Perfect-fluid higher-dimensional cosmologies

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A range of solutions to Einstein's equations in $1+d+D$ dimensions is presented for a variety of perfect-fluid energy-momentum tensors. The techniques used to obtain these solutions are explained at some length. Solutions with D dimensions eventually collapsing and d expanding are singled out and studied in detail. It is explained how the higher-dimensional universe passes into a $(1+d)$-dimensional Friedmann-Robertson-Walker phase. The modifications in the thermal history of the universe are traced. It is found that this altered scenario offers a resolution to the horizon problem.

I. INTRODUCTION

The idea that the universe we see today is only part of a higher-dimensional manifold, of which the nonvisible section is too small to be resolved at currently available energies, leads in a simple and natural way to the unification of gauge and gravitational interactions.[1] One merely writes the Einstein-Hilbert action for the extended space-time and reduces this down to $1+3$ dimensions by integrating over the extra space-time variables. The same idea permits a simple derivation of the $N=8$ supergravity Lagrangian.[2] Here one begins by putting down the $N=1$ supergravity Lagrangian in 11 dimensions and performs on this an appropriate dimensional reduction.

Within this unification scheme, internal symmetries are seen to originate in the space-time symmetries associated with the extra dimensions. Gauge invariance thus assumes the same status as space-time invariance while internal quantum numbers such as electric charge are brought onto the same footing as energy and momentum (i.e., they are seen to result from symmetry motions in the extra dimensions).

Its aesthetic appeal notwithstanding, this approach would amount to little more than a mathematical trick if the compact dimensions it postulates do not have an actual physical existence. Unfortunately, resolving these dimensions at currently available energies seems to be out of the question. Indeed, arguments linking the gravitational constant G to the gauge coupling constants and the size of the compact dimensions[3] suggest that the latter are at most a few orders of magnitude larger than the Planck length.

However, it is possible to look for effects of these extra dimensions[4] in the very early phases of the universe, for if we evolve the Friedmann-Robertson-Walker (FRW) universe back towards the big-bang singularity, we eventually reach energies at which the extra dimensions become resolvable and, in fact, come onto the same footing as the standard dimensions.

The Kaluza-Klein view of world geometry thus implies that the universe started out in a higher-dimensional phase with some dimensions eventually collapsing and stabilizing at a size close to the Planck length while three others continued to expand and are still doing so. It is precisely such a scenario[5] that will be elaborated in this paper.

The models we have in mind can be explained rather simply in terms of various two-dimensional analogs. For example, a multidimensional cosmology which collapses into a closed FRW universe with a compact S^n at each point can be visualized by considering a torus, $S_1^{(1)}(r)\times S_1^{(2)}(R)$, where $S_1^{(2)}$, say, refers to the cross-sectional circle and r and R are the respective radii. The surface of this torus is two-dimensional and, to a creature confined to it, represents the entire spatial universe. Now let this torus evolve in time in such a way that r expands while R contracts. In time, the cross section becomes unresolvable which happens for a radiation-dominated universe when the temperature T of the radiation drops below $1/R$. The torus then effectively collapses into an expanding one-dimensional ring with a compact S^1 at each point. This compact space plays no further dynamical role, but will need to be stabilized at the size of the compact dimensions today. Any creature which comes into existence after the dimensional collapse has taken place evolves as a one-dimensional being with a one-dimensional geometric intuition, little suspecting that the universe passed through a phase in which it had a larger number of spatial dimensions.

The two-dimensional analog for an open universe is likewise a cylinder whose cross section collapses to a miniscule size while the distance between any two points along the long direction increases.

As generalizations of these two-dimensional pictures, we consider spatially homogeneous universes whose surfaces of homogeneity consist of direct products of two subspaces:

$$R^3_{(r)}\times S^D_{(R)} \text{ or } S^3_{(r)}\times S^D_{(R)} \,.$$

We furthermore assume these universes to be filled with perfect fluids. The question then is: Do Einstein's equations for these $(1+3+D)$-dimensional universes predict scenarios of the form described, i.e., can we find solutions in which R eventually collapses while r continues to increase?

It is the aim of this paper to provide a fairly complete answer to this question for the first of the topologies mentioned above, viz, $R^3 \times S^D$. The $S^3 \times S^D$ case will be treated in a separate publication.

The paper is organized as follows. In Sec. II, the problem is set up, while in Sec. III A we solve, as a warm-up exercise, the relevant Einstein equations for no curvature in either of the spaces involved in the direct product but for a general perfect-fluid energy-momentum tensor. In Secs. III B and IV the equations are solved for positive curvature in the compact space. The dimensionality, D, of the latter is left arbitrary. Solutions are presented for perfect fluids described by the following equations of state: (a) $p=\rho=0$ (vacuum), (b) $p=\rho$ (Zeldovich equation of state), (c) $\rho=(3+D)p$ (pure radiation), and (d) $p=0, \rho\neq 0$ (pure "dust"). Section V contains a discussion of the horizon problem in the context of these models, while Sec. VI consists of a summary and a set of conclusions. (A reader interested only in the physics and not in all the technicalities of the underlying solutions can skip Secs. III and IV with no essential loss in continuity).

II. THE RELEVANT EQUATIONS OF MOTION

A complete delineation of the higher-dimensional origins of the universe would, ideally speaking, begin with an isotropically expanding space, which for some dynamical reason splits into a direct product of two subspaces at least one of which is three dimensional. Such a "fissioning" of the universe with its attendant change in topology would almost certainly require us to go beyond the realm of general relativity and of Riemannian geometry. In view of this, we shall, in this paper, pick up the universe's evolution at a stage at which the fissioning has already occurred.

Our ansatz for the metric accordingly is

$$g_{\mu\nu}=\begin{bmatrix} 1 & & \\ & -r^2 g_{mn} & \\ & & -R^2 g_{MN} \end{bmatrix}, \tag{2.1}$$

where the g_{mn} and g_{MN} spaces will be assumed d and D dimensional, respectively. d will eventually be set equal to 3, but the number, D, of the extra dimensions, all of which have been assumed spatial, will be left arbitrary and will be treated as a parameter of the model. Lastly, for convenience, we shall set the total number of spatial dimensions, $d+D$, equal to n.

We shall find that this direct-product topology persists as we evolve backwards in time all the way to a cosmological singularity. This is to be expected since we shall be working entirely within the framework of general relativity.

As for the matter content of the universe in its higher-dimensional phase, we shall be most interested in the cases of pure radiation and of dust, both of which are perfect fluids. We therefore take our energy-momentum tensor to correspond to a perfect fluid in n dimensions:

$$T_{\mu\nu}=-pg_{\mu\nu}+(p+\rho)u_\mu u_\nu , \tag{2.2}$$

where $u_0=1$, $u_i=0$ $(i=1,\ldots,n)$, and p will be assumed related to ρ by

$$p=(\gamma-1)\rho . \tag{2.3}$$

The cases of special interest then are (1) $\gamma=(n+1)/n$, corresponding to pure radiation, consisting of fermions which have to be put in by hand in Kaluza-Klein theories and possibly of some higher-dimensional graviton modes, and (2) $\gamma=1$, which represents a pressureless fluid. In considering this value we shall be covering for the possibility that, owing to the production of some superheavy particles, the early universe may temporarily become matter dominated. In addition to these we shall consider the value $\gamma=2$, i.e., $p=\rho$, which describes "stiff" matter and the vacuum $T_{\mu\nu}=0$, for both of which the equations of motion can be solved exactly.

Having made definitive choices, for both $g_{\mu\nu}$ and $T_{\mu\nu}$, we can put down Einstein's equations, which for arbitrary n take the form

$$R_{\mu\nu}=-8\pi\bar{G}\left[T_{\mu\nu}-\frac{g_{\mu\nu}T^\lambda_\lambda}{(n-1)}\right]. \tag{2.4}$$

Here $\bar{G}$ is the gravitational constant appropriate to n dimensions. It has dimensions of $(\text{length})^{n-1}$ and is related to the (1 + 3)-dimensional constant G by

$$\bar{G}=GV_c \tag{2.5}$$

where V_c is the volume of the compact space today. For our choice of S^D for the latter

$$V_c=\frac{2\pi^{(D+1)/2}}{\Gamma((D+1)/2)}R_c{}^D . \tag{2.6}$$

The actual value of R_c (or, equivalently that of $M_{\text{KK}}=R_c{}^{-1}$) will be treated as a parameter of the theory.

