2}}\right]=0 \tag{A14}$$

and

$$\sum_{l=0}^{\infty}\left[(2l+1)P_l(x)\frac{\partial Q_l(x)}{\partial x}+\frac{x}{2(x^2-1)^{3/2}}\right]=0\;, \tag{A15}$$

where (A15) follows from the symmetry of the right-hand side of (A5) under $x\leftrightarrow x'$.

After taking the derivative of (A13) with respect to x' we find,

$$\sum_{l=0}^{\infty}(2l+1)\frac{\partial P_l(x)}{\partial x}\frac{\partial Q_l(x')}{\partial x'}P_l(\cos\gamma) = \frac{xx'(3+\cos^2\gamma)-2\cos\gamma(x^2+x'^2)-\cos\gamma\sin^2\gamma}{(x^2+x'^2-2xx'\cos\gamma-\sin^2\gamma)^{1/2}}\;. \tag{A16}$$

For $x=x'$ and γ approaching zero, the right-hand side becomes

$$\frac{1}{\gamma^3(x^2-1)^{3/2}}+\frac{3}{8\gamma(x^2-1)^{3/2}}+\frac{3}{8\gamma(x^2-1)^{5/2}}+O(\gamma)\;.$$

Using (A3) and (A4), we see that

$$\sum_{l=0}^{\infty}\left[(2l+1)\frac{\partial P_l(x)}{\partial x}\frac{\partial Q_l(x)}{\partial x}+\frac{l(l+1)}{(x^2-1)^{3/2}}-\frac{3}{8(x^2-1)^{5/2}}\right]=0\;. \tag{A17}$$

The relation

$$\sum_{l=0}^{\infty}\left[(2l+1)l(l+1)P(x)Q(x)-\frac{l(l+1)}{(x^2-1)^{1/2}}-\tfrac{1}{8}(x^2-1)^{-3/2}\right]=0 \tag{A18}$$

may easily be proved by considering the effect of

$$\frac{\partial}{\partial x}(x^2-1)\frac{\partial}{\partial x}$$

upon (A12). After using Legendre's equation, we see that

$$\sum_{l=0}^{\infty}\left[(2l+1)\left[l(l+1)P_l(x)Q_l(x)+\frac{\partial P_l(x)}{\partial x}\frac{\partial Q_l(x)}{\partial x}(x^2-1)\right]-\frac{1}{2(x^2-1)^{3/2}}\right]=0\;. \tag{A19}$$

Termwise subtracting (x^2-1) times (A17) from (A19) proves (A18).

APPENDIX B: FORMULAS RELEVANT TO TIMELIKE POINT SEPARATION

The quantity $\sigma(x,x')$ is defined as one-half the square of the geodetic distance from x to x'. For two points $x=(t,r,\theta,\phi)$ and $x'=(t+\epsilon,r,\theta,\phi)$ in Schwarzschild spacetime one finds[13]

$$\sigma^t=\sigma^{;t}=\epsilon+\frac{M^2\epsilon^3}{6r^4}-\frac{1}{120}\left[\frac{6M^3}{r^7}\left[1-\frac{2M}{r}\right]-\frac{M^4}{r^8}\right]\epsilon^5+\cdots\;, \tag{B1}$$

$$\sigma^r=\sigma^{;r}=-\frac{1}{2}\frac{M}{r^2}\left[1-\frac{2M}{r}\right]\epsilon^2+\frac{M^2}{12r^5}\left[1-\frac{2M}{r}\right]\left[1-\frac{5M}{r}\right]\epsilon^4+\cdots \; . \tag{B2}$$

A straightforward calculation reveals

$$\frac{1}{2\sigma(x,x')}=\frac{-1}{\epsilon^2\left[1-\frac{2M}{r}\right]}+\frac{M^2}{12r^4\left[1-\frac{2M}{r}\right]}+\cdots \; , \tag{B3}$$

$$\frac{1}{4\sigma^2(x,x')}=\frac{1}{\epsilon^4\left[1-\frac{2M}{r}\right]^2}-\frac{M^2}{6r^4\epsilon^2\left[1-\frac{2M}{r}\right]^2}+\left[1-\frac{2M}{r}\right]^{-2}\left[\frac{M^3}{30r^7}-\frac{37M^4}{720r^8}\right]+\cdots \; . \tag{B4}$$

