Dirac Hamiltonian structure of $R+R^2+T^2$ Poincaré gauge theory of gravity without gauge fixing

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Dirac's Hamiltonian method for constrained systems is applied to the most general Poincaré gauge-invariant theory of gravity interacting with an arbitrary matter field. Working in the first-order formalism, it is shown that the Hamiltonian contains a linear combination of nine kinematical symmetry generators and one dynamical generator, $\mathscr{H}_\perp$. The Poisson brackets between kinematical generators are calculated in the general case. In the case of a nine-parameter theory, described by the Lagrangian of $R+R^2+T^2$ type, it is shown that some of the primary constraints appear only when parameters satisfy certain conditions, which correspond to infinite tordion masses. The explicit form of $\mathscr{H}_\perp$ valid for all values of parameters is found. Assuming massive tordions, consistency conditions of primary constraints are analyzed in detail in the case of four-parameter $R+T^2$ theory and are also discussed in the general case. As an example, the spin-1/2 matter field is examined and preliminary Dirac brackets are found.

I. INTRODUCTION

Einstein's classical theory of gravity is in agreement with all known observational facts. However, from the theoretical point of view, one can remark that the theory admits singular solutions under very general assumptions,[1] and that spin does not act as a source of the gravitational field. A serious objection against Einstein's theory is also unrenormalizability of the corresponding quantum theory of gravitation.[2]

Among many attempts to overcome some of these problems, gauge theories of gravitation are especially attractive due to their considerable success in elementary particle physics. In the Poincaré gauge theory of gravitation, gauge potentials are tetrad field ($b^k{}_\mu$) and Lorentz connection ($A^{ij}{}_\mu$), whereas the corresponding field strengths are torsion ($T^k{}_{\mu\nu}$) and curvature ($R^{ij}{}_{\mu\nu}$) tensors.[3,4] The most general, parity-conserving Lagrangian, which is at most quadratic in the torsion and the curvature, i.e., it is of $R+R^2+T^2$ type, depends on nine parameters (excluding the cosmological constant).[5]

At the moment, there are several Lagrangians of that type proposed in the literature. For one choice of parameters, the corresponding theory has a better singular behavior.[6] For other values of parameters, one can obtain a theory without ghosts and tachyons.[7–9] Although according to the results of Ref. 9 it seems impossible to have a theory which is both renormalizable and unitary, the problem is still open for some values of parameters. Besides, one can be encouraged by the fact that renormalizable[10] higher-order derivative quantum gravity has recently been proven to be also unitary.[11]

In order to leave all possibilities open, we will consider the most general case, with nine *arbitrary* parameters. Our investigation of the theory is based on the Dirac Hamiltonian method for constrained dynamical systems.[12–15] Such an analysis is necessary in order to get a clear picture about physical degrees of freedom, and to check the consistency of the classical theory. Further, it is the first step toward canonical quantization (although a covariant quantization[16] may be more useful in practice, it should be justified by a Hamiltonian analysis).

The Hamiltonian dynamics of Poincaré gauge theory has been studied in Ref. 17 from a geometric point of view. The matter field is restricted to be a *tensor field*, while the Lagrangian is left quite arbitrary, excluding the possibility of particle-spectrum investigation. Our Lagrangian, although very general, has a definite form, so that the complete dynamical structure can be studied in detail.

Dirac Hamiltonian formulation of Einstein's,[18] Einstein-Cartan,[19,20] $R+T^2$,[21] and $R+R^2+T^2$ theory of gravity[22] has already been performed in the *time gauge* (see also Ref. 23). The investigation of Einstein's and of the Einstein-Cartan theory, without gauge fixing, has been carried out in Refs. 24—27. The purpose of this work is to extend the results of Ref. 22 in a gauge-free framework. The motivation for such a work is that the time-gauge condition may not be suitable for calculations at the quantum level.[28] Besides, we are not going to modify our Lagrangian by adding a non-gauge-invariant four-divergence term, as one usually does, which could cause troubles in the quantized theory (see also Ref. 27).

Following the gauge field approach to gravity, we treat the tetrad ($b^k{}_\mu$) and the Lorentz connection ($A^{ij}{}_\mu$) as independent fields, the so-called first-order formalism (and denote the corresponding momenta by $\pi_k{}^\mu$ and $\pi_{ij}{}^\mu$, respectively). This assumption leads to at most a second-order Euler-Lagrange equation for the basic field. For that reason our results are not applicable directly to the very promising higher-order-derivative conformal theory of gravitation.[29]

In Sec. II we briefly review the basic elements of the Poincaré gauge theory of gravitation, and also introduce our notation. In Sec. III, it is shown that the total Hamiltonian of the most general Poincaré gauge-invariant theory can be written in the standard, Dirac-Arnowitt-Deser-Misner (ADM) form[18,30] (up to a three-divergence term $D^{\alpha}{}_{,\alpha}$)

$$\mathscr{H}_{\rm tot}=N\mathscr{H}_{\perp}+N^{\alpha}\mathscr{H}_{\alpha}-\tfrac{1}{2}A^{ij}{}_{0}\mathscr{H}_{ij}+D^{\alpha}{}_{,\alpha} +\dot{b}^{k}{}_{0}\pi_{k}{}^{0}+\tfrac{1}{2}\dot{A}^{ij}{}_{0}\pi_{ij}{}^{0}+(u\cdot\phi)\ , \tag{1.1}$$

where only the $\mathscr{H}_{\perp}$ and the possible primary constraints ϕ depend on the choice of the initial Lagrangian density; all other terms are purely kinematical. Besides, ϕ and secondary constraints $\mathscr{H}_{\perp}$, $\mathscr{H}_{\alpha}$, and $\mathscr{H}_{ij}$ are independent of unphysical variables $b^{k}{}_{0}$, $\pi_{k}{}^{0}$, $A^{ij}{}_{0}$, and $\pi_{ij}{}^{0}$. The Poisson brackets between kinematical terms $\mathscr{H}_{\alpha}$ and $\mathscr{H}_{ij}$ are calculated, revealing them as the generators of "shift" transformations and local Lorentz rotations. The right-hand sides of these brackets are free of the terms which are quadratic in $\mathscr{H}_{ij}$, as one could have expected according to the results of Ref. 31.

In Sec. IV we consider a nine-parameter Lagrangian which is of $R+R^2+T^2$ type. It is shown that the Hessian matrix is singular (with respect to velocities $\dot{b}^{k}{}_{\alpha}$ and $\dot{A}^{ij}{}_{\alpha}$) if the parameters take on critical values. These values of parameters coincide with the conditions for infinite tordion masses of Ref. 7 (see also Ref. 8, where tordions are defined in a different way). All corresponding primary constraints are found and presented in the form of *if-constraints* which automatically drop out of the theory if parameters are not critical. The gravitational super-Hamiltonian $\mathscr{H}_{\perp}^{G}$ is expressed in a form which is valid for all values of parameters.

In Sec. V, Dirac's procedure of finding all possible constraints is performed in the case of the four-parameter $R+T^2$ theory, assuming massive tordions. It is shown that also for noncritical values of parameters, there exist secondary constraints in the theory, which play the same role as corresponding (absent) primary constraints—they are of the second class and reduce the number of physical degrees of freedom. The final theory, after appropriate gauge fixing, contains four degrees of freedom which correspond to a massless graviton. In the $R+R^2+T^2$ case, the results of Ref. 22 concerning the consistency conditions of the if-constraints are generalized to a gauge-free formulation, which could serve as a starting point for further investigation.

Section VI is an illustration of the general method developed in Sec. III. The spin-$\frac{1}{2}$ matter field is put into the Hamiltonian form, and the preliminary Dirac brackets are derived. Similar results have been obtained in Ref. 25 in a somewhat different framework. In Sec. VII the time-gauge condition is imposed in order to make easier comparison with some previously obtained results in the literature.

A. Conventions

Our conventions are the same as in Refs. 20—22. The latin indices are the local Lorentz (anholonomic) indices, whereas the greek indices are the coordinate indices (holonomic). The first letters of both alphabets $(a,b,c,\ldots;\alpha,\beta,\gamma,\ldots)$ run over 1,2,3, whereas the rest of them run over 0,1,2,3. Furthermore, $\eta_{ij}=\mathrm{diag}(+,-,-,-)$; ϵ^{ijkl} and ϵ^{abc} are completely antisymmetric tensors and $\epsilon^{0123}=\epsilon^{123}=1$. Also, $X_{[ij]}\equiv\frac{1}{2}(X_{ij}-X_{ji})$ and $X_{(ij)}\equiv\frac{1}{2}(X_{ij}+X_{ji})$. The meaning of a bar over a latin index is explained in Appendix A. The Ricci tensor is defined by $R_{ij}\equiv R^{k}{}_{ikj}$.

II. POINCARÉ GAUGE-INVARIANT THEORY OF GRAVITATION

Let us start with an arbitrary matter field Lagrangian which is invariant under the *global* Poincaré group,

$$\mathscr{L}^{M}=\mathscr{L}^{M}(u,\partial_{k}u)\ , \tag{2.1}$$

where $\partial_{k}u\equiv u_{,k}=\partial u/\partial x^{k}$ and u is a column vector which transforms according to some representation of the Lorentz group. Such a theory has to be modified, in order to become invariant with respect to the *local* Poincaré group, by introducing two kinds of gauge potentials: $b^{k}{}_{\mu}$-tetrad field, related to the translations, and $A^{ij}{}_{\mu}$-Lorentz connection, associated to the Lorentz rotations (note that $A^{ij}{}_{\mu}=-A^{ji}{}_{\mu}$).[3,4] The covariant derivative of the matter field is defined by

$$D_{k}u=h_{k}{}^{\mu}\nabla_{\mu}u=h_{k}{}^{\mu}(\partial_{\mu}+\tfrac{1}{2}A^{ij}{}_{\mu}S_{ij})u\ , \tag{2.2}$$

where $h_{k}{}^{\mu}$ is the inverse tetrad field

$$b^{k}{}_{\mu}h_{k}{}^{\nu}=\delta_{\mu}{}^{\nu}\ , \tag{2.3}$$

and S_{ij} are the Lorentz group generators. The matter field Lagrangian density is now given by

$$\mathscr{L}^{m}=b\mathscr{L}^{M}(u,D_{k}u)\ , \tag{2.4}$$

where $b=\det b^{k}{}_{\mu}$.

In order to construct a gauge field Lagrangian, one has to introduce two kinds of gauge field strengths: torsion

$$T^{i}{}_{\mu\nu}\equiv2\nabla_{[\nu}b^{i}{}_{\mu]}=2(b^{i}{}_{[\mu,\nu]}+A^{i}{}_{l[\nu}b^{l}{}_{\mu]})\ , \tag{2.5}$$

and curvature

$$R^{ij}{}_{\mu\nu}\equiv2(A^{ij}{}_{[\mu,\nu]}+A^{i}{}_{n[\nu}A^{nj}{}_{\mu]})\ . \tag{2.6}$$

Such a gravitational Lagrangian density must be of the general form

$$\mathscr{L}^{g}=b\mathscr{L}^{G}(T_{ijk},R_{ijkl})\ , \tag{2.7}$$

where greek indices are transferred into latin with the help of the inverse tetrad field.