Written out explicitly Eqs. (2.4) read

$$d\frac{\ddot{r}}{r}+D\frac{\ddot{R}}{R}=-\frac{8\pi\bar{G}(n\gamma-2)}{(n-1)}\frac{\alpha}{\Sigma^{n\gamma}} , \tag{2.7a}$$

$$\frac{k_d}{r^2}+\frac{d}{dt}\left[\frac{\dot{r}}{r}\right]+\left[d\frac{\dot{r}}{r}+D\frac{\dot{R}}{R}\right]\left[\frac{\dot{r}}{r}\right]=8\pi\bar{G}\frac{(2-\gamma)}{(n-1)}\frac{\alpha}{\Sigma^{n\gamma}} , \tag{2.7b}$$

$$\frac{k_D}{R^2}+\frac{d}{dt}\left[\frac{\dot{R}}{R}\right]+\left[d\frac{\dot{r}}{r}+D\frac{\dot{R}}{R}\right]\left[\frac{\dot{R}}{R}\right]=8\pi\bar{G}\frac{(2-\gamma)}{(n-1)}\frac{\alpha}{\Sigma^{n\gamma}} , \tag{2.7c}$$

where we have set $\rho=\alpha/\Sigma^{n\gamma}$. α is a constant with dimensions of $(\text{length})^{n(\gamma-1)-1}$. Σ^n is the total (n-dimensional) volume ($\Sigma^n=r^dR^D$), and k_d, k_D are the curvatures of the d- and D-dimensional spaces, respectively.

As mentioned in Sec. I, we shall in this paper be primarily studying the topology $R^3\times S^D$, which corresponds to $k_d=0$. In this case we can always rescale r and R to make $k_D=+1$, 0, or -1. Of these values we shall in turn be interested principally in $k_D=+1$. Furthermore, since the absence of curvature eliminates any natural length scale from the corresponding space, we must explicitly introduce one by focusing on a region of arbitrary size at a fixed but arbitrary value of some convenient variable such as R or the temperature, T. For a region of a different size we must scale all relevant quantities by the ratio, r_0, of the latter to the one explicitly picked out above, the comparison being made at the chosen value of R or T.

For clarity, we shall write r_0 explicitly in all our expressions. Lastly, we shall find it convenient to work in terms of a quantity, A, which has dimensions of length and is given by

$$\frac{8\pi\overline{G}}{(n-1)}\rho=\frac{8\pi\overline{G}}{(n-1)}\frac{\alpha}{\Sigma^{n\gamma}}=r_0{}^{d\gamma}\frac{A^{n\gamma-2}(r_0=1)}{\Sigma^{n\gamma}} . \tag{2.8}$$

The physical significance of A will emerge clearly as the scenario is further elaborated (cf. Sec. VB).

In Secs. III and IV we shall solve the set of equations defined by (2.7) and (2.8) for a range of situations.

III. SOME EXACT SOLUTIONS

In this section, we shall solve equations (2.7) and (2.8) for $k_D=0$, $T_{\mu\nu}=0$, and $\gamma=2$. In each of these cases the equations can be integrated exactly. Although the solutions so obtained are not of much interest physically, working them through will nevertheless be useful in solving these equations for more realistic situations.

A. $k_D=0$

We shall work in terms of dimensionless quantities:

$$dt=A\,d\tau , \tag{3.1a}$$

$$r=r_0Ar(\tau) , \tag{3.1b}$$

$$R=R_0A\mathscr{R}(\tau) , \tag{3.1c}$$

$$\Sigma=R_0{}^{D/n}r_0{}^{d/n}A\sigma(\tau) . \tag{3.1d}$$

Substituting these into Eqs. (2.7) and (2.8), we get

$$d\frac{r''}{r}+D\frac{\mathscr{R}''}{\mathscr{R}}=-\frac{(n\gamma-2)}{\sigma^{n\gamma}} , \tag{3.2a}$$

$$\frac{d}{d\tau}\left[\frac{r'}{r}\right]+\left[d\frac{r'}{r}+D\frac{\mathscr{R}'}{\mathscr{R}}\right]\left[\frac{r'}{r}\right]=\frac{(2-\gamma)}{\sigma^{n\gamma}} , \tag{3.2b}$$

$$\frac{d}{d\tau}\left[\frac{\mathscr{R}'}{\mathscr{R}}\right]+\left[d\frac{r'}{r}+D\frac{\mathscr{R}'}{\mathscr{R}}\right]\left[\frac{\mathscr{R}'}{\mathscr{R}}\right]=\frac{(2-\gamma)}{\sigma^{n\gamma}} . \tag{3.2c}$$

Equations (3.2b) and (3.2c), together with

$$d\frac{r'}{r}+D\frac{\mathscr{R}'}{\mathscr{R}}=\frac{\sigma^{n'}}{\sigma^n} \tag{3.3}$$

yield an equation for σ^n alone:

$$\frac{d}{d\tau}\left[\frac{\sigma^{n'}}{\sigma^n}\right]+\left[\frac{\sigma^{n'}}{\sigma^n}\right]^2=\frac{\sigma^{n''}}{\sigma^n}=\frac{n(2-\gamma)}{\sigma^{n\gamma}} \tag{3.4}$$

whence

$$\tau=\int_0^{\sigma}\frac{\sigma^{n-1}d\sigma}{(\sigma^{n(2-\gamma)}+\sigma_0{}^{n(2-\gamma)})^{1/2}} . \tag{3.5}$$

Equations (3.2b) and (3.2c) can now be individually integrated to give

$$\frac{r'}{r}=\frac{1}{\sigma^n}\left[(\sigma^{n(2-\gamma)}+\sigma_0{}^{n(2-\gamma)})^{1/2}\pm\left[\frac{(n-1)D}{d}\right]^{1/2}\sigma_0{}^{n(2-\gamma)/2}\right] , \tag{3.6a}$$

$$\frac{\mathscr{R}'}{\mathscr{R}}=\frac{1}{\sigma^n}\left[(\sigma^{n(2-\gamma)}+\sigma_0{}^{n(2-\gamma)})\mp\left[\frac{(n-1)d}{D}\right]^{1/2}\sigma_0{}^{n(2-\gamma)/2}\right] , \tag{3.6b}$$

where we have used Eqs. (3.2a) and (3.3) to fix the constants of integration to be $\pm[(n-1)D/d]^{1/2}\sigma_0{}^{n(2-\gamma)/2}$ and $\mp[(n-1)d/D]^{1/2}\sigma_0{}^{n(2-\gamma)/2}$, respectively. A further integration turns Eqs. (3.6) into

$$\frac{r}{r_0}=\sigma^{1\mp[(n-1)D/d]^{1/2}}[(\sigma^{n(2-\gamma)}+\sigma_0{}^{n(2-\gamma)})-\sigma_0{}^{n(2-\gamma)/2}]^{\pm2[(n-1)D/d]^{1/2}/n(2-\gamma)} , \tag{3.7a}$$

$$\frac{\mathscr{R}}{\mathscr{R}_0}=\sigma^{1\pm[(n-1)d/D]^{1/2}}[(\sigma^{n(2-\gamma)}+\sigma_0{}^{n(2-\gamma)})-\sigma_0{}^{n(2-\gamma)/2}]^{\mp2[(n-1)d/D]^{1/2}/n(2-\gamma)} . \tag{3.7b}$$