The bivectors of parallel transport (7) may be found by exploiting the symmetries of the Schwarzschild metric. For two points whose separation does not involve the angular coordinates, the connecting geodesic lies entirely in the r,t plane. We may construct an orthonormal tetrad at the point x' such that its timelike leg is aligned with $\sigma^{;\mu'}$,

$$e_0^{\mu'}=\frac{\sigma^{\mu'}}{(-\sigma^{\mu}\sigma_{\mu})^{1/2}} \; . \tag{B5}$$

The angular legs may be chosen to lie along the Schwarzschild angular coordinate directions. The fourth leg of this right-handed tetrad is uniquely specified by requiring orthonormality with respect to the other three legs. A second orthonormal tetrad may be constructed similarly at the point x. The properties of geodesics and the spherical symmetry of the Schwarzschild metric ensure that this second tetrad is the same as the first tetrad, parallel transported to the point x. Any vector may be parallel transported from x' to x by finding its components with respect to the tetrad at x' and constructing the vector at x with the same components with respect to the tetrad at x. Symbolically,

$$g^{\mu}{}_{\nu'}=\eta^{ab}e_a{}^{\mu}e_{b\nu'} \; , \tag{B6}$$

where the tetrad indices a and b run from 0 to 3 and η^{ab} is the Minkowski metric.

For our case, this procedure yields

$$g^{\theta\theta'}=r^{-2} \; , \tag{B7}$$

$$g^{\phi\phi'}=r^{-2}\sin^{-2}\theta \; , \tag{B8}$$

$$g^{tt'}=-\left[1-\frac{2M}{r}\right]^{-1}\left[1+\frac{M^2\epsilon^2}{2r^4}-\epsilon^4\left[\frac{M^3}{6r^7}-\frac{3M^4}{8r^8}\right]+\cdots\right] , \tag{B9}$$

$$g^{rr'}=\left[1-\frac{2M}{r}\right]\left[1+\frac{M^2\epsilon^2}{2r^4}-\epsilon^4\left[\frac{M^3}{6r^7}-\frac{3M^4}{8r^8}\right]+\cdots\right] , \tag{B10}$$

$$g^{tr'}=-g^{rt'}=\frac{M\epsilon}{r^2}-\frac{M^2}{6r^5}\left[1-\frac{3M}{r}\right]\epsilon^3+\cdots \; . \tag{B11}$$

For small, nonzero ϵ we may express inverse powers of ϵ in the following way:

$$\epsilon^{-2}=-\kappa^2\sum_{n=1}^{\infty}n\cos(n\kappa\epsilon)+\frac{\kappa^2}{12}+O(\epsilon^2) \; , \tag{B12}$$

$$\epsilon^{-4}=\frac{\kappa^4}{6}\sum_{n=1}^{\infty}n^3\cos(n\kappa\epsilon)-\frac{\kappa^4}{720}+O(\epsilon^2) \; , \tag{B13}$$

$$\epsilon^{-1}=\kappa\sum_{n=1}^{\infty}\sin(n\kappa\epsilon)+O(\epsilon) \; , \tag{B14}$$

$$\epsilon^{-3}=-\frac{\kappa^3}{3}\sum_{n=1}^{\infty}n^2\sin(n\kappa\epsilon)+O(\epsilon) \; . \tag{B15}$$

The proof of (B14) is as follows:

$$\begin{aligned}\sum_{n=1}^{\infty}\sin(n\kappa\epsilon)&=\frac{1}{2i}\left[\sum_{n=1}^{\infty}e^{in\kappa\epsilon}-\sum_{n=1}^{\infty}e^{-in\kappa\epsilon}\right]\\&=\frac{1}{2i}\left[\frac{e^{i\kappa\epsilon}}{1-e^{i\kappa\epsilon}}-\frac{e^{i\kappa\epsilon}}{1-e^{-i\kappa\epsilon}}\right]\\&=\tfrac{1}{2}\cot\left[\frac{\kappa\epsilon}{2}\right]\underset{\epsilon\to 0}{\sim}\frac{1}{\kappa\epsilon}+O(\epsilon) \; .\end{aligned} \tag{B16}$$

The results (B12), (B13), and (B15) are easily obtained by differentiating (B14) with respect to $\kappa\epsilon$.