One usually assumes that the matter field is minimally coupled to gravity

$$\mathscr{L}=\mathscr{L}^{m}+\mathscr{L}^{g}\ , \tag{2.8}$$

and that the gravitational Lagrangian is a parity-conserving scalar which is at most quadratic in the field strengths.[5] It can be written as the sum of a three-parameter torsion part

$$\mathscr{L}^{T}=AT_{ijk}T^{ijk}+BT_{ijk}T^{jik}+CT^{i}{}_{ik}T_{j}{}^{jk} \equiv\beta_{ijk}(T)T^{ijk}\ , \tag{2.9}$$

and a six-parameter curvature Lagrangian

$$\mathcal{L}^R = aR + b_1 R_{ijkl}R^{ijkl} + b_2 R_{ijkl}R^{klij} + b_3 R_{ij}R^{ij} + b_4 R_{ij}R^{ji} + b_5 R^2 + b_6(\epsilon_{ijkl}R^{ijkl})^2 \equiv aR + \beta_{ijkl}(R)R^{ijkl}, \tag{2.10}$$

where R_{ij} and R denote the Ricci tensor and scalar curvature. According to the Bach-Lanczos identity,[32] only five of the six parameters $b_1, \ldots, b_6$ are independent; thus, our gravitational action depends on nine arbitrary parameters.[33] In Secs. IV and V we will use another, more convenient, set of parameters,

$$A = \alpha/2 - \gamma/18, \quad B = \alpha/2 + \gamma/9, \quad C = -\alpha/2 + \beta, \tag{2.11}$$

$$\begin{aligned} &b_1 = (3a_2 + 4a_3)/8, \quad b_2 = (3a_2 - 4a_3)/8, \\ &b_3 = (-3a_2 - 4a_3 + 2a_4 + 2a_5)/4, \\ &b_4 = (-3a_2 + 4a_3 - 2a_4 + 2a_5)/4, \\ &b_5 = (a_2 - a_5 + 4a_6)/4, \quad b_6 = -(4a_1 - 3a_2)/96, \end{aligned} \tag{2.12}$$

which appear when the gravitational Lagrangian is expressed in terms of the irreducible components of the torsion and the curvature.[5]

Due to the presence of $(A^{ij}{}_\mu)^2$ terms in the $R+T^2$ part of the Lagrangian, the irreducible components of the Lorentz field describe particles (called *tordions*) which are not necessarily massless, as in ordinary gauge theories.[7,8] According to the results of Ref. 7, if the parameters do not take on degenerate values, i.e., if

$$\begin{aligned} &a \neq 0, \quad a - 3\alpha/2 \neq 0, \quad a + 3\beta/2 \neq 0, \\ &a - 2\gamma/3 \neq 0, \\ &\alpha, \beta, \gamma, a, a_1, \ldots, a_6 < \infty, \end{aligned} \tag{2.13}$$

all tordions are massive. On the other hand, if the parameters take on critical values (see Sec. IV), the masses of the tordions become infinite. Both sets of parameters, degenerate and critical, are very important for a Hamiltonian analysis, as will be shown in Sec. V.

III. GENERAL FORM OF THE TOTAL HAMILTONIAN

The basic dynamical variables in our theory are u, $b^k{}_\mu$, and $A^{ij}{}_\mu$. Let us denote the corresponding momenta by π, $\pi_k{}^\mu$, and $\pi_{ij}{}^\mu$, respectively. Due to the fact that the torsion and the curvature are defined through the antisymmetric derivatives of $b^k{}_\mu$ and $A^{ij}{}_\mu$ they do not involve velocities $b^k{}_{0,0}$ and $A^{ij}{}_{0,0}$. As a consequence, one immediately obtains the *primary constraints*

$$\pi_k{}^0 \approx 0, \tag{3.1}$$

$$\pi_{ij}{}^0 \approx 0. \tag{3.2}$$

The situation is very similar to the case of a gauge theory based on an internal-symmetry group. If the Lagrangian (2.9) is singular with respect to the remaining variables, u, $b^k{}_\alpha$, and $A^{ij}{}_\alpha$, one obtains further primary constraints, which will be generically denoted by ϕ.

In this section we are going to show that the canonical Hamiltonian density,[12–15] is linear in the fields $b^k{}_0$ and $A^{ij}{}_0$, up to a three-divergence term

$$\mathcal{H}_{\mathrm{can}} = b^k{}_0 \mathcal{H}_k - \tfrac{1}{2} A^{ij}{}_0 \mathcal{H}_{ij} + D^\alpha{}_{,\alpha}, \tag{3.3}$$

and that the other possible primary constraints, ϕ, are independent of $b^k{}_0$ and $A^{ij}{}_0$. Thus, the total Hamiltonian density[12–15] is given by

$$\mathcal{H}_{\mathrm{tot}} = \mathcal{H}_{\mathrm{can}} + u^k{}_0 \pi_k{}^0 + \tfrac{1}{2} u^{ij}{}_0 \pi_{ij}{}^0 + (u \cdot \phi), \tag{3.4}$$

where u's are arbitrary multipliers. Using the form of the total Hamiltonian in the consistency conditions for the primary constraints (3.1) and (3.2),[12–15] one immediately obtains the *secondary constraints*

$$\mathcal{H}_k \approx 0, \tag{3.5}$$

$$\mathcal{H}_{ij} \approx 0. \tag{3.6}$$

At the end of this section, we will give arguments that $\mathcal{H}_k$ and $\mathcal{H}_{ij}$ are the symmetry generators, thus, the first-class constraints. As a consequence, variables $b^k{}_0$ and $A^{ij}{}_0$ are arbitrary functions of time. For that reason, we will call them and their momenta $\pi_k{}^0$ and $\pi_{ij}{}^0$ *unphysical* variables. Using the Hamiltonian equations of motion one can easily infer that multipliers $u^k{}_0$ and $u^{ij}{}_0$ are equal to $\dot{b}^k{}_0$ and $\dot{A}^{ij}{}_0$, respectively, and therefore arbitrary functions of time. That means that constraints (3.1) and (3.2) should also be first-class constraints.

A. Decomposition of the inverse tetrad field

Before we proceed to prove that the canonical Hamiltonian can be written in the form (3.3) it is important to realize that the Lagrangian (2.9) depends on an unphysical variable $b^k{}_0$ also through the inverse tetrad field $h_k{}^\mu$. To recognize this dependence clearly, it is convenient to pass from $h_k{}^\mu$ to the set of variables $\{n_k, h_{\bar{k}}{}^\alpha, N, N^\alpha\}$ as follows. Let us first note that the components of the unit normal $\vec{n}$ to the $x^0 = \text{const}$ hypersurface, with respect to the local Lorentz basis are given by

$$n_k \equiv h_k{}^0/\sqrt{g^{00}}, \quad g^{\mu\nu} \equiv h_k{}^\mu h^{k\nu}, \tag{3.7}$$

and they are independent of $b^k{}_0$, as can easily be inferred from the orthogonality relations (2.3). Further, one can decompose $h_k{}^\mu$ into the orthogonal and parallel components with respect to the local Lorentz indices (see Appendix A):

$$h_k{}^\mu = h_{\bar{k}}{}^\mu + n_k h_\perp{}^\mu, \tag{3.8}$$

$$h_{\bar{k}}{}^\mu \equiv \delta_{\bar{k}}{}^l h_l{}^\mu = (\delta_k{}^l - n_k n^l) h_l{}^\mu, \tag{3.9}$$

$$h_\perp{}^\mu \equiv n^k h_k{}^\mu = n^\mu. \tag{3.10}$$

Let us now observe that

$$h_{\bar{k}}{}^0 \equiv 0, \quad h_{\bar{k}}{}^\alpha = {}^3g^{\alpha\beta} b_{k\beta}, \tag{3.11}$$

where ${}^3g^{\alpha\beta}$ is the three-dimensional contravariant metric. From (3.11) it follows that $h_{\bar{k}}{}^\alpha$ does not depend on $b^k{}_0$, too. Introducing now, as usual,[30] lapse and shift functions

$$N \equiv 1/\sqrt{g^{00}} = n_k b^k{}_0, \tag{3.12}$$

$$N^{\alpha}\equiv -g^{0\alpha}/g^{00}=h_{\bar{k}}{}^{\alpha}b^{k}{}_{0}\ , \tag{3.13}$$

respectively, we can write (3.10) in the form

$$h_{\perp}{}^{0}=1/N,\ \ h_{\perp}{}^{\alpha}=-N^{\alpha}/N\ . \tag{3.14}$$

It is now clear that we can always pass from $\{h_k{}^{\mu}\}$ to the more suitable set $\{n_k, h_{\bar{k}}{}^{\alpha}; N, N^{\alpha}\}$ and vice versa.[34]

One can now rewrite the canonical Hamiltonian (3.3) in the Dirac-ADM form,[18,30] using the fact that N and N^{α} are linear functions of $b^{k}{}_{0}$:

$$\mathscr{H}_{\text{can}}=N\mathscr{H}_{\perp}+N^{\alpha}\mathscr{H}_{\alpha}-\tfrac{1}{2}A^{ij}{}_{0}\mathscr{H}_{ij}+D^{\alpha}{}_{,\alpha}\ , \tag{3.15}$$

where lapse ($\mathscr{H}_{\perp}$) and shift ($\mathscr{H}_{\alpha}$) Hamiltonians are related to $\mathscr{H}_k$ as

$$\mathscr{H}_{\perp}=n^{k}\mathscr{H}_{k},\ \ \mathscr{H}_{\alpha}=b^{k}{}_{\alpha}\mathscr{H}_{k}\ , \tag{3.16}$$

$$\mathscr{H}_{k}=n_{k}\mathscr{H}_{\perp}+h_{\bar{k}}{}^{\alpha}\mathscr{H}_{\alpha}\ . \tag{3.17}$$

Equations (3.15) and (3.4) are equivalent to Eq. (1.1).

B. Matter Hamiltonian

Let us assume, for a moment, that coupling between matter and gauge fields is minimal, so that the canonical Hamiltonian is a sum of the matter ($\mathscr{H}^{M}$) and gravitational ($\mathscr{H}^{G}$) parts.

The initial matter Lagrangian (2.4) depends on the time derivative $u_{,0}$ only through the covariant derivatives $D_k u$. Decomposition of $D_k u$ into the orthogonal and parallel components

$$D_k u = n_k D_{\perp} u + D_{\bar{k}} u \equiv h_{\perp}{}^{\mu}\nabla_{\mu} u + h_{\bar{k}}{}^{\alpha}\nabla_{\alpha} u\ , \tag{3.18}$$

is very convenient, because $D_{\bar{k}}u$ does not depend on velocities as well as on unphysical variables [see Eqs. (2.2) and (3.11)]. Thus, expressing the initial matter Lagrangian in terms of $D_{\perp}u$, $D_{\bar{k}}u$, and n_k, instead of $D_k u$

$$\mathscr{L}^{M}=\bar{\mathscr{L}}^{M}(u, D_{\perp}u, D_{\bar{k}}u; n^{k})\ , \tag{3.19}$$

has the advantage that complete dependence on velocities and unphysical variables is through $D_{\perp}u$. Using that form of the matter Lagrangian in the standard definition of the matter field momenta, and the usual factorization property of the determinant b,[35]

$$b\equiv \det b^{k}{}_{\mu}=\frac{N}{n_0}\det b^{a}{}_{\alpha}\equiv NJ \tag{3.20}$$

(note that J is independent of $b^{k}{}_{0}$) lead to the result

$$\pi \equiv \frac{\partial(b\mathscr{L}^{M})}{\partial u_{,0}}=J\frac{\partial \bar{\mathscr{L}}^{M}}{\partial D_{\perp}u}\ . \tag{3.21}$$

Let us now replace velocities in the definition of the (canonical) matter field Hamiltonian

$$\mathscr{H}^{M}{}_{\text{can}}=\pi u_{,0}-b\mathscr{L}^{M}\ , \tag{3.22}$$

by the expression

$$u_{,0}=ND_{\perp}u+N^{\alpha}\nabla_{\alpha}u-\tfrac{1}{2}A^{ij}{}_{0}S_{ij}u \tag{3.23}$$

[see Eqs. (2.2), (3.11), and (3.18)]. The result can be written in the Dirac-ADM form (3.15), where

$$\mathscr{H}^{M}{}_{ij}=\pi S_{ij}u,\ \ \mathscr{H}^{M}{}_{\alpha}=\pi\nabla_{\alpha}u\ , \tag{3.24}$$

$$\mathscr{H}^{M}{}_{\perp}=\pi D_{\perp}u-J\bar{\mathscr{L}}^{M}=J\left[\frac{\partial\bar{\mathscr{L}}^{M}}{\partial D_{\perp}u}D_{\perp}u-\bar{\mathscr{L}}^{M}\right]\ , \tag{3.25}$$

and $D^{M\alpha}{}_{,\alpha}\equiv 0$. From the last equality in Eq. (3.25) it is evident that $(\mathscr{H}^{M}{}_{\perp}/J)$ is just the Legendre transformation of the function $\bar{\mathscr{L}}^{M}$ with respect to $D_{\perp}u$; therefore, it can be expressed as a function of u, $D_{\bar{k}}u$, n^k, and π/J [according to Eq. (3.21)]. Thus $\mathscr{H}^{M}{}_{\perp}$ is independent of unphysical variables.

If the matter Lagrangian $\bar{\mathscr{L}}^{M}$ is singular with respect to $D_{\perp}u$, the system of equations for momenta (3.21) gives rise to further primary constraints:

$$\phi^{(M)}=\phi^{(M)}(u, D_{\bar{k}}u, \pi/J)\ ,$$

which are again independent of unphysical variables. As a consequence, the

$$(u\cdot\phi)^{M}\equiv\sum_{(M)}u^{(M)}\phi^{(M)}$$

term in the total matter Hamiltonian density is also independent of them.