We can now look at these solutions for various limiting values of τ or equivalently of σ. For concreteness, we shall stick to the first of the solutions above and will set $r_0=\mathscr{R}_0=1$ and $d=3$ where a specific choice is called for. Since $[(n-1)D/d]^{1/2}$, $[(n-1)d/D]^{1/2}>1$ for $d>1,D>0$, we have as $\sigma\to0$,

$$r\to\text{const}\times\sigma^{1+[(n-1)D/d]^{1/2}}\to0 , \tag{3.8a}$$

$$\mathscr{R}\to\text{const}\times\sigma^{1-[(n-1)d/D]^{1/2}}\to\infty , \tag{3.8b}$$

$$r'\to\text{const}\times\left[1+\left[\frac{(n-1)D}{d}\right]^{1/2}\right]\sigma^{-(n-1)\{1-[D/(n-1)d]^{1/2}\}}\to\infty , \tag{3.8c}$$

$$\mathscr{R}'\to\text{const}\times\left[1-\left[\frac{(n-1)d}{D}\right]^{1/2}\right]\sigma^{-(n-1)\{1+[d/(n-1)D]^{1/2}\}}\to-\infty , \tag{3.8d}$$

while for $\sigma\to\infty$,

$$r,\mathscr{R}\to\sigma\to\infty , \tag{3.9a}$$

$$r',\mathscr{R}'\to\sigma^{1-n\gamma/2}\to0 . \tag{3.9b}$$

Thus, the universe begins from a "line-like"[6] singularity. The scale factor, r, for the d-dimensional space starts out from zero with an infinite slope and increases monotonically for all values of τ, while that for the D-dimensional space decreases from infinity [with $R'(0)=-\infty$], reaches a minimum at

$$\sigma_{\min}=(2n/D)^{1/n(2-\gamma)}\sigma_0$$

(for $d=3$), reverses and then begins to increase. At a much later stage, the universe expands isotropically with an ever-decreasing expansion rate.

B. $T_{\mu\nu}=0$ (vacuum) and $\gamma=2, D>1$

As we shall see below both these cases can be treated simultaneously.

The substitutions

$$dt=A\mathscr{R}(\tau)d\tau\,, \tag{3.10a}$$

$$r(t)=r_0Ar(\tau)\,, \tag{3.10b}$$

$$R(t)=A\mathscr{R}(\tau)\,, \tag{3.10c}$$

$$\Sigma^n(t)=r_0{}^dA^n\mathscr{R}^D(\tau)r^d(\tau)=r_0{}^dA^n\sigma^n(\tau)\,, \tag{3.10d}$$

turn Eqs. (2.7) into

$$\frac{d}{d\tau}\left[\frac{r^{d'}}{r^d}\right]-\frac{1}{(D-1)}\frac{\mathscr{R}^{D-1'}}{\mathscr{R}^{D-1}}\frac{r^{d'}}{r^d}+\frac{1}{d}\frac{r^{d'}}{r^d}+\frac{D}{(D-1)}\frac{d}{d\tau}\left[\frac{\mathscr{R}^{D-1'}}{\mathscr{R}^{D-1}}\right]=\begin{cases}0, & \text{vacuum}\\ -\dfrac{2(n-1)}{(r^d)^2(\mathscr{R}^{D-1})^2}, & \gamma=2\,,\end{cases} \tag{3.11a}$$

$$\frac{r^{d''}}{r^d}+\frac{\mathscr{R}^{D-1'}}{\mathscr{R}^{D-1}}\frac{r^{d'}}{r^d}=0\,, \tag{3.11b}$$

$$(D-1)+\frac{\mathscr{R}^{D-1'}}{\mathscr{R}^{D-1}}\frac{r^{d'}}{r^d}+\frac{\mathscr{R}^{D-1''}}{\mathscr{R}^{D-1}}=0\,. \tag{3.11c}$$

From Eqs. (3.11b) and (3.11c) it immediately follows that

$$\frac{d}{d\tau}(r^{d'}\mathscr{R}^{D-1})=0 \tag{3.12a}$$

and

$$\frac{d^2}{d\tau^2}(r^d\mathscr{R}^{D-1})=-(D-1)r^d\mathscr{R}^{D-1}\,, \tag{3.12b}$$

which in turn implies that

$$r^{d'}\mathscr{R}^{D-1}=c_1 \tag{3.13a}$$

and

$$r^d\mathscr{R}^{D-1}=c_2\sin[\tau(D-1)^{1/2}]\,. \tag{3.13b}$$

Thus,

$$\frac{r^{d'}}{r^d}=\frac{c_1}{c_2\sin\{\tau[(D-1)]^{1/2}\}} \tag{3.14a}$$

and

$$\frac{\mathscr{R}^{D-1'}}{\mathscr{R}^{D-1}}=\frac{(r^d\mathscr{R}^{D-1})'}{r^d\mathscr{R}^{D-1}}-\frac{r^{d'}}{r^d}=\frac{(D-1)^{1/2}\cos\{\tau[(D-1)]^{1/2}\}-(c_1/c_2)}{\sin\{\tau[(D-1)]^{1/2}\}}\,. \tag{3.14b}$$

The substitution of Eqs. (3.14) into Eq. (3.11a) produces a relation between the constants of integration,

$$D+\left[\frac{c_1}{c_2}\right]^2\left[\frac{(d-1)}{d}-\frac{D}{(D-1)}\right]=0,\quad\text{vacuum} \tag{3.15a}$$

$$D+\left[\frac{c_1}{c_2}\right]^2\left[\frac{(d-1)}{d}-\frac{D}{(D-1)}-\frac{2(n-1)}{c_1{}^2}\right]=0\,,\quad \gamma=2\,. \tag{3.15b}$$

Finally, we can further integrate Eq. (3.14a) to get

$$r^d=c_3\tan^{c_1/\{c_2[(D-1)]^{1/2}\}}[(D-1)^{1/2}\tau/2]\,, \tag{3.16}$$

which, together with Eq. (3.13b), implies that

$$\mathscr{R}^{D-1}=\frac{c_2}{c_3}\frac{\sin\{\tau[(D-1)]^{1/2}\}}{\tan^{c_1\{c_2[(D-1)]^{1/2}\}}[(D-1)^{1/2}\tau/2]}\,. \tag{3.17}$$

We note that for both the vacuum and for $\gamma=2$ we have only two independent constants of integration. This is to be expected since, of the four constants corresponding to the two coupled second-order differential Eqs. (2.7b) and (2.7c), one is fixed by Eq. (2.7a) and a second by requiring that $\tau=0$ correspond to the initial singularity.

The properties of these solutions can now be easily delineated. From Eqs. (3.15) it is clear that $a=c_1/\{c_2[(D-1)]^{1/2}\}$ can be either positive or negative. We shall examine in detail only the physically more relevant case $a>0$. (The value $a<0$ can be treated in a similar manner.)

For $\gamma=2$, we can have $a>1$, $a=1$, or $a<1$.

If $a>1$, the universe begins in a line-like singularity $R\to\infty, r\to 0$. The extra dimensions then contract monotonically while the normal ones continuously expand until the universe terminates at $\tau_0=\pi/(D-1)^{1/2}$ in the line-like singularity $R=0$, $r\to\infty$.

For $a=1$, R shrinks from $(2c_2/c_2)^{1/(D-1)}$ at $\tau=0$ to zero at $\tau=\tau_0$, while r behaves as before.

Finally for $a<1$, R expands from the "point-like" singularity $r=R=0$ at $\tau=0$ to a certain maximum value and then shrinks to zero at $\tau_0=\pi/(D-1)^{1/2}$. The qualitative behavior of r once again remains unaltered.

In all three cases $\Sigma\to 0$ at both singularities and the universe lives for a time

$$t_{\text{univ}}=A\int_0^{\pi/(D-1)^{1/2}}\mathscr{R}\,d\tau=\frac{A}{(D-1)^{1/2}}B\left[\frac{1}{2}\frac{(D-a)}{(D-1)},\frac{1}{2}\frac{(D+a)}{(D-1)}\right]\,, \tag{3.18}$$

where B is the β function.

Finally, for the vacuum $d>a>1$, and thus, here only the first of the above behaviors is possible.