We also note that

$$\sum_{n=1}^{\infty}\cos(n\kappa\epsilon)=-\tfrac{1}{2} \tag{B17}$$

when $\epsilon\neq 0$. This is a restatement of the Fourier decomposition of a Dirac δ function of $\kappa\epsilon$.

We require products of bivectors of parallel transport and sums involving powers of n and $\sin(n\kappa\epsilon)$ or $\cos(n\kappa\epsilon)$. It is straightforward to express these sums as inverse powers of ϵ using Eqs. (B12)–(B17), complete the required multiplication, and reexpress the surviving inverse powers of ϵ as n sums. An example, useful in the renormalization of $\langle T_\theta{}^\theta\rangle$, is

$$g^{rr'} \sum_{n=1}^{\infty} n^2 \sin(n\kappa\epsilon) = \left[\kappa\epsilon \left[\frac{2}{\xi+1} \right]^2 - \frac{(\xi-2)32\kappa^3\epsilon^3}{3} \left[\frac{2}{\xi+1} \right]^6 \right] \left[\frac{-2}{(\kappa\epsilon)^3} + O(\epsilon) \right]$$

$$= \frac{-2}{(\kappa\epsilon)^2} \left[\frac{2}{\xi+1} \right]^2 + \frac{64(\xi-2)}{3} \left[\frac{2}{\xi+1} \right]^6 + O(\epsilon)$$

$$= -2 \left[\frac{2}{\xi+1} \right]^2 \sum_{n=1}^{\infty} n \cos(n\kappa\epsilon) - \frac{1}{6} \left[\frac{2}{\xi+1} \right]^2 + \frac{64(\xi-2)}{3} \left[\frac{2}{\xi+1} \right]^6 + O(\epsilon) \,. \qquad \text{(B18)}$$

TABLE IV. *WS* and *WP*.

$$\chi^2 = (l+\tfrac{1}{2})^2(\xi^2-1) + \frac{n^2}{16}(1+\xi)^4, \quad \chi_0 = \frac{n}{4}(1+\xi)^2 \,,$$

$$WS^{(1)} = \chi \,,$$

$$WS^{(2)} = -\frac{1}{8\chi} + \frac{(2\xi^2-6\xi+7)\chi_0{}^2}{4\chi^3} - \frac{5(\xi-2)^2\chi_0{}^2}{8\chi^5} \,,$$

$$WS^{(3)} = -\frac{1105(\xi-2)^4\chi_0{}^8}{128\chi^{11}} + \frac{221(\xi-2)^2(2\xi^2-6\xi+7)\chi_0{}^6}{32\chi^9} - \frac{(16\xi^2+9)}{128\chi^3}$$
$$- \frac{(376\xi^4-2144\xi^3+5145\xi^2-6764\xi+4426)\chi_0{}^4}{64\chi^7} + \frac{(16\xi^4-60\xi^3+84\xi^2-58\xi+157)\chi_0{}^2}{32\chi^5}$$

$$WS^{(4)} = \frac{414\,125(\xi-2)^6\chi_0{}^{12}}{1024\chi^{17}} - \frac{248\,475(\xi-2)^4(2\xi^2-6\xi+7)\chi_0{}^{10}}{512\chi^{15}}$$
$$+ \frac{(822\,128\xi^4-4\,797\,728\xi^3+12\,085\,883\xi^2-15\,827\,868\xi+9\,782\,088)(\xi-2)^2\chi_0{}^8}{1024\chi^{13}}$$
$$- \frac{\chi_0{}^6(67\,648\xi^6-570\,276\xi^5+2\,124\,232\xi^4-4\,603\,830\xi^3+6\,398\,415\xi^2-5\,721\,462\xi+2\,645\,630)}{256\chi^{11}}$$
$$+ \frac{(29\,824\xi^6-197\,472\xi^5+574\,944\xi^4-985\,216\xi^3+1\,133\,991\xi^2-1\,095\,852\xi+963\,472)\chi_0{}^4}{1024\chi^9}$$
$$- \frac{(256\xi^6-1008\xi^5+1216\xi^4-344\xi^3+1450\xi^2-12\,883\xi+11\,105)\chi_0{}^2}{512\chi^7} + \frac{(128\xi^4+792\xi^2+153)}{1024\chi^5} \,,$$