Note that $\mathscr{H}^{M}{}_{ij}$ and $\mathscr{H}^{M}{}_{\alpha}$ are purely kinematical terms depending only on the Lorentz transformation properties of the matter field. On the other hand, the "super-Hamiltonian" $\mathscr{H}_{\perp}$ is dynamical: it depends on the choice of the initial matter field Lagrangian.[36]

C. Gravitational Hamiltonian

Construction of the gravitational Hamiltonian density can be performed in a way, very similar to the case of the matter field; the role of $D_k u$ should be taken over by $T^{k}{}_{ij}$ and $R^{ij}{}_{kl}$.[37] First, one can decompose the torsion and the curvature tensor as

$$T^{k}{}_{lm}=2T^{k}{}_{[\bar{l}\perp}n_{m]}+T^{k}{}_{\bar{l}\bar{m}} \equiv \mathscr{T}^{k}{}_{lm}+\bar{T}^{k}{}_{lm}\ , \tag{3.26}$$

$$R^{ij}{}_{kl}=2R^{ij}{}_{[\bar{k}\perp}n_{l]}+R^{ij}{}_{\bar{k}\bar{l}} \equiv \mathscr{R}^{ij}{}_{kl}+\bar{R}^{ij}{}_{kl}\ , \tag{3.27}$$

so that $T^{k}{}_{\bar{l}\bar{m}}$ and $R^{ij}{}_{\bar{k}\bar{l}}$ are independent of velocities and unphysical variables [see Eqs. (2.5), (2.6), and (A.5)]. If we now define the following convenient "parallel" gravitational momenta (see Appendix A)

$$\pi_k{}^{\bar{l}}=\pi_k{}^{\alpha}b^{l}{}_{\alpha},\ \ \pi_{ij}{}^{\bar{k}}\equiv\pi_{ij}{}^{\alpha}b^{k}{}_{\alpha}\ , \tag{3.28}$$

which satisfy $\pi_k{}^{\bar{l}}n_l=0$, $\pi_{ij}{}^{\bar{k}}n_k=0$, we can easily obtain

$$\pi_k{}^{\bar{l}}\equiv b^{l}{}_{\alpha}\frac{\partial(b\mathscr{L}^{G})}{\partial b^{k}{}_{\alpha,0}}=J\frac{\partial\bar{\mathscr{L}}^{G}}{\partial T^{k}{}_{\bar{l}\perp}}\ ,$$

$$\pi_{ij}{}^{\bar{k}}\equiv b^{k}{}_{\alpha}\frac{\partial(b\mathscr{L}^{G})}{\partial A^{ij}{}_{\alpha,0}}=J\frac{\partial\bar{\mathscr{L}}^{G}}{\partial R^{ij}{}_{\bar{k}\perp}}\ , \tag{3.29}$$

where $\mathscr{L}^{G}=\bar{\mathscr{L}}^{G}(T_{k\bar{l}\perp}, T_{k\bar{l}\bar{m}}; R_{ij\bar{k}\perp}, R_{ij\bar{k}\bar{l}}; n_k)$.

The canonical gravitational Hamiltonian density

$$\mathscr{H}^G{}_{\rm can}=\pi_k{}^\alpha b^k{}_{\alpha,0}+\tfrac{1}{2}\pi_{ij}{}^\alpha A^{ij}{}_{\alpha,0}-b\mathscr{L}^G\ , \tag{3.30}$$

can be written in the Dirac-ADM form (3.15) after one replaces velocities by the expressions

$$b^k{}_{\alpha,0}=Nb^l{}_\alpha T^k{}_{\bar{l}\perp}+N^\beta T^k{}_{\alpha\beta}-T^k{}_{\alpha 0}(0)\ ,$$
$$A^{ij}{}_{\alpha,0}=Nb^l{}_\alpha R^{ij}{}_{\bar{l}\perp}+N^\beta R^{ij}{}_{\alpha\beta}-R^{ij}{}_{\alpha 0}(0)\ , \tag{3.31}$$

where

$$T^k{}_{\alpha 0}(0)=-b^k{}_{0,\alpha}+b^i{}_\alpha A^k{}_{i0}-b^i{}_0 A^k{}_{i\alpha}\ ,$$
$$R^{ij}{}_{\alpha 0}(0)=-A^{ij}{}_{0,\alpha}+A^i{}_{l0}A^{lj}{}_\alpha-A^i{}_{l\alpha}A^{lj}{}_0\ . \tag{3.32}$$

Note the presence of spatial derivatives of the unphysical variables $b^k{}_0$ and $A^{ij}{}_0$, which cause the appearance of the three-divergence term in the canonical Hamiltonian. After a little algebra one obtains the result

$$D^{G\alpha}{}_{,\alpha}=(b^k{}_0\pi_k{}^\alpha+\tfrac{1}{2}A^{ij}{}_0\pi_{ij}{}^\alpha)_{,\alpha}\ , \tag{3.33}$$

$$\mathscr{H}^G{}_{ij}=2\pi_{[i}{}^\alpha b_{j]\alpha}+\nabla_\alpha\pi_{ij}{}^\alpha\ , \tag{3.34}$$

$$\mathscr{H}^G{}_\alpha=\pi_k{}^\beta T^k{}_{\beta\alpha}-b^k{}_\alpha\nabla_\beta\pi_k{}^\beta+\tfrac{1}{2}\pi_{ij}{}^\beta R^{ij}{}_{\beta\alpha}\ , \tag{3.35}$$

$$\mathscr{H}^G{}_\perp=J\left[\frac{1}{J}\pi_k{}^{\bar{l}}T^k{}_{\bar{l}\perp}+\frac{1}{2J}\pi_{ij}{}^{\bar{l}}R^{ij}{}_{\bar{l}\perp}-\bar{\mathscr{L}}^G\right]-n^k\nabla_\alpha\pi_k{}^\alpha\ . \tag{3.36}$$

The expression in parentheses in the last equation can be expressed as a function of $T^k{}_{\bar{l}\bar{m}},\pi_k{}^{\bar{l}}/J;R^{ij}{}_{\bar{k}\bar{l}}$ and $\pi_{ij}{}^{\bar{k}}/J$, with the help of Eq. (3.29). It is independent of unphysical variables and represents the only dynamical part of the gravitational canonical Hamiltonian.

From Eq. (3.29) it is also clear that the other possible gravitational primary constraints do not involve unphysical variables. This completes our proof about the general form of the total Hamiltonian from the beginning of this section.

D. General case

Our construction of the Hamiltonian is based only on the fact that velocities $u_{,0}$, $b^k{}_{\mu,0}$, and $A^{ij}{}_{\mu,0}$ appear in the Lagrangian only through covariant combinations $D_k u$, T_{ijk}, and $R^{ij}{}_{kl}$, respectively. Therefore, generalization to a derivative coupling theory, which can be described by a Lagrangian of the general form

$$\mathscr{L}=\mathscr{L}(u,D_k u,T_{ijk},R_{ijkl}) \tag{3.37}$$

is straightforward. In that case, the Hamiltonian can again be written in the Dirac-ADM form, in which the kinematical generators are simply equal to a sum of already obtained expressions:

$$D^\alpha{}_{,\alpha}=(b^k{}_0\pi_k{}^\alpha+\tfrac{1}{2}A^{ij}{}_0\pi_{ij}{}^\alpha)_{,\alpha}\ , \tag{3.38}$$

$$\mathscr{H}_{ij}=\pi S_{ij}u+2\pi_{[i}{}^\alpha b_{j]\alpha}+\nabla_\alpha\pi_{ij}{}^\alpha\ , \tag{3.39}$$

$$\mathscr{H}_\alpha=\pi\nabla_\alpha u+\pi_k{}^\beta T^k{}_{\beta\alpha}-b^k{}_\alpha\nabla_\beta\pi_k{}^\beta+\tfrac{1}{2}\pi_{ij}{}^\beta R^{ij}{}_{\beta\alpha}\ , \tag{3.40}$$

whereas the "super-Hamiltonian" $\mathscr{H}_\perp$ is given by

$$\mathscr{H}_\perp=(\pi D_\perp u+\pi_k{}^{\bar{l}}T^k{}_{\bar{l}\perp}+\tfrac{1}{2}\pi_{ij}{}^{\bar{k}}R^{ij}{}_{\bar{k}\perp}-J\bar{\mathscr{L}})-n^k\nabla_\alpha\pi_k{}^\alpha\ , \tag{3.41}$$

where $\bar{\mathscr{L}}$ is defined in the same way as $\bar{\mathscr{L}}^M$ and $\bar{\mathscr{L}}^G$.

Note that $\mathscr{H}_\perp$ is again the only part of the Hamiltonian which carries dynamical aspects of the theory. This is a remarkable advantage of the Dirac-ADM form of the Hamiltonian, compared to the form (3.3), in which all $\mathscr{H}_k$ depend on the choice of the initial gravitational Lagrangian [see Eq. (3.16)].

The above written parts of the Hamiltonian $\mathscr{H}_\perp$, $\mathscr{H}_\alpha$, and $\mathscr{H}_{ij}$ generalize corresponding previously obtained expressions in the literature, due to the fact that we have not imposed any gauge-fixing condition and confined ourselves to a specific form of the Lagrangian. Such assumptions *a priori* diminish the number of physical degrees of freedom in the theory, and thus change the form of $\mathscr{H}_\perp$, $\mathscr{H}_\alpha$, and $\mathscr{H}_{ij}$. Furthermore, we have not added a nongauge invariant four-divergence term to the gravitational Lagrangian density, which should have altered the "canonical" form (momentum × field) of the kinematical parts [see, for example, Eq. (D4) in Ref. 22].

E. Symmetry generators of the theory

According to Dirac's general arguments,[12–15] there have to be (at least) ten first-class constraints in the theory, as it is invariant under the ten-parameter local Poincaré group. It has already been shown that $\mathscr{H}_\perp$ and $\mathscr{H}_\alpha$ represent the generators of the orthogonal and parallel x^0=const hypersurface deformations in the purely metric theory of gravity, and the corresponding Poisson brackets have been derived as a consequence of the path independence of dynamical evolution[30,38] (see also Ref. 12). After that, in the second-order tetrad formulation of Einstein's theory, it has been shown that $\mathscr{H}_{ij}$ represents the generators of the local Lorentz transformations,[18,24] and the complete algebra has been established:[25]

$$\{\mathscr{H}_{ij},\mathscr{H}'_{kl}\}=\tfrac{1}{2}f_{ij}{}^{mn}{}_{kl}\mathscr{H}_{mn}\delta(\vec{x}-\vec{x}\,')\ , \tag{3.42}$$

$$\{\mathscr{H}_{ij},\mathscr{H}'_\alpha\}=0\ , \tag{3.43}$$

$$\{\mathscr{H}_\alpha,\mathscr{H}'_\beta\}=(\mathscr{H}'_\alpha\partial_\beta+\mathscr{H}_\beta\partial_\alpha+\tfrac{1}{2}R^{ij}{}_{\alpha\beta}\mathscr{H}_{ij})\delta(\vec{x}-\vec{x}\,')\ , \tag{3.44}$$

$$\{\mathscr{H}_{ij},\mathscr{H}'_\perp\}=0\ , \tag{3.45}$$

$$\{\mathscr{H}_\alpha,\mathscr{H}'_\perp\}=(\mathscr{H}_\perp\partial_\alpha+\tfrac{1}{2}R^{ij}{}_{\alpha\perp}\mathscr{H}_{ij})\delta(\vec{x}-\vec{x}\,')\ , \tag{3.46}$$

$$\{\mathscr{H}_\perp,\mathscr{H}'_\perp\}=-({}^3g^{\alpha\beta}\mathscr{H}_\alpha+{}^3g'^{\alpha\beta}\mathscr{H}'_\alpha)\partial_\beta\delta(\vec{x}-\vec{x}\,')\ , \tag{3.47}$$

where $f_{ij}{}^{mn}{}_{kl}$ are structure constants of the Lorentz group [in the above formulas a condensed notation is used; for example, $\{\mathscr{H}_\perp,\mathscr{H}'_\perp\}$ is a shorthand for the equal-time Poisson brackets $\{\mathscr{H}_\perp(x),\mathscr{H}_\perp(x')\}$]. These results have been verified in Refs. 26 and 27 in the first-order tetrad formulation of Einstein's theory.