It should be mentioned that Eqs. (2.7) can be solved with equal ease for $D=1$. For completeness we give the solution below:

$$dt = A\mathscr{R}(\tau)d\tau\ , \tag{3.19a}$$

$$\left[\frac{r}{r_0}\right]^d = \tau\ , \tag{3.19b}$$

$$\mathscr{R} = \mathscr{R}_0 \tau^c e^{-\tau^2/4}\ , \tag{3.19c}$$

where

$$c = \begin{cases} \frac{1}{2}\left[\frac{1-d}{d}\right], & \text{vacuum} \quad (3.19\text{d}) \\ \frac{1}{2}\left[\frac{1-d}{d}\right] + \frac{(n-1)}{r_0^{2d}}, & \gamma=2 \quad (3.19\text{e}) \end{cases}$$

and $r_0, \mathscr{R}_0$ are constants.

IV. $\gamma=(n+1)/n$ (PURE RADIATION) AND $\gamma=1$ (DUST)

For both these cases, it is rather difficult to solve Eqs. (2.7) exactly. However, the following alternative strategy accords us all the information that an exact solution would have provided.

We begin by putting down a power-series expansion for the solution at $\tau=0$. The expansion can be made as accurate as desired by going to a sufficiently high order in τ. We shall find that exactly two coefficients (which correspond to the two independent constants of integration in all exact solutions) are left undetermined by the equations of motion. On giving specific values to these arbitrary coefficients, we get a complete set of initial data which can then be integrated out to a second singularity at τ_0 with the help of any standard numerical integration routine. Finally, we make a second expansion at $\tau=\tau_0$. Once again we find that some of the coefficients are not determined by Eqs. (2.7). For a given choice of initial data at $\tau=0$, these coefficients are, however, fixed and a program can easily be written to match the coefficients at the two ends.

We thus have a numerical solution away from the singularities and power-series expansions at the latter, where, of course, a computer is not too convenient a tool of study.

Let us implement this strategy in sequence for the values $\gamma=(n+1)/n$ and $\gamma=1$.

A. $\gamma=(n+1)/n$

It proves expedient to make the following set of substitutions:

$$\Sigma(\tau) = r_0^{d/n} A\sigma(\tau)\ , \tag{4.1a}$$

$$\mathrm{r}(\tau) = r_0 A r(\tau)\ , \tag{4.1b}$$

$$dt = \left[\frac{n}{n-1}\right]^{1/2} A\sigma(\tau)d\tau\ , \tag{4.1c}$$

$$\frac{d}{dt} \to \left[\frac{n-1}{n}\right]^{1/2} \frac{1}{A\sigma(\tau)}\frac{d}{d\tau}\ ,\qquad \frac{d^2}{dt^2} \to \frac{(n-1)}{n}\frac{1}{A^2\sigma^2}\frac{d^2}{d\tau^2}\ . \tag{4.1d}$$

Equation (2.7b) then reads

$$\frac{d}{d\tau}\left[\frac{r'}{r}\right] + \frac{\sigma^{n-1'}}{\sigma^{n-1}}\frac{r'}{r} = \frac{1}{\sigma^{n-1}}\ , \tag{4.2}$$

which can be immediately integrated to give

$$\frac{r'}{r} = \frac{\tau+k}{\sigma^{n-1}}\ , \tag{4.3}$$

where k is a constant of integration. Equation (2.7c) gives

$$\frac{n}{(n-1)}\frac{\Sigma^2}{R^2} + \frac{d}{d\tau}\left[\frac{R'}{R}\right] + \frac{\sigma^{n-1'}}{\sigma^{n-1}}\frac{R'}{R} = \frac{1}{\sigma^{n-1}}\ . \tag{4.4}$$

Adding d times Eq. (4.2) to D times Eqs. (4.4) and writing Σ^2/R^2 as $[r/\sigma]^{2d/D}$, we can eliminate R altogether:

$$D\left[\frac{r}{\sigma}\right]^{2d/D} = \frac{(n-1)-\sigma^{n-1''}}{\sigma^{n-1}}\ . \tag{4.5}$$

Furthermore, by differentiating Eq. (4.5) and using Eq. (4.3), we can derive the following equation containing σ^{n-1} exclusively:

$$\sigma^{n-1}\sigma^{n-1'''} + \left[\left[\frac{2d}{D(n-1)}-1\right]\sigma^{n-1'} - \frac{2d}{D}(\tau+k)\right][\sigma^{n-1''}-(n-1)] = 0\ . \tag{4.6}$$

Similarly by writing the left-hand side of Eq. (2.7a) as

$$d\frac{\ddot{r}}{r} + D\frac{\ddot{R}}{R} = \frac{d}{d\tau}\left[\frac{\dot{\Sigma}^n}{\Sigma^n}\right] + d\left[\frac{\dot{r}}{r}\right]^2 + D\left[\frac{1}{D}\frac{\dot{\Sigma}^n}{\Sigma^n} - \frac{d}{D}\frac{\dot{r}}{r}\right]^2 \tag{4.7}$$

and substituting Eqs. (4.1), we get yet another equation for σ^{n-1}:

$$\sigma^{n-1}\sigma^{n-1''} + \left[\frac{d}{D(n-1)}-1\right](\sigma^{n-1'})^2 - \frac{2d}{D}(\tau+k)\sigma^{n-1'} + (n-1)\sigma^{n-1} + \frac{d(n-1)}{D}(\tau+k)^2 = 0\ . \tag{4.8}$$

It can easily be checked that Eq. (4.8) is the first integral of Eq. (4.6) with the arbitrary constant of integration set equal to $d(n-1)k^2/D$. This is in accord with the general property of Einstein's equations that the $R_{0\mu}$ equations are always consistent with solutions of the R_{ij} equations and act only to restrict the initial data.

Our next step is to expand the solutions to Eq. (4.8) as the power series around $\tau=0$. We get two different power series, one for $k=0$ and another for $k\neq 0$. These two series are not related by any obvious limiting procedure, a situation not uncommon for nonlinear systems of equations.

The series for $k=0$ is

$$\sigma^{n-1}=\frac{(n-1)}{2}\tau^2+a_0\tau^4+a_1\tau^6+a_2\tau^8+a_3\tau^{10}+O(\tau^{12})\,, \tag{4.9}$$

where

$$a_1=\frac{4}{5}\frac{a_0^2}{(n-1)}\left[1-\frac{4d}{D(n-1)}\right], \tag{4.10a}$$

$$a_2=\frac{3}{7}\frac{a_0a_1}{(n-1)}\left[1-\frac{8d}{D(n-1)}\right], \tag{4.10b}$$

$$a_3=\frac{1}{27}\frac{1}{(n-1)}\left[6\left[\frac{6d}{D(n-1)}\right]a_1^2 -4\left[1+\frac{16d}{D(n-1)}\right]a_0a_2\right], \tag{4.10c}$$

etc., while that for $k\neq 0$ is

$$\sigma^{n-1}=c\tau+\frac{(n-1)}{2}\tau^2+\tau^m[a_0+a_1\tau+a_2\tau^2+\cdots]+\tau^{2m-1}[b_0+\cdots]\,, \tag{4.11}$$

where

$$c=\frac{(2d/D)(n-1)k}{2d/D+[4d(n-1)/D]^{1/2}}\,, \tag{4.12a}$$

$$a_1=-\frac{a_0}{c}\frac{(m-1)(m-4)}{2(m+1)}(n-1)\,, \tag{4.12b}$$

$$a_2=-\frac{a_1}{c}\frac{m(m-3)}{4(m+2)}(n-1)\,, \tag{4.12c}$$

$$b_0=-\frac{a_0^2}{c}\frac{m\{md/[D(n-1)]-1\}}{(2m-1)(m-1)}\,, \tag{4.12d}$$

and

$$m=3+\left[\frac{4d}{D(n-1)}\right]^{1/2}. \tag{4.12e}$$

[In both cases, a_0 is left undetermined by Eq. (4.8).]

A numerical integration of Eqs. (4.9) and (4.11) reveals that σ^{n-1} has a second zero at a finite value, τ_0, of τ (cf. Fig. 1, where σ^{n-1} is plotted for some sample values of c and a_0). The precise value of τ_0 depends on the specific choice of the arbitrary coefficients.