$$WP^{(1)} = \frac{\xi^2}{4(\xi^2-1)^2\chi} + \frac{\xi(\xi-2)\chi_0{}^2}{2(\xi^2-1)^2\chi^3} + \frac{(\xi-2)^2\chi_0{}^4}{4(\xi^2-1)^2\chi^5} \,,$$

$$WP^{(2)} = \frac{65(\xi-2)^4\chi_0{}^8}{32(\xi^2-1)^2\chi^{11}} - \frac{(\xi-2)^2(3\xi^2-44\xi+133)\chi_0{}^6}{16(\xi^2-1)^2\chi^9} - \frac{(59\xi^4-292\xi^3+543\xi^2-292\xi-196)\chi_0{}^4}{32(\xi^2-1)^2\chi^7}$$
$$+ \frac{\xi(3\xi^3-11\xi^2+19\xi-27)\chi_0{}^2}{8(\xi^2-1)^2\chi^5} + \frac{5\xi^2}{32(\xi^2-1)^2\chi^3} \,,$$

$$WP^{(3)} = \frac{30\,685(\xi-2)^6\chi_0{}^{12}}{512(\xi^2-1)^2\chi^{17}} - \frac{3(\xi-2)^4(5263\xi^2-20\,794\xi+35\,938)\chi_0{}^{10}}{256(\xi^2-1)^2\chi^{15}}$$
$$- \frac{(18\,925\xi^4-64\,708\xi^3-5282\xi^4+324\,120\xi-464\,194)(\xi-2)^2\chi_0{}^8}{512(\xi^2-1)^2\chi^{13}}$$
$$+ \frac{(6546\xi^6-50\,358\xi^5+164\,792\xi^4-291\,348\xi^3+268\,641\xi^2-53\,320\xi+81\,300)\chi_0{}^6}{128(\xi^2-1)^2\chi^{11}}$$
$$- \frac{(6000\xi^6-37\,696\xi^5+103\,362\xi^4-165\,320\xi^3+186\,713\xi^2-132\,228\xi+14\,828)\chi_0{}^4}{512(\xi^2-1)^2\chi^9}$$
$$+ \frac{(192\xi^6-752\xi^5+1344\xi^4-1208\xi^3+1990\xi^2-5820)\chi_0{}^2}{512(\xi^2-1)^2\chi^7} + \frac{(80\xi^4+171\xi^2)}{512(\xi^2-1)^2\chi^5} \,.$$

APPENDIX C: DERIVATION OF THE WKB APPROXIMATION TO $(1/n)p_l^n(\xi)q_l^n(\xi)$

We set

$$\alpha^2(\xi)=\frac{1}{n}p_l^n(\xi)q_l^n(\xi)\ , \tag{C1}$$

choose a new radial variable z such that

$$\frac{d}{dz}=(\xi^2-1)\frac{d}{d\xi}\ , \tag{C2}$$

and write

$$\chi^2(\xi)=(l+\tfrac{1}{2})^2(\xi^2-1)+\frac{n^2}{16}(\xi+1)^4\ . \tag{C3}$$

The radial equation becomes

$$\left[\frac{d^2}{dz^2}-[\chi^2-\tfrac{1}{4}(\xi^2-1)]\right]p_l^n(\xi)=0\ . \tag{C4}$$

It follows from (C4) and the Wronskian relation

$$p_l^n(\xi)\frac{dq_l^n(\xi)}{d\xi}-q_l^n(\xi)\frac{dp_l^n(\xi)}{d\xi}=\frac{-2n}{\xi^2-1} \tag{C5}$$

that α satisfies the nonlinear equation

$$\frac{d^2\alpha}{dz^2}-[h^2\chi^2-\tfrac{1}{4}(\xi^2-1)]\alpha+\alpha^{-3}=0 \tag{C6}$$

with h an expansion parameter that will ultimately be set to unity.