Recently it has been noticed[31] that the method used in Ref. 25, although very general, can determine the right-hand sides of the above brackets only up to such terms which are quadratic functions of the constraint $\mathscr{H}_{ij}$. For that reason, one cannot take the relations (3.42)—(3.46) for granted, and has to calculate the Poisson brackets before attempts to quantize the theory.

What can be anticipated without explicit verification, for the purpose of this work, is the fact that $\mathscr{H}_{\perp}$, $\mathscr{H}_{\alpha}$, and $\mathscr{H}_{ij}$ are the first-class constraints (see also Ref. 39), so that their consistency conditions are trivially satisfied. Nevertheless, we have calculated the kinematical part of the Poisson brackets, and our results show that Eqs. (3.42)—(3.44) hold as they are; thus, there are no quadratic terms in them. We leave explicit verification of the remaining Poisson brackets (which involve dynamical super-Hamiltonian $\mathscr{H}_{\perp}$) for one of our forthcoming papers.

IV. PRIMARY CONSTRAINTS AND SUPER-HAMILTONIAN IN THE $R+R^2+T^2$ CASE

We are now going to investigate the nine-parameter gravitational Lagrangian (2.9)—(2.12)

$$\mathscr{L}^g=\mathscr{L}^t+\mathscr{L}^r=b\,[\beta_{ijk}(T)T^{ijk}+aR+\beta_{ijkl}(R)R^{ijkl}]\ , \tag{4.1}$$

in more detail. We have already obtained the general form of the total Hamiltonian [see Eqs. (1.1) and (3.38)—(3.41)], therefore we have to find only the gravitational super-Hamiltonian $\mathscr{H}^{G}{}_{\perp}$ and $(u\cdot\phi)^G$ term explicitly. In our case, they are the sum of the torsion and the curvature parts, respectively,

$$\mathscr{H}^{G}{}_{\perp}=\mathscr{H}^{T}{}_{\perp}+\mathscr{H}^{R}{}_{\perp},\quad (u\cdot\phi)^G=(u\cdot\phi)^T+(u\cdot\phi)^R\ . \tag{4.2}$$

A. The torsion part

Let us consider equations for the torsion momenta first [see Eq. (3.29)]. Using the fact that $\overline{\mathscr{L}}^{\,T}$ depends on $T^k{}_{\bar{l}\perp}$ only through combination $\mathscr{T}^k{}_{lm}\equiv 2T^k{}_{[\bar{l}\perp}n_{m]}$, according to Eq. (3.26), one easily obtains

$$\pi_k{}^{l}/J=n^m\frac{\partial\mathscr{L}^T}{\partial T^k{}_{lm}}=4\beta_k{}^{\bar{l}\perp}(T) \tag{4.3}$$

[the factor 4 appears as a consequence that $\beta(T)\cdot T$ is a quadratic function of the antisymmetric torsion tensor[40]]. Now it is convenient to use the fact that β is a linear function of T to insert "velocities" $T^k{}_{\bar{l}\perp}$ on the left-hand side of the above equation

$$4\beta^{k\bar{l}\perp}(\mathscr{T})=P^{k\bar{l}}\equiv\pi^{k\bar{l}}/J-4\beta^{k\bar{l}\perp}(\bar{T})\ , \tag{4.4}$$

where we have introduced generalized momenta $P^{k\bar{l}}$,[21,41] which are convenient functions of fields and momenta. The explicit form of the above equation can be easily obtained with the help of Eq. (2.9):

$$\begin{aligned}4AT_{k\bar{l}\perp}+2BT_{\bar{l}\bar{k}\perp}+2C\eta_{\bar{k}\bar{l}}T^{\bar{m}}{}_{\bar{m}\perp}+\tfrac{1}{2}(B+C)n_kT_{\perp\bar{l}\perp}&\\ =P_{k\bar{l}}&\\ =\pi_{k\bar{l}}/J+2BT_{\perp\bar{k}\bar{l}}+2Cn_kT^{\bar{m}}{}_{\bar{m}\bar{l}}\ .&\end{aligned} \tag{4.5}$$

Now, using the method of Appendix A, one can easily diagonalize the system by decomposing it into the irreducible parts with respect to the group of three-dimensional rotations in the $x^0=$constant plane.[42] Introducing also parameters α,β,γ instead of A,B,C [see Eq. (2.11)] one obtains

$$2(\alpha+\beta)T_{\perp\bar{l}\perp}=P_{\perp\bar{l}}\equiv\pi_{\perp\bar{l}}/J-(\alpha-2\beta)T^{\bar{m}}{}_{\bar{m}\bar{l}}\ , \tag{4.6}$$

$$\begin{aligned}(\alpha-4\gamma/9)\,{}^AT_{\bar{k}\bar{l}\perp}&={}^AP_{\bar{k}\bar{l}}\\ &\equiv{}^A\pi_{\bar{k}\bar{l}}/J+(\alpha+2\gamma/9)T_{\perp\bar{k}\bar{l}}\ ,\end{aligned} \tag{4.7}$$

$$3\alpha\,{}^TT_{\bar{k}\bar{l}\perp}={}^TP_{\bar{k}\bar{l}}\equiv{}^T\pi_{\bar{k}\bar{l}}/J\ , \tag{4.8}$$

$$6\beta T^{\bar{k}}{}_{\bar{k}\perp}=P^{\bar{k}}{}_{\bar{k}}\equiv\pi^{\bar{k}}{}_{\bar{k}}/J\ . \tag{4.9}$$

We see that the torsion Lagrangian (2.9) is singular with respect to velocities $\dot{b}^k{}_{\alpha}$ (or, equivalently, with respect to $T^k{}_{\bar{l}\perp}$) if the parameters take on critical values: $\alpha+\beta=0$, $\alpha-4\gamma/9=0$, $\alpha=0$, and (or) $\beta=0$. In that case, one obtains the following primary constraints: $P^{\perp\bar{k}}\approx 0$, ${}^AP^{\bar{k}\bar{l}}\approx 0$, ${}^TP^{\bar{k}\bar{l}}\approx 0$, and (or) $P^{\bar{k}}{}_{\bar{k}}\approx 0$. In order to be able to treat all such possibilities in a unique way, we introduce if-constraints[41] in the theory

$$\phi_{\perp\bar{k}}\equiv[1-\lambda(\alpha+\beta)]P_{\perp\bar{k}}\approx 0\ , \tag{4.10}$$

$${}^A\phi_{\bar{k}\bar{l}}\equiv[1-\lambda(\alpha-4\gamma/9)]\,{}^AP_{\bar{k}\bar{l}}\approx 0\ , \tag{4.11}$$

$${}^T\phi_{\bar{k}\bar{l}}\equiv[1-\lambda(\alpha)]\,{}^TP_{\bar{k}\bar{l}}\approx 0\ , \tag{4.12}$$

$$\phi^{\bar{k}}{}_{\bar{k}}\equiv[1-\lambda(\beta)]P^{\bar{k}}{}_{\bar{k}}\approx 0\ , \tag{4.13}$$

where we have used the singular function

$$\lambda(x)/x\equiv\begin{cases}1/x, & x\neq 0\\ 0, & x=0\ .\end{cases} \tag{4.14}$$

Let us explain this definition. If, for example, $\alpha+\beta=0$, then $1-\lambda(\alpha+\beta)=1$, and from (4.10) it follows that $P_{\perp\bar{k}}\approx 0$, as it should be, according to Eq. (4.6). Contrarily, if $\alpha+\beta\neq 0$, $1-\lambda(\alpha+\beta)=0$, Eq. (4.10) results in the trivial identity $0\approx 0$. It should be noted here that if-constraints do not represent any modification in the standard Dirac Hamiltonian formulation. They are a trivial consequence of the fact that our theory depends on arbitrary parameters, and thus, the primary constraints may exist only when some of the critical values are satisfied (otherwise they do not appear as primary constraints in the theory at all).

In order to exploit generalized momenta in the torsion super-Hamiltonian [see Eq. (3.36)],

$$\mathscr{H}^{T}{}_{\perp}=\pi_k{}^{\bar{l}}T^k{}_{\bar{l}\perp}-J\beta_{klm}(T)T^{klm}-n^k\nabla_{\alpha}\pi_k{}^{\alpha}\ , \tag{4.15}$$

one has to decompose T^{klm} according to Eq. (3.26), and then to use Eqs. (4.3) and (4.4) as well as the identities[40]

$$\beta_{klm}\mathscr{T}^{klm}\equiv 2\beta_{k\bar{l}\perp}T^{k\bar{l}\perp}\ , \tag{4.16}$$

$$\beta_{klm}(\mathscr{T})T^{k\bar{l}\bar{m}}\equiv\beta_{klm}(\bar{T})\mathscr{T}^{klm}\ , \tag{4.17}$$

to get

$$\mathscr{H}^{T}{}_{\perp}=\tfrac{1}{2}JP^{k\bar{l}}T_{k\bar{l}\perp}-J\overline{\mathscr{L}}^{\,T}(\bar{T})-n^k\nabla_{\alpha}\pi_k{}^{\alpha}\ , \tag{4.18}$$

where $\overline{\mathscr{L}}^{T}(\bar{T}) \equiv \overline{\mathscr{L}}^{T}(T_{k\bar{l}\perp}=0, T_{k\bar{l}\bar{m}}; n_k)$. The "velocities" $T_{k\bar{l}\perp}$ can now be eliminated, after one decomposes $P^{k\bar{l}}T_{k\bar{l}\perp}$ according to Eqs. (A8) and (A13). Let us consider, for example, the term of the sum

$$\tfrac{1}{2}JP^{\perp\bar{l}}T_{\perp\bar{l}\perp} .$$

If $\alpha+\beta\neq 0$, we can eliminate $T_{\perp\bar{l}\perp}$, according to Eq. (4.6) and obtain

$$\tfrac{1}{2}J\frac{1}{2(\alpha+\beta)}P^{\perp\bar{l}}P_{\perp\bar{l}} .$$

Contrarily, if $\alpha+\beta=0$, $\phi^{\perp\bar{l}} \equiv P^{\perp\bar{l}} \approx 0$, and we can absorb this term into the $(u\cdot\phi)^T$ part of the total torsion Hamiltonian to get

$$(\tfrac{1}{2}JT^{\perp\bar{l}\perp}+u^{\perp\bar{l}})\phi_{\perp\bar{l}} \to u'^{\perp\bar{l}}\phi_{\perp\bar{l}} ,$$

where u' is an arbitrary multiplier. Both possibilities can easily be handled by exploiting the λ function (4.14) again. The final result is

$$\mathscr{H}^{T}{}_{\perp}=\tfrac{1}{2}P_T{}^2-J\overline{\mathscr{L}}^{T}(\bar{T})-n^k\nabla_\alpha\pi_k{}^\alpha , \tag{4.19}$$

$$P_T{}^2 \equiv J\left[\frac{\lambda(\alpha+\beta)}{2(\alpha+\beta)}(P^{\perp\bar{k}})^2+\frac{\lambda(\alpha-4\gamma/9)}{\alpha-4\gamma/9}({}^{A}P^{\bar{k}\bar{l}})^2 + \frac{\lambda(\alpha)}{3\alpha}({}^{T}P^{\bar{k}\bar{l}})^2+\frac{\lambda(\beta)}{18\beta}(P^{\bar{k}}{}_{\bar{k}})^2\right] , \tag{4.20}$$

and the $(u\cdot\phi)^T$ term is simply a sum

$$(u\cdot\phi)^T=u^{\perp\bar{k}}\phi_{\perp\bar{k}}+{}^{A}u^{\bar{k}\bar{l}A}\phi_{\bar{k}\bar{l}}+{}^{T}u^{\bar{k}\bar{l}T}\phi_{\bar{k}\bar{l}}+\tfrac{1}{3}u^{\bar{k}}{}_{\bar{k}}\phi^{\bar{l}}{}_{\bar{l}} , \tag{4.21}$$

where u denotes the arbitrary multipliers (we have omitted primes).