As a final step in our program, we put down the expansion for σ^{n-1} at τ_0. In terms of $\xi=\tau_0-\tau$, this reads

$$\sigma^{n-1}=c_1\xi+\frac{(n-1)}{2}\xi^2+c_2\xi^p+c_3\xi^{p+1}+c_4\xi^{p+2}+d_0\xi^{2p-1}+\cdots\,, \tag{4.13}$$

where

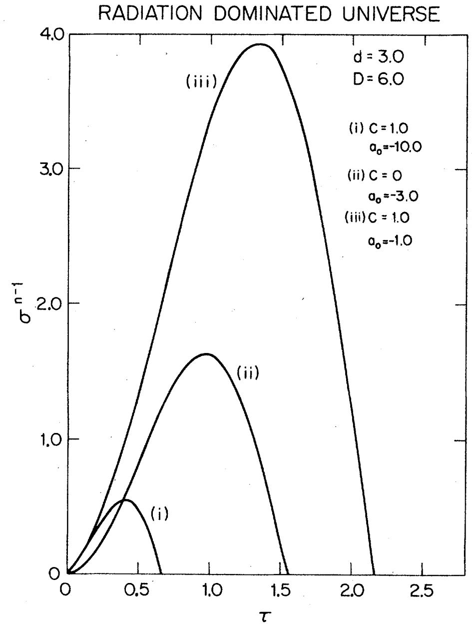

FIG. 1. σ^{n-1} for various values of the arbitrary constants of integration c and a_0, as a function of the modified (dimensionless) time variable τ. (i) $c=1.0$, $a_0=-10.0$, (ii) $c=0$, $a_0=-3.0$, (iii) $c=1.0$, $a_0=-1.0$.

$$c_1=(\tau_0+k)\left[\frac{d}{D}+\left[\frac{d(n-1)}{D}\right]^{1/2}\right]\Big/\left[1-\frac{d}{D(n-1)}\right] =\frac{(\tau_0+k)}{\beta}\,, \tag{4.14a}$$

$$c_3=-\frac{\beta c_2}{(\tau_0+k)}\frac{(n-1)}{2}\frac{(m-1)(m-4)}{(m+1)}\,, \tag{4.14b}$$

$$c_4=-\frac{\beta c_3}{(\tau_0+k)}\frac{(n-1)}{4}\frac{m(m-3)}{(m+2)}\,, \tag{4.14c}$$

$$d_0=-\frac{\beta c_2^2}{(\tau_0+k)}\frac{m[m-D(n-1)/d]}{(2m-1)(m-1)}\frac{d}{D(n-1)}\,, \tag{4.14d}$$

$$p=3-\frac{2d}{D(n-1)}-\frac{2d}{D}\beta\,, \tag{4.14e}$$

etc., and c_2 has to be matched to the choice of a_0 in Eq. (4.9) or (4.11). The expansion for $k=0$ is obtained by simply setting $k=0$ in Eqs. (4.14).

Numerical integration based on the expansions (4.9), (4.11), (4.13), and the program for matching coefficients constitutes a complete set of solutions to the system (2.7) and (2.8) of equations for $\gamma=(n+1)/n$. The results are plotted in Figs. 1–3 for some sample values of the coefficients c and a_0.

It is worth noting that for $d=1, D=2$, $a_i=0$ $(i\geq 1)$ and the expansion (4.11) turns into an exact solution, which on being integrated completely yields[7]

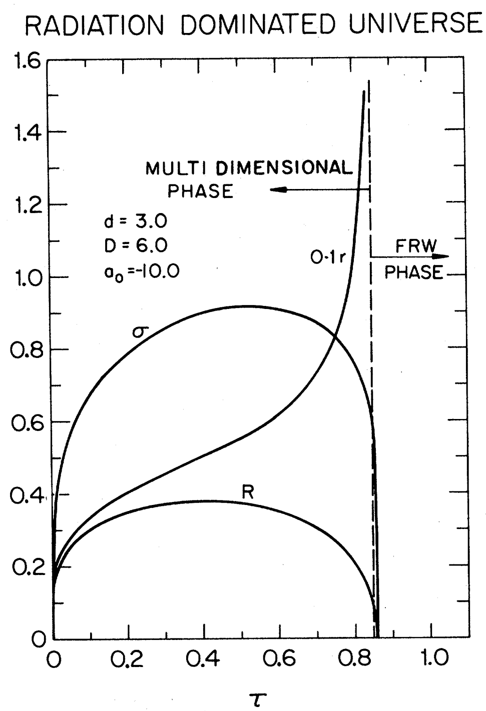

FIG. 2. The behavior of r, the scale factor of the ordinary dimensions, R, the radius of compact ones, and σ, the average scale factor, as functions of τ for $d=3$, $D=6$, $c=0$ ($a_0=-10.0$) and for a universe assumed radiation dominated. This is the first of the two types of behaviors displayed by r. The shapes are generic for the class $c=0$, but the values and slopes at various points depend on d and D. The universe does not follow this solution out to the singularity at τ_0. Instead, at some value of τ, which depends on the parameters of the model and which is indicated by the dashed line, the temperature T falls below $1/R$ and the universe passes from the multidimensional phase to a FRW one, in which it continues to the present day. The behavior of $T \propto 1/\sigma$ can be deduced by inspection: The universe cools from $T=\infty$, reheats until it changes phase, and then cools again but in the FRW manner.

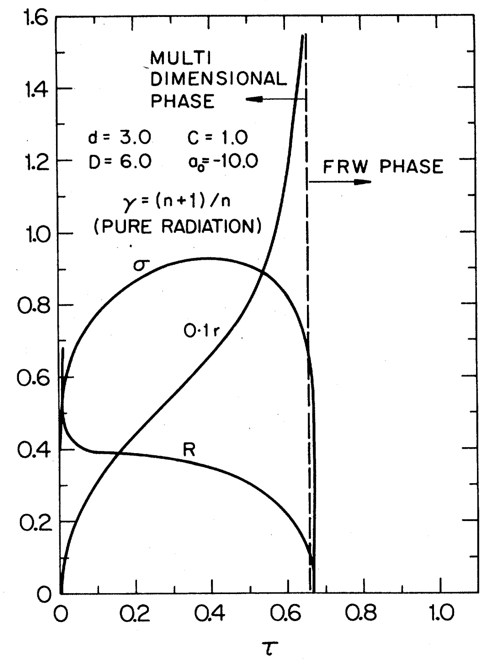

FIG. 3. The second class of solutions. These correspond to $c \neq 0$. Here R decreases from infinity with infinite negative slope. At the dashed lines the universe once again becomes FRW-like. The behavior of r and σ is qualitatively the same as in Fig. 2.

$$\sigma^2 = c\tau + \tau^2 + a_0\tau^4 , \tag{4.15a}$$

$$dt = \left[\frac{3}{2}\right]^{1/2} A\sigma(\tau)d\tau , \tag{4.15b}$$

$$\Sigma = r_0^{1/3} A\sigma(\tau) , \tag{4.15c}$$

$$r = r_0 A\tau^2/\sigma , \tag{4.15d}$$

$$R = A\sigma^2/\tau , \tag{4.15e}$$

$$A^2 = \frac{3\pi F_3}{M_p^2}\left[\frac{S}{F_3}\right]^{4/3} . \tag{4.15f}$$

F_n is the numerical factor linking the entropy density of radiation in an n-dimensional space with its temperature: $S(r_0=1)=\{F_n[\Sigma(r_0=1)T]^n\}$.

It can be easily checked that

$$F_n = \frac{(n+1)}{n}\frac{\Gamma(n+1)\zeta(n+1)}{2^{n-1}\pi^{n/2}\Gamma(n/2)} \times \left[\left[1-\frac{1}{2^n}\right]N_f(n)+N_b(n)\right] , \tag{4.15g}$$

where $N_f(n)[N_b(n)]$ are the number of effectively massless fermionic [bosonic] degrees of freedom in n spatial dimensions.