It is convenient to rewrite (C6) as

$$\alpha=\chi^{-1/2}\left[1-\left[\frac{1}{\alpha}\frac{d^2\alpha}{dz^2}+\frac{1}{4}(\xi^2-1)\right]\Big/h^2\chi^2\right]^{-1/4}\ . \tag{C7}$$

We solve (C7) iteratively, taking $\chi^{-1/2}$ as a first approximation. This procedure yields

$$\alpha^2\sim\sum_{k=1}^{\infty}\frac{W_l^{(k)n}(\xi)}{h^{2k-2}}\ . \tag{C8}$$

The first four $W^{(k)}$ are displayed in Table II.

To evaluate the sum S_2 of Sec. III, we need an approximation to the product

$$\frac{dp_l^n(\xi)}{d\xi}\frac{dq_l^n(\xi)}{d\xi}\ .$$

From (C1) and (C5) it is easy to show that

$$\frac{1}{n}\frac{dp_l^n(\xi)}{d\xi}\frac{dq_l^n(\xi)}{d\xi}=\left[\frac{d\alpha}{d\xi}\right]^2-\frac{\alpha^{-2}}{(\xi^2-1)^2}\ , \tag{C9}$$

from which we may define

$$\frac{dp_l^n(\xi)}{d\xi}\frac{dq_l^n(\xi)}{d\xi}\sim\sum_{k=1}^{\infty}\frac{WP_l^{(k)n}(\xi)}{h^{2k-2}}-\sum_{k=1}^{\infty}\frac{WS_l^{(k)n}(\xi)}{h^{2k-2}(\xi^2-1)^2}\ .$$

The first four $WS^{(k)}$ and the first three $WP^{(k)}$ appear in Table IV.

APPENDIX D: ON SPEEDING THE CONVERGENCE OF SUMS

Here we briefly discuss two methods to speed the convergence of sums. This discussion draws heavily from Chap. 8 of the book by Bender and Orszag.[14]

Suppose the nth term in the sequence of partial sums of a series takes the form

$$A_n=A+\alpha q^n \tag{D1}$$

with $|q|<1$. A is the sum of the series and αq^n is a transient. It is easy to show that

$$A=\frac{A_{n+1}A_{n-1}-A_n{}^2}{A_{n+1}+A_{n-1}-2A_n}\ . \tag{D2}$$

This is known as the Shanks transformation and is exact if (D1) is precisely correct. If the A_n have several transients, and (D1) is only an approximation, then (D2) gives the nth term in a sequence which converges faster than the original A_n. This transformation may be iterated to remove several leading transients. Unfortunately, this method suffers a loss of accuracy in numerical computation. Roughly speaking, if the A_n are known to k digits, the Shanks transformation will be effected by roundoff errors at $k/2$ digits.

The second method is known as the Richardson extrapolation. It assumes the A_n take the forms

$$\begin{aligned}A_n&=Q_0+Q_1n^{-1}+Q_2n^{-2}+\cdots+Q_Nn^{-N}\ ,\\ A_{n+1}&=Q_0+Q_1(n+1)^{-1}+Q_2(n+1)^{-2}\\ &\quad+\cdots+Q_N(n+1)^{-N}\ ,\\ &\cdots\\ A_{n+N}&=Q_0+Q_1(n+N)^{-1}+Q_2(n+N)^{-2}\\ &\quad+\cdots+Q_N(n+N)^{-N}\ ,\end{aligned} \tag{D3}$$

from which we see

$$Q_0=\lim_{n\to\infty}A_n\ . \tag{D4}$$

The expression for Q_0 in terms of the A_n is

$$Q_0=\sum_{k=0}^{N}\frac{A_{n+k}(n+k)^n(-1)^{k+N}}{k!(N-k)!}\ . \tag{D5}$$

Unfortunately this method is also limited in numerical accuracy, somewhat more so than the Shanks method. The greater sensitivity to roundoff error for large n of the Richardson method is due to the large coefficients of alternating sign in (D5).

The Shanks transformation and Richardson extrapolation are both used in this work, depending upon the suitability of the forms (D1) and (D3). Each is most useful in different circumstances.

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