B. The curvature part

The curvature super-Hamiltonian $\mathscr{H}^{R}{}_{\perp}$ can be obtained in a very similar way. The equations for momenta (3.29), lead to the system of equations

$$8\beta^{mn\bar{k}\perp}(\mathscr{R})=P^{mn\bar{k}}\equiv\pi^{mn\bar{k}}/J-8\beta^{mn\bar{k}\perp}(\bar{R})+4an^{[m}\eta^{n]\bar{k}} , \tag{4.22}$$

which can be diagonalized again by the method of Appendix A:

$$2(2a_3+a_4)^{A}R^{\perp\bar{k}\bar{l}\perp}={}^{A}P^{\perp\bar{k}\bar{l}} ,$$
$${}^{A}P^{\perp\bar{k}\bar{l}}={}^{A}\pi^{\perp\bar{k}\bar{l}}/J-2(2a_3-a_4)^{A}\underline{R}^{\bar{k}\bar{l}} , \tag{4.23}$$

$$(3a_2+2a_5)^{T}R^{\perp\bar{k}\bar{l}\perp}={}^{T}P^{\perp\bar{k}\bar{l}} ,$$
$${}^{T}P^{\perp\bar{k}\bar{l}}={}^{T}\pi^{\perp\bar{k}\bar{l}}/J-(3a_2-2a_5)^{T}\underline{R}^{\bar{k}\bar{l}} , \tag{4.24}$$

$$2(a_5+12a_6)R^{\perp\bar{k}}{}_{\bar{k}\perp}=P^{\perp\bar{k}}{}_{\bar{k}} ,$$
$$P^{\perp\bar{k}}{}_{\bar{k}}=\pi^{\perp\bar{k}}{}_{\bar{k}}/J-(a_5-12a_6)R^{\bar{k}\bar{l}}{}_{\bar{k}\bar{l}}+6a , \tag{4.25}$$

$$4(a_1+a_3)^{P}R^{\perp}={}^{P}P$$
$$={}^{P}\pi/J-4(a_1-a_3)\epsilon_{\bar{k}\bar{l}\bar{m}\perp}R^{\perp\bar{k}\bar{l}\bar{m}} , \tag{4.26}$$

$$2(a_4+a_5)^{V}R^{\bar{k}\perp}={}^{V}P^{\bar{k}}={}^{V}\pi^{\bar{k}}/J+2(a_4-a_5)R^{\perp\bar{l}\bar{k}}{}_{\bar{l}} , \tag{4.27}$$

$$(3a_2+4a_3)^{T}R^{\bar{k}\bar{l}\bar{m}\perp}={}^{T}P^{\bar{k}\bar{l}\bar{m}} ,$$
$${}^{T}P^{\bar{k}\bar{l}\bar{m}}={}^{T}\pi^{\bar{k}\bar{l}\bar{m}}/J+(3a_2-4a_3)^{T}R^{\perp\bar{m}\bar{k}\bar{l}} , \tag{4.28}$$

where ${}^{A}\underline{R}^{\bar{k}\bar{l}}$ and ${}^{T}\underline{R}^{\bar{k}\bar{l}}$ are the irreducible parts obtained from $\underline{R}^{\bar{k}\bar{l}}\equiv R^{\bar{m}\bar{k}}{}_{\bar{m}}{}^{\bar{l}}$. The super-Hamiltonian $\mathscr{H}^{R}{}_{\perp}$ [see Eq. (3.36)] can be written in terms of the generalized momenta and $\overline{\mathscr{L}}^{R}(\bar{R})=\overline{\mathscr{L}}^{R}(R_{ij\bar{k}\perp}=0, R_{ij\bar{k}\bar{l}}; n_k)$

$$\mathscr{H}^{R}{}_{\perp}=\tfrac{1}{4}(P_{1R})^2+\tfrac{1}{4}(P_{2R})^2-J\overline{\mathscr{L}}^{R}(\bar{R}) , \tag{4.29}$$

$$(P_{1R})^2=J\left[\frac{\lambda(2a_3+a_4)}{2a_3+a_4}({}^{A}P^{\perp\bar{k}\bar{l}})^2+\frac{2\lambda(3a_2+2a_5)}{3a_2+2a_5}({}^{T}P^{\perp\bar{k}\bar{l}})^2+\frac{\lambda(a_5+12a_6)}{3(a_5+12a_6)}(P^{\perp\bar{k}}{}_{\bar{k}})^2\right] , \tag{4.30}$$

$$(P_{2R})^2=J\left[\frac{-\lambda(a_1+a_3)}{24(a_1+a_3)}({}^{P}P)^2+\frac{\lambda(a_4+a_5)}{2(a_4+a_5)}({}^{V}P^{\bar{k}})^2+\frac{4}{3}\frac{\lambda(3a_2+4a_3)}{(3a_2+4a_3)}({}^{T}P^{\bar{k}\bar{l}\bar{m}})^2\right] , \tag{4.31}$$

and the $(u\cdot\phi)^R$ term in the total Hamiltonian is

$$(u\cdot\phi)^R=2{}^{A}u^{\perp\bar{k}\bar{l}A}\phi_{\perp\bar{k}\bar{l}}+2{}^{T}u^{\perp\bar{k}\bar{l}T}\phi_{\perp\bar{k}\bar{l}}+\tfrac{2}{3}u^{\bar{k}}{}_{\bar{k}}\phi^{\bar{l}}{}_{\bar{l}} -\tfrac{1}{6}{}^{P}u^{P}\phi+{}^{V}u^{\bar{k}V}\phi_{\bar{k}}+\tfrac{4}{3}{}^{T}u^{\bar{k}\bar{l}\bar{m}T}\phi_{\bar{k}\bar{l}\bar{m}} , \tag{4.32}$$

where u denotes the arbitrary multipliers and ϕ's are if-constraints, which are determined by the system of equations (4.23)–(4.28). For example, the first of them reads

$${}^{A}\phi^{\perp\bar{k}\bar{l}}=[1-\lambda(2a_3+a_4)]^{A}P^{\perp\bar{k}\bar{l}}\approx 0 . \tag{4.33}$$

Note that $\mathscr{H}^{T}{}_{\perp}$, $\mathscr{H}^{R}{}_{\perp}$, and $(u\cdot\phi)$ terms depend on momenta only through the generalized momenta $P_k{}^{\bar{l}}$ and $P_{kl}{}^{\bar{m}}$, that is why we have introduced them.

At this point, let us summarize our results. First, we have found all the values of parameters which diminish the rank of the Hessian matrices $\partial^2\mathscr{L}^T/\partial\dot{b}^{k}{}_{\alpha}\partial\dot{b}^{l}{}_{\beta}$ and $\partial^2\mathscr{L}^R/\partial\dot{A}^{ij}{}_{\alpha}\partial\dot{A}^{kl}{}_{\beta}$—critical values of parameters. As one could have expected, such values of parameters result in infinite masses of the tordions.[7] Further, we have found all the possible primary constraints, which appear when parameters take on the critical values. They can be written in the form of if-constraints, which automatically drop out from the theory when the corresponding critical values of the parameters are not fulfilled. At the end, we have found expression for the super-Hamiltonian $\mathscr{H}^{G}{}_{\perp}$, which is valid for all values of parameters.

Before investigating the consistency conditions of the if-constraints, let us consider the special, "most dynamical" case of the theory, when parameters are not critical. Although we know that such a choice is of no physical importance due to the presence of ghosts and tachyons, it is interesting to see what dynamical degrees of freedom are in that case. The only primary constraints $\pi_k{}^0 \approx 0$ and $\pi_{ij}{}^0 \approx 0$ result in ten first-class secondary constraints $\mathscr{H}_k \approx 0$ and $\mathscr{H}_{ij} \approx 0$; therefore Dirac's procedure is finished after one imposes ten gauge-fixing constraints. After that, the unphysical variables $b^k{}_0$, $\pi_k{}^0$, $A^{ij}{}_0$, and $\pi_{ij}{}^0$ can be dropped out, whereas ten gauge-fixing constraints, together with ten secondary constraints reduce the number of 24 tetrad degrees of freedom to four—which correspond to the massless graviton. All other fields $A^{ij}{}_\alpha$ and $\pi_{ij}{}^\alpha$ remain dynamical, i.e., all tordions propagate in the theory. Note that if some tordions are massless (for example, $a=0$, α,β, $\gamma\neq 0$), our conclusion still holds. That means that there are no extra first-class constraints (and thus extra symmetries in the theory) connected with the zero-mass tordions, as one could have naively expected.[22]

V. CONSISTENCY CONDITIONS OF THE IF-CONSTRAINTS

In order to keep all parameters arbitrary, up to the end of Dirac's procedure, we have to investigate the consistency conditions of all the possible primary constraints. Such an analysis has already been performed in Ref. 21 for the $R+T^2$ type of the theory and in Ref. 22 for the general $R+R^2+T^2$ theory. In these papers, Einstein's Lagrangian aR has been modified by the addition of a four-divergence term (the time gauge condition imposed in these papers is not very important for the following discussion). We know that such a modification does not alter equations of motion in the theory but very well affects the structure of the constraints (curvature generalized momenta $P_{ij}{}^{\bar k}$ ceases to depend on a, whereas $P_k{}^{\bar l}$ also includes terms of the $a\cdot A^{ij}{}_\alpha$ type). Therefore, it is necessary to check whether the scheme, proposed in these references, works in our case, since we do not want to add such a four-divergence for the reasons explained in the Introduction.

We confine ourselves to check the scheme (and generalize it to a gauge-free framework) in the case of the four-parameter $R+T^2$ theory of gravitation. This example is very important as it includes the standard Einstein-Cartan theory and also exhibits all important features of the proposed scheme, being relatively easily calculable at the same time.

A. $R+T^2$ case

As a result of the assumption that all a_i vanish ($i=1,\dots,6$), all curvature if-constraints become nontrivial, and they can be written in the compact form

$$\phi_{kl}{}^{\bar m} = \pi_{kl}{}^{\bar m}/J + 4an_{[k}\delta^{\bar m}{}_{l]} \approx 0 , \qquad (5.1)$$

according to Eq. (4.22). Nevertheless, it is more convenient to investigate the consistency conditions of the irreducible components of the above primary constraints, as will be shown in the sequel.

Assuming now, as usual, that the matter field Lagrangian $\mathscr{L}^M$ is linear in derivatives, and thus in Lorentz connection $A^{ij}{}_\alpha$, one can decompose the super-Hamiltonian $\mathscr{H}_\perp$ (which we denote here by $\mathscr{H}^{ET}{}_\perp$) as

$$\mathscr{H}^{ET}{}_\perp = \tfrac{1}{2}(P_T)^2 + \mathscr{H}_\perp(A^2) - A^{\perp l}{}_{\bar k}\pi_l{}^{\bar k} + \mathscr{H}_\perp{}^{\text{rest}} , \qquad (5.2)$$

where $\frac{1}{2}(P_T)^2$ is given by Eq. (4.20), $\mathscr{H}_\perp(A^2)$ is quadratic in $A^{ij}{}_{\bar k}$:

$$\mathscr{H}_\perp(A^2) = -J[\bar{\mathscr{L}}^T(A^2) + \bar{\mathscr{L}}^R(A^2)] \qquad (5.3)$$

[note that $\mathscr{H}^R{}_\perp$, in our case, contains only Einstein's term $-J\bar{\mathscr{L}}^R(\bar R) \equiv -aJR^{\bar k\bar l}{}_{\bar k\bar l}$], whereas

$$A^{ij}{}_{\bar k} \equiv A^{ij}{}_\alpha h_{\bar k}{}^\alpha , \qquad (5.4)$$

is "canonically conjugate" to $\pi_{ij}{}^{\bar k}$ (see Appendix A). The last term $\mathscr{H}_\perp{}^{\text{rest}}$ is at most linear in $A^{ij}{}_\alpha$ and $\pi_k{}^\beta$; variables which have nontrivial brackets with the primary constraints (5.1).