B. $\gamma=1$

We shall now repeat this entire exercise for $\gamma=1$. We start with the following set of substitutions:

$$dt = A\,d\tau , \tag{4.16a}$$

$$\Sigma(t) = r_0^{d/n} A\sigma(\tau) , \tag{4.16b}$$

$$\mathrm{r}(t) = r_0 A r(\tau) , \tag{4.16c}$$

$$\Sigma^n = r_0^d A^n \sigma^n \Longrightarrow R = A\left[\frac{\sigma^n}{r^d}\right]^{1/D} . \tag{4.16d}$$

Equation (2.7b) then becomes

$$\frac{d}{d\tau}\left[\frac{r'}{r}\right] + \frac{\sigma^{n'}}{\sigma^n}\frac{r'}{r} = \frac{1}{\sigma^n} , \tag{4.17}$$

which integrates to

$$\frac{r'}{r} = \frac{\tau+\kappa}{\sigma^n} , \tag{4.18}$$

where κ is a constant of integration. Equation (2.7c) gives:

$$\frac{r^{2d/D}}{(\sigma^n)^{2/D}}+\frac{d}{d\tau}\left[\frac{R'}{R}\right]+\frac{\sigma^{n'}}{\sigma^n}\frac{R'}{R}=\frac{1}{\sigma^n}\,, \tag{4.19}$$

while Eqs. (4.16) and (4.18) jointly imply

$$D\frac{r^{2d/D}}{(\sigma^n)^{2/D}}=\frac{n-\sigma^{n''}}{\sigma^n}\,. \tag{4.20}$$

Eliminating r between Eqs. (4.18) and (4.20), we get

$$\left[\frac{2d}{D}(\tau+\kappa)+\left[1-\frac{2}{D}\right]\sigma^{n'}\right](n-\sigma^{n''})+\sigma^n\sigma^{n'''}=0\,. \tag{4.21}$$

Equation (2.7a) gives the first integral of Eq. (4.21) with the constant of integration set equal to $dn\kappa^2/D$:

$$\sigma^n\sigma^{n''}+\left[\frac{1}{D}-1\right](\sigma^{n'})^2-\frac{2d}{D}(\tau+\kappa)\sigma^{n'}+\frac{dn}{D}(\tau+\kappa)^2+(n-2)\sigma^n=0\,. \tag{4.22}$$

The solution to Eq. (4.22) has the following power-series expansions at $\tau=0$:

$$\sigma^n=c\tau+\frac{n}{2}\tau^2+a_0\tau^m+a_1\tau^{m+1}+a_2\tau^{m+2}+b_0\tau^{2m-1}+\cdots\,, \tag{4.23}$$

where

$$c=n\kappa/\{1+[1+n(D-1)/d]^{1/2}\}\,, \tag{4.24a}$$

$$m=3-\frac{2}{D}+\frac{2d}{D}\frac{\kappa}{c}\,, \tag{4.24b}$$

$$a_1=-\frac{a_0}{c}\frac{(m-1)}{(m+1)}\left[\frac{nm}{2}-2n+2\right], \tag{4.24c}$$

$$a_2=-\frac{a_1}{c}\frac{m}{2(m+2)}\left[\frac{nm}{2}-\frac{3n}{2}+2\right], \tag{4.24d}$$

$$b_0=\frac{{a_0}^2}{c}\frac{m(1-m/D)}{(m-1)(2m-1)}\,, \tag{4.24e}$$

etc., and for $\kappa=0$:

$$\sigma^n=\frac{n}{2}\tau^2+a_0\tau^m+a_1\tau^{2m-2}+a_2\tau^{3m-4}+\cdots\,, \tag{4.25}$$

where

$$m=4(n-1)/n\,, \tag{4.26a}$$

$$a_1=\frac{{a_0}^2m(1-m/D)}{(2m-3)(n-2)}\,, \tag{4.26b}$$

$$a_2=-\frac{a_0a_1(m-1)[m(1+4/D)-6]}{2(n-2)(3m-5)}\,. \tag{4.26c}$$

In both cases a_0 is not determined by the equations.

Finally, the expansion at τ_0, the second zero of σ^n, in terms of $\xi=\tau_0-\tau$ is

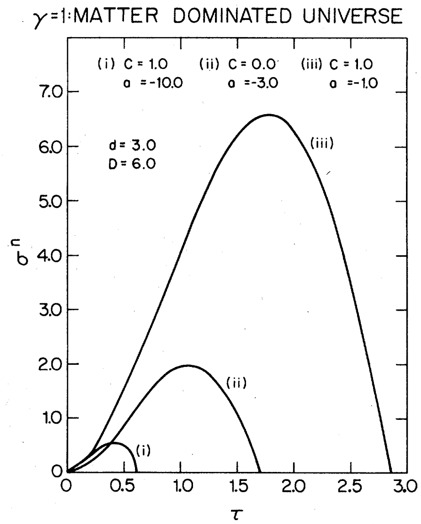

FIG. 4. σ^n for a matter-dominated universe. The arbitrary coefficients have the following values: (i) $c=1.0$, $a_0=-10.0$, (ii) $c=0$, $a_0=-3.0$, (iii) $c=1.0$, $a_0=-1.0$.

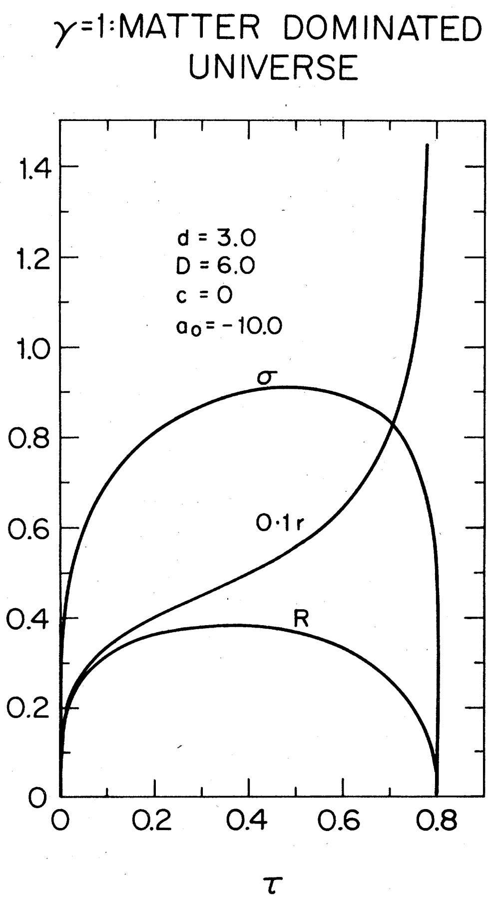

FIG. 5. r, R, and σ for the matter-dominated universe (case 1: $c=0$) as functions of the dimensionless time variable τ. N.B. The behavior of these quantities is qualitatively the same as for the radiation-dominated case (Fig. 2).

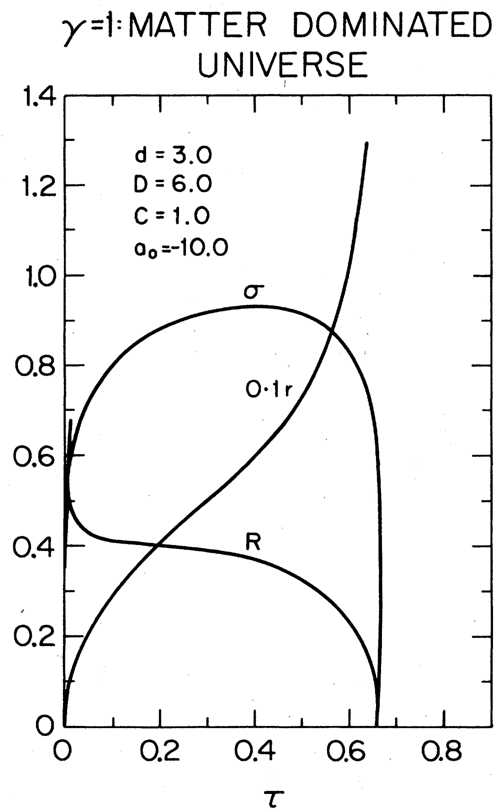

FIG. 6. r, R, and σ for the matter-dominated universe (case 2: $c\neq 0$). Note the similarity to Fig. 3.

$$\sigma^n = c_1\xi + \frac{n}{2}\xi^2 + c_2\xi^p + c_3\xi^{p+1} + \cdots , \tag{4.27}$$

where

$$c_1 = \frac{\alpha(\tau_0+\kappa)}{\beta}\left[1+\left[1+\frac{n(D-1)}{d}\right]^{1/2}\right] = \frac{\tau_0+\kappa}{\beta} , \tag{4.28a}$$

$$c_3 = \frac{-c_2\beta}{(\tau_0+\kappa)}\frac{(p-1)}{(p+1)}\left[\frac{np}{2}-2n+2\right] , \tag{4.28b}$$

$$p = 3 - \frac{2}{D} - \frac{2d}{D}\beta , \tag{4.28c}$$

and c_2 has to be matched to the choice of coefficients at $\tau=0$.