Turning now to the constraints ${}^T\phi_{\bar k\bar l\bar m}$ and ${}^P\phi$, one can easily show that they have weakly vanishing Poisson brackets with all torsion generalized momenta $P_k{}^{\bar l}$, and therefore with all possible torsion primary constraints. As a result, their consistency conditions lead to the following secondary constraints:

$$\begin{aligned} {}^T\chi_{\bar k\bar l\bar m} \equiv d\,{}^T\phi_{\bar k\bar l\bar m}/dt &\approx \left\{ {}^T\phi_{\bar k\bar l\bar m}, \int N'\mathscr{H}'^{ET}{}_\perp d^3x' \right\} \\ &\approx 2N(a-3\alpha/2)\,{}^TA_{\bar k\bar l\bar m} + {}^Tf_{\bar k\bar l\bar m} \approx 0 , \end{aligned} \qquad (5.5)$$

$${}^P\chi \equiv d\,{}^P\phi/dt \approx 4N(a-2\gamma/3)\,{}^PA + {}^Pf \approx 0 , \qquad (5.6)$$

where ${}^Tf_{\bar k\bar l\bar m} \equiv \{ {}^T\phi_{\bar k\bar l\bar m}, \int N'\mathscr{H}'_\perp{}^{\text{rest}}d^3x' \}$ and ${}^Pf$ is defined in an analogous manner; thus f's are independent of $A^{ij}{}_{\bar k}$ and $\pi_k{}^{\bar l}$. Now, it is clear that the above secondary constraints are of the second class:

$$\{ {}^T\phi_{\bar k\bar l\bar m}, {}^{T'}\chi^{\bar n\bar p\bar q} \} = -3\frac{N}{J}(a-3\alpha/2)\,{}^T\delta^{\bar n\bar p\bar q}_{klm}\delta(\vec{x}-\vec{x}\,') , \qquad (5.7)$$

$$\{ {}^P\phi, {}^P\chi' \} = 48\frac{N}{J}(a-2\gamma/3)\delta(\vec{x}-\vec{x}\,') , \qquad (5.8)$$

if the tordions are massive [see Eqs. (2.13) and (A19)]. For what follows, we will assume that tordions are massive.

In order to examine the consistency conditions of the other curvature primary constraints, we first note that nonvanishing Poisson brackets between them and the torsion generalized momenta $P_k{}^{\bar l}$ are given by

$$\{ \phi_{\bar k\bar m}{}^{\bar m}, P'_\perp{}^{\bar l} \} \approx -\frac{4}{J}[(a-3\alpha/2)+(\alpha+\beta)] \times \delta^{\bar l}{}_k\delta(\vec{x}-\vec{x}\,') , \qquad (5.9)$$

$$\{ {}^A\phi_{\perp\bar k\bar l}, {}^AP'^{\bar m\bar n} \} \approx \frac{2}{J}[(a-2\gamma/3)-(\alpha-4\gamma/9)] \times \delta_{[k}{}^{\bar m}\delta^{\bar n}{}_{l]}\delta(\vec{x}-\vec{x}\,') , \qquad (5.10)$$

$$\{ {}^T\phi_{\perp\bar k\bar l}, {}^TP'^{\bar m\bar n} \} \approx -\frac{2}{J}a\,(\delta^{\bar m}{}_{(k}\delta^{\bar n}{}_{l)} - \tfrac{1}{3}\eta^{\bar m\bar n}\eta_{\bar k\bar l})\delta(\vec{x}-\vec{x}\,') , \qquad (5.11)$$

$$\{\phi_{\perp\bar{k}}{}^{\bar{k}},P'_{\bar{m}}{}^{\bar{m}}\}\approx\frac{12}{J}a\,\delta(\vec{x}-\vec{x}\,')\ . \tag{5.12}$$

They take such a simple form, due to the fact that we have used irreducible components instead of the constraints (5.1), as one usually does (see, for example, Ref. 27). Note that nontrivial brackets appear only between torsion generalized momenta and curvature primary constraints which are of the same spin and parity. One should be aware of the fact that the decomposition into the irreducible parts of the constraints $\phi_{kl}{}^{\bar{m}}$ should be performed before taking the Poisson brackets with $\pi_k{}^T$, as the following relation holds:

$$\{\pi_\perp{}^T,n'_m\}=\delta^T{}_m\delta(\vec{x}-\vec{x}\,') \tag{5.13}$$

(which reveals $-\pi_\perp{}^T$ as a variable canonically conjugate to normal n_l).

From the above Poisson brackets we see that all possible torsion primary constraints are of the second class. If, for example, $\alpha+\beta=0$, primary constraint $\phi_\perp{}^T=P_\perp{}^T$ does not commute with $\phi_{T\bar{m}}{}^{\bar{m}}(a-3\alpha/2\neq0)$. The maximal number of torsion primary constraints appear when $\alpha=\beta=\gamma=0$, i.e., in the case of Einstein-Cartan theory. In that case the Dirac's procedure is finished after one constructs Dirac brackets based on the primary constraints as well as on the secondary constraints (5.5) and (5.6). Note that the components of the Lorentz connection $A^{\bar{k}\bar{m}}{}_{\bar{m}}$ and $A_{\perp\bar{k}T}$ do not appear in these second-class constraints. Still, it makes sense to say that pairs of constraints $(\phi_{\bar{k}\bar{m}}{}^{\bar{m}},\phi_\perp{}^{\bar{k}})$ and $(\phi_{\perp\bar{k}T},\phi_{\bar{k}T})$ determine variables $(\pi_{\bar{k}\bar{m}}{}^{\bar{m}},A^{\bar{k}T}{}_T)$ and $(\pi_{\perp\bar{k}T},A_{\perp\bar{k}T})$, respectively, because it is well known that $A^{\bar{k}\bar{m}}{}_{\bar{m}}$ and $A^{\perp\bar{k}T}$ can be expressed in terms of other variables and their velocities by using equations of motion for tetrad fields as well as these constraints.[21,27]

What happens if $\alpha+\beta\neq0$, $\alpha-4\gamma/9\neq0$, $\alpha\neq0$, and (or) $\beta\neq0$? Using Eqs. (4.20), (5.2), and (5.3) one can easily obtain that consistency conditions for $\phi_{\bar{k}\bar{m}}{}^{\bar{m}}$ and $\phi_{\perp\bar{k}T}$ result in the secondary constraints, which can be written (after multiplication with appropriate factors) as

$$\chi_{\bar{k}\bar{m}}{}^{\bar{m}}\equiv(-a+\alpha/2-\beta)P_{\perp\bar{k}} +\frac{\alpha+\beta}{2}[2(a+\alpha/2+2\beta)A_{\bar{k}\bar{m}}{}^{\bar{m}}+f'_{\bar{k}\bar{m}}{}^{\bar{m}}]\approx0\ , \tag{5.14}$$

$${}^A\chi_{\perp\bar{k}T}\equiv(2a-\alpha-8\gamma/9)\,{}^AP_{\bar{k}T} +(\alpha-4\gamma/9)[2(\alpha-4\gamma/9)\,{}^AA_{\perp\bar{k}T}+{}^Af'_{\perp\bar{k}T}]\approx0\ , \tag{5.15}$$

$${}^T\chi_{\perp\bar{k}T}\equiv-2(a-3\alpha/2)\,{}^TP_{\bar{k}T}+3\alpha\,{}^Tf'_{\perp\bar{k}T}\approx0\ , \tag{5.16}$$

$$\chi_{\perp\bar{k}}{}^{\bar{k}}\equiv12(a+3\beta/2)P^{\bar{k}}{}_{\bar{k}}+18\beta f'^{\bar{k}}{}_{\bar{k}}\approx0\ , \tag{5.17}$$

where f''s are independent of $A^{ij}{}_\alpha$ and $\pi_k{}^\alpha$. Note that these constraints reduce to the corresponding possible torsion primary constraints given by Eqs. (4.10)–(4.13) if $\alpha+\beta$, $\alpha-4\gamma/9$, α, and (or) β vanish. Therefore, the above constraints are always present in the theory and are of the second class, as one can easily prove that

$$\{\phi^{\bar{k}\bar{m}}{}_{\bar{m}},\chi'_{T\bar{n}}{}^{\bar{n}}\}=(4/J)(a+3\beta/2)(a-3\alpha/2) \times\delta^{\bar{k}}{}_l\delta(\vec{x}-\vec{x}\,')\ , \tag{5.18}$$

$$\{{}^A\phi_{\perp\bar{k}T},{}^A\chi'^{\perp\bar{m}\bar{n}}\}=(4/J)(a-2\gamma/3)(a-3\alpha/2) \times\delta_{[k}{}^{\bar{m}}\delta^{\bar{n}}{}_{l]}\delta(\vec{x}-\vec{x}\,')\ , \tag{5.19}$$

$$\{{}^T\phi_{\perp\bar{k}T},{}^T\chi^{\perp\bar{m}\bar{n}}\}=(4/J)a(a-3\alpha/2) \times(\delta_{(k}{}^{\bar{m}}\delta^{\bar{n}}{}_{l)}-\tfrac{1}{3}\eta^{\bar{m}\bar{n}}\eta_{\bar{k}T})\delta(\vec{x}-\vec{x}\,')\ , \tag{5.20}$$

$$\{\phi_{\perp\bar{k}}{}^{\bar{k}},\chi'_{\perp\bar{m}}{}^{\bar{m}}\}=(144/J)a(a+3\beta/2)\delta(\vec{x}-\vec{x}\,')\ . \tag{5.21}$$

In this way, there are no tertiary constraints due to the fact that we assume massive tordions. As in the Einstein's case, after imposing ten gauge-fixing conditions there remain only four physical degrees of freedom in the theory, which correspond to the massless graviton. The Lorentz connection field is undynamical, and can be determined by the second-class constraints (or through the equations of motion).

B. $R+R^2+T^2$ case

From the preceding example, we see that the only essential difference between our results, and the results of Refs. 21 and 22 lies in the fact that the torsion if-constraints cannot always be solved in terms of $A^{\bar{k}\bar{m}}{}_{\bar{m}}$ and $A_{\perp\bar{k}T}$ as should have been the case if we had added the already mentioned four-divergence term to the Lagrangian. Instead, one has to use the equations of motion for the tetrad field in order to determine $A^{\bar{k}\bar{m}}{}_{\bar{m}}$ and $A_{\perp\bar{k}T}$ explicitly (in such a way one actually reproduces the constraints which exist in the Lagrangian formulation of the theory, and which involve velocities). On the other hand, the second-class property of the if-constraints and their consistency conditions [see Eqs. (5.7), (5.8), and (5.17)–(5.20)] is exactly the same in both approaches, as one can easily verify. Therefore, we adopt the scheme proposed in Ref. 22 here, being aware of the subtlety concerning determination of $A^{\bar{k}\bar{m}}{}_{\bar{m}}$ and $A^{\perp\bar{k}T}$.

Following the scheme, we first decompose all if-constraints into six groups according to their spin and parity. The first four groups contain the pairs of if-constraints, say ϕ_1 and ϕ_2. If both if-constraints are nontrivial, they are of the second class and the corresponding components of the Lorentz connection cease to be dynamical. If, on the other hand, only one if-constraint in the pair exists as an ordinary primary constraint, say, ϕ_1, its consistency condition leads to the secondary constraints χ_1, which generalizes the expression for ϕ_2 [in the same sense as the constraints (5.14)–(5.17) are a generalization of the torsion if-constraints (4.10)–(4.13)]. Then ϕ_1 and χ_1 serve to freeze the corresponding components of the Lorentz connection (and their momenta). The last two groups contain only one if-constraint ϕ, which, if it appears as a primary constraint, leads to a secondary constraint χ. Then ϕ and χ can be used to determine the corresponding Lorentz connection degrees of freedom. These

TABLE I. If-constraints, their consistency conditions, and dynamical consequences in the theory.

Group	J^P	Critical values of parameters	Constraints	Undynamical variables
I		$\beta=0$	$\phi^{\bar{k}}{}_{\bar{k}},\ \chi^{\bar{k}}{}_{\bar{k}}$	
	0^+	$a_5+12a_6=0$	$\phi_{\perp T}{}^{T},\ \chi_{\perp T}{}^{T}$	$A_{\perp\bar{k}}{}^{\bar{k}},\ \pi_{\perp\bar{k}}{}^{\bar{k}}$
II		$\alpha-4\gamma/9=0$	${}^{A}\phi_{\bar{k}T},\ {}^{A}\chi_{\bar{k}T}$	
	1^+	$2a_3+a_4=0$	${}^{A}\phi_{\perp\bar{k}T},\ {}^{A}\chi_{\perp\bar{k}T}$	${}^{A}A_{\perp\bar{k}T},\ {}^{A}\pi_{\perp\bar{k}T}$
III		$\alpha=0$	${}^{T}\phi_{\bar{k}T},\ {}^{T}\chi_{\bar{k}T}$	
	2^+	$3a_2+2a_5=0$	${}^{T}\phi_{\perp\bar{k}T},\ {}^{T}\chi_{\perp\bar{k}T}$	${}^{T}A_{\perp\bar{k}T},\ {}^{T}\pi_{\perp\bar{k}T}$
IV		$\alpha+\beta=0$	$\phi_{\perp\bar{k}},\ \chi_{\perp\bar{k}}$	
	1^-	$a_4+a_5=0$	$\phi_{\bar{k}\bar{m}}{}^{\bar{m}},\ \chi_{\bar{k}\bar{m}}{}^{\bar{m}}$	$A_{\bar{k}\bar{m}}{}^{\bar{m}},\ \pi_{\bar{k}\bar{m}}{}^{\bar{m}}$
V				
	0^-	$a_1+a_3=0$	${}^{P}\phi,\ {}^{P}\chi$	${}^{P}A,\ {}^{P}\pi$
VI				
	2^-	$3a_2+4a_3=0$	${}^{T}\phi_{\bar{k}T\bar{m}},\ {}^{T}\chi_{\bar{k}T\bar{m}}$	${}^{T}A_{\bar{k}T\bar{m}},\ {}^{T}\pi_{\bar{k}T\bar{m}}$

results are summarized in Table I.

In this paper we have nothing to add to the scheme. Still one should be aware of the problems which may arise. The right-hand sides of brackets (5.9)—(5.12), in the general case, besides the already written terms, involve $R^{ij}{}_{\bar{k}T}$ and $\pi_{ij}{}^{\bar{k}}$ linearly. The secondary constraints χ could, in principle, depend on higher powers of $A^{ij}{}_{\bar{k}}$ and $\pi_{ij}{}^{\bar{k}}$ (up to the third). Even more, brackets of the type $\{\phi,\chi'\}$ may involve derivatives of the δ functions, requiring a careful analysis and fixing of the boundary conditions.[15] For that reason one can accept our scheme (which is valid at least in the massive-tordion weak-field approximation, as shown in Ref. 22) as the first step toward a more detailed analysis.

VI. SPIN-$\frac{1}{2}$ MATTER FIELD

Let us consider a spin-$\frac{1}{2}$ matter field minimally coupled to gravity in the Poincaré gauge-invariant framework. This case is important since we believe that most of the matter in the universe (quarks and leptons) are described by the Dirac field. Besides, the spin tensor of the Dirac field is nontrivial and can play an important role as a source of the gravitational field.

We start with the special-relativistic Lagrangian density in which independent variables ψ and $\bar{\psi}$ play symmetric roles:

$$\mathscr{L}^S=\tfrac{1}{2}\bar{\psi}i\gamma^k\overleftrightarrow{\partial}_k\psi-m\bar{\psi}\psi\ , \qquad (6.1)$$

where $\bar{\psi}\overleftrightarrow{\partial}\psi\equiv\bar{\psi}(\partial\psi)-(\partial\bar{\psi})\psi$ and γ^k are Dirac's matrices. The generators of the Lorentz group in the bispinor representation are given by

$$s_{ij}=\tfrac{1}{4}[\gamma_i,\gamma_j]\equiv\tfrac{1}{4}(\gamma_i\gamma_j-\gamma_j\gamma_i)\ . \qquad (6.2)$$

Using the minimal substitution rule in the Lagrangian (6.1) one gets

$$\mathscr{L}^s=b\,(\tfrac{1}{2}\bar{\psi}i\gamma^k\overleftrightarrow{D}_k\psi-m\bar{\psi}\psi)\ . \qquad (6.3)$$

If we want to apply here the results obtained for an arbitrary matter field directly, we should use an eight-dimensional column vector $u^s=(\begin{smallmatrix}\psi\\ \bar{\psi}T\end{smallmatrix})$ in the Lagrangian (6.3). Instead, we prefer to keep standard variables ψ and $\bar{\psi}$, and their momenta $\bar{\pi}$ and π, respectively. Repeating the few steps which have led us to Eqs. (3.24) and (3.25), one easily finds

$$\mathscr{H}^S{}_{ij}=\bar{\pi}s_{ij}\psi-\bar{\psi}s_{ij}\pi\ , \qquad (6.4)$$

$$\mathscr{H}^S{}_{\alpha}=\bar{\pi}\nabla_\alpha\psi+(\nabla_\alpha\bar{\psi})\pi\ , \qquad (6.5)$$

$$\mathscr{H}^S{}_{\perp}=-J(\tfrac{1}{2}\bar{\psi}i\gamma^{\bar{k}}\overleftrightarrow{D}_{\bar{k}}\psi-m\bar{\psi}\psi)\ . \qquad (6.6)$$

The last equation is a consequence of the fact that the Lagrangian (6.3) is linear in the velocities; therefore, $\mathscr{H}^S{}_\perp$ is simply equal to $-J\bar{\mathscr{L}}^S(D_\perp\psi=D_\perp\bar{\psi}=0)$. The $(u\cdot\phi)^S$ term, which appears in the total spin-$\frac{1}{2}$ Hamiltonian, is given by

$$(u\cdot\phi)^S=\bar{u}\phi+\bar{\phi}u\ , \qquad (6.7)$$

where u's are the arbitrary multipliers and ϕ and $\bar{\phi}$ the primary constraints in the theory,

$$\phi\equiv\pi+\frac{i}{2}J\gamma^\perp\psi\approx 0\ , \qquad (6.8)$$

$$\bar{\phi}\equiv\bar{\pi}-\frac{i}{2}J\bar{\psi}\gamma^\perp\approx 0\ . \qquad (6.9)$$

It is easy to see that ϕ and $\bar{\phi}$ are second-class constraints:

$$\{\phi,\bar{\phi}'\}=iJ\gamma^\perp\delta(\vec{x}-\vec{x}\,'),\quad \det\gamma^\perp=1\ . \qquad (6.10)$$

Therefore, the consistency conditions of ϕ and $\bar{\phi}$ determine four-multipliers $\bar{u}$ and u, respectively, and do not lead to any secondary constraints.

One can now easily construct the preliminary Dirac brackets based on constraints (6.8) and (6.9).[12–15] In our case they are given by

$$\{F,G'\}^*=\{F,G'\}+\frac{i}{J}\int d^3x''[\{F,\bar{\phi}''\}\gamma^{\perp''}\{\phi'',G'\}-\{F,\phi^{T''}\}(\gamma^\perp)^{T''}\{\bar{\phi}^{T''},G'\}]\ , \qquad (6.11)$$

where F and G are arbitrary variables and "T" denotes the transpose matrix [see also the text below Eq. (3.47)].

After that, one can use (6.8) and (6.9) as strong equalities to decrease the number of physical degrees of freedom from the theory. If one decides to eliminate $\bar{\psi}$ and π, it is enough to find the preliminary Dirac brackets for the remaining set of variables.[43] From Eq. (6.11) it is clear that the basic preliminary Dirac brackets[12–15] for two variables can differ from the corresponding Poisson brackets, only if both of them belong to the set $\{\psi,\bar{\pi},\pi_k{}^\alpha\}$. These brackets are given by

$$\{\psi,\bar{\pi}'\}^*=\tfrac{1}{2}I\delta(\vec{x}-\vec{x}\,')\ , \qquad (6.12)$$

$$\{\pi_k{}^\alpha,\psi'\}^*=\tfrac{1}{2}(h_{\bar{k}}{}^\alpha+n_k h_{\bar{l}}{}^\alpha\gamma^{\bar{l}}\gamma^\perp)\psi\delta(\vec{x}-\vec{x}\,')\ , \qquad (6.13)$$

$$\{\pi_k{}^\alpha, \bar\pi'\}^* = -\tfrac{1}{2}\bar\pi(h_{\bar k}{}^\alpha + n_k h_{\bar l}{}^\alpha \gamma^{\bar l}\gamma^\perp)\delta(\vec{x}-\vec{x}') , \quad (6.14)$$

$$\{\pi_k{}^\alpha, \pi'_l{}^\beta\}^* = iJ n_k n_l h_{\bar m}{}^\alpha h_{\bar n}{}^\beta \bar\psi \gamma^\perp s^{mn}\psi\delta(\vec{x}-\vec{x}') \quad (6.15)$$

[where s^{mn} is given by Eq. (6.2)] and they reduce to the brackets which have been found in Refs. 18 and 20, after imposing the time-gauge condition.

VII. TIME-GAUGE CONDITION

At the end of this work, let us say something about the time-gauge condition (TGC) which is commonly used in the literature. It is usually imposed from the very beginning, eliminating $b^0{}_\alpha$ degrees of freedom from the Lagrangian:

$$b^0{}_\alpha = 0, \quad b^0{}_{\alpha,\mu} = 0 . \quad (7.1)$$

A somewhat modified method of the TGC is used in Ref. 22, where only velocities $b^0{}_{\alpha,0}$ are not restricted to vanish in the Lagrangian. In such an approach, it is necessary to treat $b^0{}_\alpha \approx 0$ as an ordinary primary constraint which has to satisfy the consistency condition $\dot b^0{}_\alpha \equiv \{b^0{}_\alpha, \int \mathscr{H}'_{\rm tot} d^3x'\} \approx 0$.

Both methods of imposing the TGC are equivalent to the standard method,[44] which we are going to explain here. In the standard approach, gauge-fixing constraints

$$b^0{}_\alpha \approx 0 \quad (7.2)$$

should be imposed after the whole Dirac's procedure is finished. They have the nonvanishing Poisson brackets with boost generators $\mathscr{H}_{0b}$, given by Eq. (3.48), which can be explicitly solved in terms of $\pi_0{}^\beta$:

$$\pi_0{}^\beta - {}^3h^{b\beta}(b^0{}_\gamma \pi_b{}^\gamma - \mathscr{H}^M{}_{0b} - \mathscr{H}^R{}_{0b}) \approx 0 , \quad (7.3)$$

where ${}^3h_b{}^\beta$ are the inverse triads, and $\mathscr{H}^M{}_{0b}$ and $\mathscr{H}^R{}_{0b}$ are the matter and curvature boost generators. Thus, constraints (7.2) and (7.3) are of the second class, and serve to eliminate $b^0{}_\alpha$ and $\pi_0{}^\alpha$ from the theory.

Before that, we have to construct the preliminary Dirac brackets, using the constraints. But, in this case, the basic preliminary Dirac brackets are simply equal to the standard Poisson brackets, as far as the remaining set of variables is concerned. That means that we can simply eliminate $b^0{}_\alpha$ and $\pi_0{}^\alpha$ from the theory, using (7.2) and (7.3) as strong equations.

As a consequence of the breakdown of the boost invariance, the multiplier $A^{0b}{}_0$ is determined by the consistency conditions for the TGC, $\dot b^0{}_\alpha \approx 0$, and can be expressed in terms of other variables.

The decomposition of the inverse tetrad field $h_k{}^\mu$, in the time gauge, is simpler than in the general case:

$$n^k \to \delta^k{}_0, \quad h_{\bar 0}{}^\alpha \to 0 ,$$
$$h_{\bar b}{}^\alpha \to h_b{}^\alpha , \quad (7.4)$$
$$N \to b^0{}_0, \quad N^\alpha \to -b^0{}_0 h_0{}^\alpha .$$

From the above expressions for n^k, we see that TGC fixes the first tetrad leg to coincide with the normal to the x^0=const hypersurface. This is the geometrical meaning of the TGC. Besides, there is no need to use n^k and $h_{\bar k}{}^\alpha$ at all, and the decomposition described in Appendix A should be replaced by the standard "space + time" decomposition:

$$\text{“}\perp\text{”} \to \text{“}0\text{”}, \quad \text{“}\bar k\text{”} \to \text{“}a\text{”} \quad a = 1,2,3 . \quad (7.5)$$

It is now a trivial exercise to rewrite all the if-constraints and the total Hamiltonian in the time gauge. They are somewhat different from the corresponding expressions of Ref. 22, where a four-divergence term has been added to the Lagrangian.

To conclude, the advantage of the TGC lies in the fact that it gives a physical meaning to the first general coordinate x^0, and simplifies the theory, on account of destroying the boost invariance of the theory (see also Refs. 23 and 25).

VIII. CONCLUDING REMARKS

As has already been mentioned in Sec. III E, we have left verification of Eqs. (3.45)–(3.47) for one of our next papers. This problem is interesting not only because we want to see whether there are terms which are quadratic in the constraints, but also for the following reason. Such brackets are usually derived by exploiting the symmetry of the theory under general coordinate transformations (and local Lorentz rotations) and also imposing some kind of consistency requirements.[12,38] But, it has already proved by Schwinger[45] that the theories of spins higher than 2 do not satisfy similar requirements, even in flat space-time. Thus, it seems to be worthwhile to check these equations in a straightforward manner and to find which are the consistency requirements in our case (if there are any).