The results of numerically integrating Eqs. (4.22) and (4.24) are displayed in Figs. 4—6. We now have all the solutions we need and can turn to an analysis of their properties.

V. ELABORATION OF THE SCENARIO

A. The qualitative picture

We have seen that the solutions to Eqs. (2.7) and (2.8) for $k_d=0$ have a series of properties which are common to all the energy-momentum tensors we have considered. In particular, the universe begins in a singularity at $\tau=0$ and ends in one at $\tau=\tau_0$. At both singularities, the n-dimensional volume $\Sigma^n \to 0$. If we introduce no mechanism for generating entropy, the evolution of this universe is adiabatic and the temperature T is simply given by

$$T = \frac{[S(r_0)/F_n]^{1/n}}{\Sigma(r_0)} , \tag{5.1}$$

where $S(r_0)$ is the entropy contained in a region characterized by the size factor r_0 and F_n is defined in Eq. (4.15g). The temperature T is then infinite at both singularities.

We, furthermore, have a whole class of solutions in which r, the scale factor for the flat space, starts from 0 at $\tau=0$ and increases monotonically to ∞ at $\tau=\tau_0$.

The radius, R, for the extra compact dimensions, on the other hand, displays one of two possible behaviors. It either begins from ∞ at $\tau=0$ and shrinks monotonically to 0 at $\tau=\tau_0$, or it begins from $R=0$ at $\tau=0$, and increases to a certain maximum value before collapsing back to zero at $\tau=\tau_0$.

Having convinced ourselves that these behaviors are generic, we can now specialize our considerations to a radiation-dominated universe. In doing so, we shall not follow the relevant solutions all the way into the singularity at $\tau=\tau_0$. Recall that Eqs. (2.7) were obtained on the assumption that the pressure is isotropic in all n dimensions. This is so only as long as the size of each and every dimension is large compared to the propagation wavelength of the radiation, i.e., large compared to $1/T$. As we shall soon see, close to $\tau=\tau_0$, RT becomes equal to one, and this assumption clearly breaks down.

Since $1/R$ is the minimum energy required to excite a mode in the curved compact space, $T<1/R$ implies that the temperature has dropped below the threshold for propagating any particle in the compact space. If there is no conservation law preventing the decay of these excitations,[8] they will dump all their energy into lighter particles, i.e., particles propagating exclusively along the flat directions. Henceforth, the extra dimensions will no longer sustain modes and will for all practical purposes, assume the role of spectators. We shall, however, require that they stabilize at some value close to R_{deex} so as not to cause a conflict with the current limits[9] on the variation of G. The precise mechanism for achieving this stability is a topic for future research. Casimir energies computed in the presence of fermions or zero-point gravitational energy effects could conceivably solve the problem.[10] At this point, we shall merely note that if this stabilizing mechanism generates a constant effective pressure,[11] $\lambda=1/R_c{}^2$, for the compact space, the universe will make a transition to a (1 + d)-dimensional FRW cosmology. Indeed, beyond the point at which the extra dimensions "deexcite," the energy-momentum tensor changes and the equations of motion read

$$d\frac{\ddot{r}}{r} + D\frac{\ddot{R}}{R} = 8\pi G\left[\frac{R_c}{R}\right]^{n-3}\rho^{(d)} , \tag{5.2a}$$

$$\frac{d}{dt}\left[\frac{\dot{r}}{r}\right]+\left[d\frac{\dot{r}}{r}+D\frac{\dot{R}}{R}\right]\left[\frac{\dot{r}}{r}\right]=8\pi G\left[\frac{R_c}{R}\right]^{n-3}p^{(d)}, \tag{5.2b}$$

$$\frac{1}{R^2}+\frac{d}{dt}\left[\frac{\dot{R}}{R}\right]+\left[d\frac{\dot{r}}{r}+D\frac{\dot{R}}{R}\right]\left[\frac{\dot{R}}{R}\right]=\lambda=\frac{1}{R_c{}^2}, \tag{5.2c}$$

where $p^{(d)}$ and $\rho^{(d)}$ refer to the d-dimensional pressure and density, respectively ($\rho^{(d)}=dp^{(d)}$) since the radiation is now effectively confined to a (1 + d)-dimensional space. Setting $R=R_c,\dot{R}=0$ these equations clearly reduce to those for a d-dimensional FRW universe.

B. Thermal history and the horizon problem

The existence of a multidimensional phase prior to $T_{\rm deex}$ alters radically the thermal history of the early universe. Had all dimensions been expanding, the radiation could not but cool. However, with some dimensions expanding and others shrinking, the n-dimensional volume Σ^n, and hence the temperature T, can and, in fact, do have more complicated profiles. We have seen that the universe starts out infinitely hot, cools to a certain $T_{\rm min}$, and then reheats until it becomes effectively 1 + 3 dimensional. Beyond this point, the radiation temperature drops in the usual FRW manner.

The tremendous increase in the scale factor r, during the latter half of the multidimensional phase, accompanied by an actual increase (rather than a drop) in temperature, provides within this framework an explanation for the horizon problem of the standard big-bang cosmology.

The essential features of the problem[12] are as follows. Light reaching us from the time of photon decoupling indicates that the temperature of the microwave background is the same at every point of this sphere of last scatter to one part in 10^4 or better.[13] Within the standard cosmology, the universe has a finite lifetime, $t_{\rm univ}$, which is long enough to allow us to receive signals from this surface, but not sufficiently long to allow diametrically[14] opposite points to come into causal contact. Furthermore, as we evolve the universe back in time, we find that the horizon contracts faster than the scale factor, r, so that the size of a causally connected region becomes increasingly smaller. The conclusion then is that not only are the diametrically opposite points not in contact today, they have never been in causal contact at all. It is then nothing short of remarkable that they have come out looking so nearly identical.

In view of the adiabatic evolution of the universe assumed in the standard cosmology, the above question may be conveniently rephrased in terms of entropy: "Does the entropy falling within a causal horizon equal at any stage the entropy of the observed universe?"

For a flat FRW space-time the entropy, $S_{\rm hor}^{\rm FRW}$, inside a causal horizon for the radiation-dominated phase, is easily calculated. Indeed, $r(t)$ evolves according to

$$\left[\frac{\dot{r}}{r}\right]^2=\frac{8\pi G}{3}\frac{\alpha}{r^4} \tag{5.3}$$

(α=constant) which implies that the horizon

$$l(t)=r(t)\int_0^t\frac{dt'}{r(t')}=\left[\frac{3}{8\pi G\alpha}\right]^{1/2}r(t). \tag{5.4}$$

Furthermore, the entropy and density are related to the temperature T by

$$S(r)=F_3(rT)^3, \tag{5.5a}$$

$$\rho=\tfrac{3}{4}F_3T^4, \tag{5.5b}$$

whence

$$\alpha=\tfrac{3}{4}F_3\left[\frac{S}{F_3}\right]^{4/3}. \tag{5.6}$$

Putting (5.4) and (5.6) together we find that the entropy in the volume $l(t)^3$ is

$$\begin{aligned}S_{\rm hor}^{\rm FRW}&=S(r)\left[\frac{l(t)}{r(t)}\right]^3\\&=S(r)\left[\frac{1}{(2\pi F_3)^{1/2}}\left[\frac{F_3}{S(r)}\right]^{1/3}\frac{M_p}{T}\right]^3\\&=\frac{F_3{}^{-1/2}}{(2\pi)^{3/2}}\left[\frac{M_p}{T}\right]^3.\end{aligned} \tag{5.7}$$

The entropy $S_{\rm obs}$ of the observed universe receives contributions principally from the backgrounds of cosmic photons and neutrinos. Assuming that neutrinos decouple at a stage where the only other massless particles are electrons and photons, $S_\nu=(21/22)S_\gamma$, whence

$$S_{\rm obs}\simeq 2F_3(3\times10^9\ {\rm yr}\times 2.7^\circ\ {\rm K})^3\gtrsim 10^{86}. \tag{5.8}$$

Comparing this to $S_{\rm hor}^{\rm FRW}$ at $T=10^{17}$ GeV, e.g., we find that the latter falls short by a factor $\simeq 10^{83}$.