The next problem we have not completely solved is the problem of consistency conditions of the if-constraints, especially in the massless-tordion case. Still, the results of Secs. IV and V may serve as a starting point for further investigation of these problems. Especially, the massless-tordion case can be very well studied in the $R+T^2$ framework, as condition (2.13) does not involve parameters from the R^2 sector of the theory.[46]

At the end we emphasize again that our initial Lagrangian is not altered by adding a non-gauge-invariant four-divergence term. According to the results of Refs. 20 and 14, one could expect that our total Hamiltonian density must be improved by adding some suitable surface terms which reveal the Hamiltonian as the total energy of the system under consideration. This problem is also left for further investigation.

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APPENDIX A: "3 + 1" DECOMPOSITION WITH RESPECT TO x^0=CONST HYPERSURFACE

We are going to explain here how to decompose any tensor field with respect to the subgroup of three-dimensional rotations in the x^0=const hypersurface. As in the main part of the text, we prefer to work in the local Lorentz basis. In the vector representation, projectors to the normal, $P_\perp$, and to the x^0=const plane, $P_{||}$, are given by

$$(P_\perp)^k{}_l = n^k n_l, \quad (P_{||})^k{}_l \equiv \delta^{\bar k}{}_{\bar l} = \delta^k{}_l - n^k n_l \ , \tag{A1}$$

where $n_k = h_k{}^0/\sqrt{g^{00}}$. We adopt the convention that an overbar above an index "k" does not mean a different index "$\bar k$" but denotes the fact that contraction with n_k vanishes. Thus one can equally well write

$$\delta^{\bar k}{}_{\bar l} = \delta^k{}_{\bar l} = \delta^{\bar k}{}_l \ , \tag{A2}$$

as $\delta^{\bar k}{}_{\bar l}$ is symmetrical in its indices. The projectors (A1) are orthogonal to each other and define a complete set

$$\delta^{\bar k}{}_{\bar l} n^l = 0, \quad \delta^k{}_l = \delta^{\bar k}{}_{\bar l} + n^k n_l \ . \tag{A3}$$

Using (A3) we can express any vector field, say, D_k, in terms of its orthogonal and parallel components:

$$D_k = n_k(n^l D_l) + \delta_{\bar k}{}^l D_l \equiv n_k D_\perp + D_{\bar k} \ , \tag{A4}$$

where the notation is self-evident. A second-rank antisymmetrical tensor $X_{kl} = -X_{lk}$ can be decomposed as

$$X_{kl} = X_{\bar k \bar l} + 2X_{[\bar k \perp} n_{l]} \ ,$$
$$X_{\bar k \bar l} \equiv \delta^m{}_{\bar k} \delta^n{}_{\bar l} X_{mn}, \quad X_{\bar k \perp} \equiv \delta^n{}_{\bar k} n^m X_{nm} \ , \tag{A5}$$

where $2X_{[\bar k \perp} n_{l]} \equiv X_{\bar k \perp} n_l - X_{\bar l \perp} n_k$. Similarly, the tetrad and Lorentz connection "parallel" momenta $\pi_k{}^{\bar l}$ and $\pi_{kl}{}^{\bar m}$ can be further decomposed in the following way:

$$\pi^{k\bar l} = \pi^{\bar k \bar l} + n^k \pi^{\perp \bar l} \ , \tag{A6}$$

$$\pi^{kl\bar m} = \pi^{\bar k \bar l \bar m} + 2n^{[k}\pi^{\perp \bar l]\bar m} \tag{A7}$$

[see the text below (3.28)]. Besides, using Eq. (A3) one easily obtains that

$$\pi^{k\bar l} T_{k\bar l} = \pi^{\bar k \bar l} T_{\bar k \bar l} + \pi^{\perp \bar l} T_{\perp \bar l} \ , \tag{A8}$$

$$\pi^{kl\bar m} R_{kl\bar m} = \pi^{\bar k \bar l \bar m} R_{\bar k \bar l \bar m} + 2\pi^{\perp \bar l \bar m} R_{\perp \bar l \bar m} \ , \tag{A9}$$

for any tensors T_{kl} and R_{klm}.

The parallel tensors lie in the hypersurface x^0=const, and can be further decomposed into the irreducible parts using the standard method of symmetrization, antisymmetrization, and contraction. This decomposition should be performed by using $\eta_{\bar k \bar l}$ and $\epsilon_{\bar k \bar l \bar m \perp}$ instead of η_{ij} and ϵ_{ijkl}, in order to remain in the plane. These tensors satisfy the useful identities

$$\eta_{kl}\eta^{\bar l \bar m} \equiv \eta_{\bar k \bar l}\eta^{\bar l \bar m} = \delta_{\bar k}{}^{\bar m} \ ,$$
$$\delta^{\bar k}{}_{\bar k} = 3, \quad \epsilon_{\bar k \bar l \bar m \perp}\epsilon^{\bar k \bar l \bar m \perp} = -6 \ . \tag{A10}$$

(Note that $\epsilon_{\bar k \bar l \bar m \bar n} \equiv 0$ as well as $\epsilon_{\perp\perp mn} \equiv 0$.)

Starting from the parallel tetrad momenta $\pi_{\bar k \bar l}$, we can form the antisymmetric, traceless-symmetric, and scalar part, respectively,

$${}^A\pi_{\bar k \bar l} \equiv \pi_{[\bar k \bar l]}, \quad {}^T\pi_{\bar k \bar l} \equiv \pi_{(\bar k \bar l)} - \tfrac{1}{3}\eta_{\bar k \bar l}\pi^{\bar m}{}_{\bar m}, \quad {}^S\pi = \pi^{\bar k}{}_{\bar k} \ . \tag{A11}$$

The inverse relation is

$$\pi_{\bar k \bar l} = {}^A\pi_{\bar k \bar l} + {}^T\pi_{\bar k \bar l} + \tfrac{1}{3}\eta_{\bar k \bar l}{}^S\pi \ , \tag{A12}$$

whereas the $\pi^{\bar k \bar l} T_{\bar k \bar l}$ term in Eq. (A8) becomes

$$\pi^{\bar k \bar l} T_{\bar k \bar l} = {}^A\pi^{\bar k \bar l\,A} T_{\bar k \bar l} + {}^T\pi^{\bar k \bar l\,T} T_{\bar k \bar l} + \tfrac{1}{3}{}^S\pi^S T \ . \tag{A13}$$

The same decomposition can be performed for $\pi^{\perp \bar l \bar m}$, which is present in Eqs. (A7) and (A9).

The antisymmetric parallel tensor $\pi^{\bar k \bar l \bar m} = -\pi^{\bar l \bar k m}$ can be expressed as a sum of the pseudoscalar, vector, and tensor parts, respectively,

$$\pi^{\bar k \bar l \bar m} = -\tfrac{1}{6}\epsilon^{\bar k \bar l \bar m \perp\, P}\pi + {}^V\pi^{[\bar k}\eta^{\bar l]\bar m} + \tfrac{4}{3}{}^T\pi^{[\bar k \bar l]\bar m} \ , \tag{A14}$$

where

$${}^P\pi \equiv \epsilon^{\bar k \bar l \bar m \perp}\pi_{\bar k \bar l \bar m}, \quad {}^V\pi^{\bar k} \equiv \pi^{\bar k \bar l}{}_{\bar l} \ ,$$
$${}^T\pi^{\bar k \bar l \bar m} = \pi^{\bar k(\bar l \bar m)} - \tfrac{1}{2}{}^V\pi^{\bar k}\eta^{\bar l \bar m} + \tfrac{1}{2}\eta^{\bar k(\bar l\,V}\pi^{\bar m)} \ . \tag{A15}$$

The sum $\pi^{\bar k \bar l \bar m} R_{\bar k \bar l \bar m}$ in (A9) can be written as

$$\pi^{\bar k \bar l \bar m} R_{\bar k \bar l \bar m} = \tfrac{4}{3}{}^T\pi^{\bar k \bar l \bar m\,T} R_{\bar k \bar l \bar m} + {}^V\pi^{\bar k\,V} R_{\bar k} - \tfrac{1}{6}{}^P\pi^P R \ . \tag{A16}$$

Note that the tensor part satisfies the useful cyclic identity

$${}^T\pi_{\bar k \bar l \bar m} + {}^T\pi_{\bar m \bar k \bar l} + {}^T\pi_{\bar l \bar m \bar k} \equiv 0 \ . \tag{A17}$$

The Lorentz connection field $A^{ij}{}_{\bar k}$, defined by Eq. (5.4), can be decomposed analogously as $\pi_{ij}{}^{\bar k}$. Note that passing to parallel components is not a canonical transformation as n_k does not commute with $\pi_\perp{}^{\bar k}$. Still, the following "basic" Poisson brackets within the set $\{A^{ij}{}_{\bar k}, \pi_{ij}{}^{\bar k}\}$ holds:

$$\{A^{ij}{}_{\bar k}, \pi'_{mn}{}^{\bar l}\} = 2\delta^i{}_{[m}\delta^j{}_{n]}\delta^{\bar l}{}_{\bar k}\delta(\vec{x} - \vec{x}\,') \ , \tag{A18}$$

from which it is clear that the brackets between irreducible components of $A^{ij}{}_{\bar k}$ and $\pi_{mn}{}^{\bar l}$ are nontrivial if both of them are of the same spin and parity, for example,

$$\{{}^T A_{\bar k \bar l \bar m}, {}^{T'}\pi^{\bar n \bar p \bar q}\} = \tfrac{3}{2}{}^T\delta^{\bar n \bar p \bar q}_{klm}\delta(\vec{x} - \vec{x}\,') \ ,$$
$${}^T\delta^{\bar n \bar p \bar q}_{klm} \equiv \delta^{\bar n}{}_k \delta^{\bar p}{}_{(l}\delta^{\bar q}{}_{m)} - (\text{all necessary traces}) \ . \tag{A19}$$

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[33]The cosmological constant is neglected in this paper, as it plays a trivial role from the point of view of the Dirac method. However, it can easily be included.
[34]The geometrical meaning of the set $\{n_k, h_{\bar{k}}{}^\alpha; N, N^\alpha\}$ is obvious from the relations

$$\vec{u}_k = n_k \vec{n} + h_{\bar{k}}{}^\alpha \vec{e}_\alpha, \quad \vec{e}_0 = N\vec{n} + N^\alpha \vec{e}_\alpha ,$$

where $\vec{u}_k$ ($\vec{e}_\alpha$) denotes the basis coordinate vectors in the local Lorentz frame (general coordinate frame).
[35]Note that n_0, in Eq. (3.20), denotes the first component of the normal in the local Lorentz frame.
[36]The relation between the special-relativistic Hamiltonian $\mathcal{H}^{M'} = \mathcal{H}^{M'}(u, \partial_a u, \pi)$ [which corresponds to the Lagrangian (2.1)] and $\mathcal{H}^M{}_\perp$ is not trivial. In the time gauge, $\mathcal{H}^{M'}$ and $\mathcal{H}^M{}_\perp$ are related by the "minimal substitution" rule $\mathcal{H}^M{}_\perp = J\mathcal{H}^{M'}(u, D_a u, \pi/J)$, as shown in Ref. 21.
[37]Although the torsion can be expressed as a function of the covariant derivative $\nabla_\mu b^k{}_\nu$, we cannot directly apply the results obtained for the matter field part, since $\nabla_\mu b^k{}_\nu$ depends also on $b^k{}_0$, contrary to $\nabla_\mu u$. Furthermore, the curvature tensor cannot be expressed as a function which depends only on $\nabla_\mu A^{ij}{}_\nu$, at all. Thus, we have to investigate the gravitational sector separately.
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[40]Equations (4.3) and (4.17) can be most easily verified if one defines

$$\beta_{ijk}(T) \equiv \beta_{ijk\,|\,lmn} T^{lmn} ,$$

where $\beta_{ijk\,|\,lmn} = \beta_{lmn\,|\,ijk}$ without loss of generality.
[41]In Ref. 21, generalized momenta are called if-constraints. Our terminology is supposed to be more suitable for a discussion of the consistency conditions of the possible primary constraints, which are called if-constraints in this paper. Of course, they have nothing to do with generalized momenta of classical mechanics.
[42]One may, of course, try to solve the system (4.5) with respect to the velocities $\dot{b}^k{}_\alpha$ in a straightforward manner, but will find it a problem even to calculate the 12×12 determinant of the system. Thus, it is more suitable to "change the variables," i.e., to put the system into a block-diagonal form (4.6)—(4.9), exploiting the symmetry of the theory.
[43]It has been shown in Ref. 25 that the elimination of π and $\bar{\pi}$ leads to simpler basic Dirac brackets in the time gauge.
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