For the multidimensional scenario (and for τ not too close to zero),

$$\begin{aligned}\frac{l(t)}{r(t)}&=\int_0^t\frac{dt'}{r(t')}\\&=\frac{D}{r_0}\left[\frac{n}{(n-1)}\right]^{1/2}\int_0^t\left[\frac{\sigma^{n-1}}{(n-1)-\sigma^{n-1}}\right]^{D/2d}d\tau\\&\simeq O(1).\end{aligned} \tag{5.9}$$

Thus the entropy, $S_{\rm hor}^{\rm MD}$, inside the horizon at $t_{\rm deex}\simeq S(r_0=1)$. Now $S(r_0=1)$ is genuinely a free parameter of the model. It fixes, e.g., the temperature curve, $T(t)$, of the radiation, and determines in particular the lowest temperature, $T_{\rm min}$, to which the universe cools:

$$\begin{aligned}T_{\rm min}&=\frac{[S(r_0=1)/F_n]^{1/n}}{\Sigma_{\rm max}(r_0=1)}\simeq\frac{S(r_0=1)^{1/n}}{A\sigma_{\rm max}}\\&\sim\frac{MS(r_0=1)^{-2/n(n-1)}}{\sigma_{\rm max}},\end{aligned} \tag{5.10}$$

where

$$M^{n-1}=M_p{}^2 M_{KK}{}^{n-3} . \qquad (5.11)$$

[This is to be contrasted with the flat radiation-dominated FRW cosmology where $T(t)$ is independent of $S(r_0=1)$ and the latter is really not a free parameter.]

It is useful at this stage to return to Eq. (5.9) and to trace precisely where the large factor needed to solve the horizon problem is coming from. To this end we first examine the exact solution (4.15) which corresponds to a reduction from three dimensions to one. Furthermore, we shall, for increased transparency, set $c=0$ and $a_0=-1$.

In this case, the "deexcitation" condition $R_{\text{deex}}T_{\text{deex}}=1$ implies that

$$(1-\tau_{\text{deex}}{}^2)^{1/2}=[F_3/S(r_0=1)]^{1/3} , \qquad (5.12)$$

whence

$$T_{\text{deex}}=\left[\frac{S(r_0=1)}{F_3}\right]^{1/3}\frac{1}{A\tau_{\text{deex}}(1-\tau_{\text{deex}}{}^2)^{1/2}}$$
$$=\frac{M_p}{(3\pi F_3)^{1/2}} . \qquad (5.13)$$

The horizon at this stage has the value

$$l(t_{\text{deex}})=r(t_{\text{deex}})\int_0^{t_{\text{deex}}}\frac{dt}{r(t)}$$
$$=\frac{A\tau_{\text{deex}}}{(1-\tau_{\text{deex}}{}^2)^{1/2}}\left[\frac{2}{3}\right]^{1/2}$$
$$\simeq\left[\frac{3\pi F_3}{M_p}\right]^{1/2}\left[\frac{S}{F_3}\right]^{2/3}\left[\frac{S}{F_3}\right]^{1/3}\left[\frac{2}{3}\right]^{1/2} , \qquad (5.14)$$

whence $F_1 l(t_{\text{deex}})T_{\text{deex}}\simeq S(r_0=1)$.

Thus, the large factor in question arises from two sources. We have first the factor, A, which measures the lifetime of the higher-dimensional phase, and hence the progress made by the horizon over its duration. Second, as the universe moves towards deexcitation, $r(t)$ [and consequently $l(t)$] approaches a pole singularity. The factor $(1-\tau_{\text{deex}})^{-1/2}$ thus introduced accounts for the rest of the required large number.

The situation for arbitrary d and D is similar. We can use expansion (4.13) to make the relevant estimates, since for the interesting (large) values of $S(r_0=1)$, deexcitation occurs very close to τ_0. Keeping only the first term in (4.13) and inserting this into Eq. (4.3) we get

$$l(t_{\text{deex}})\sim r(t_{\text{deex}})\times O(1)$$
$$\simeq A(\tau_0-\tau_{\text{deex}})^{-\beta} . \qquad (5.15)$$

τ_{deex}, in turn, is the solution to

$$R(\tau_{\text{deex}})T(\tau_{\text{deex}})=\frac{\mathscr{R}(\tau_{\text{deex}})}{\sigma(\tau_{\text{deex}})}\left[\frac{S(r_0=1)}{F_n}\right]^{1/n}=1 , \qquad (5.16)$$

which implies that

$$\frac{\mathscr{R}(\tau_{\text{deex}})}{\sigma(\tau_{\text{deex}})}=\left[\frac{\sigma(\tau_{\text{deex}})}{r(\tau_{\text{deex}})}\right]^{d/D}$$
$$\simeq[(c_1\xi_{\text{deex}})^{1/(n-1)}\Sigma_{\text{deex}}^{\beta}]^{d/D}$$
$$=\left[\frac{S(r_0=1)}{F_n}\right]^{-1/n} . \qquad (5.17)$$

Thus,

$$\xi_{\text{deex}}=\tau_0-\tau_{\text{deex}}\sim S(r_0=1)^{-[2/(3-p)]/n} \qquad (5.18)$$

and

$$r(\tau_{\text{deex}})\simeq AS(r_0=1)^{2/n[\beta/(3-p)]}$$
$$\sim MS(r_0=1)^{n+1/n(n-1)+D/nd-[2/n(n-1)]/(3-p)} . \qquad (5.19)$$

The deexcitation temperature is similarly determined to be

$$T_{\text{deex}}=\frac{[S(r_0=1)/F_n]^{1/n}}{A\sigma(\tau_{\text{deex}})}$$
$$\simeq MS^{-[(n+1)/n(n-1)]+1/n}\xi_{\text{deex}}^{1/n-1}$$
$$\sim MS(r_0=1)^{[2/n(n-1)]\{[1/(3-p)]-1\}} . \qquad (5.20)$$

From (5.17) and (5.19) it is immediate that

$$[r(\tau_{\text{deex}})T_{\text{deex}}]^d\sim S(r_0=1) . \qquad (5.21)$$

For T_{deex} close to M, we see that it is once again the rapid increase in r near τ_0 following upon the expansion of the horizon during the lifetime

$$\tau_{\text{univ}}^{\text{MD}}=A\int_0^{\tau_{\text{deex}}}\sigma(\tau)d\tau\sim A \qquad (5.22)$$

of the multidimensional phase which causes an entropy $\sim S(r_0=1)$, to fall into a causal horizon at T_{deex}. [The integral in (5.22) is evaluated numerically.]

The horizon problem is therefore resolved if we choose $S(r_0=1)\geq S_{\text{obs}}$, which is an embarrassingly large number. This is admittedly unaesthetic, but notice that the parameter to really compare with the natural mass scale M in these models is

$$A^{-1}\sim M[S(r_0=1)/F_n]^{-[(n+1)n/(n-1)]} .$$

If $S(r_0=1)=10^{83}F_n$, $A^{-1}/M=10^{-54}$, 10^{-10}, $10^{-3.6}$ for $n=3$, 10, and 25, respectively. Clearly on reduction to $1+3$ dimensions, the relative effect of $S(r_0=1)$ is vastly amplified.

It is enlightening to compare and contrast the above resolution to the horizon problem with that offered by the inflationary scenario.[15] In the latter, we begin with a small causally connected region, R, of the FRW universe. As this universe cools to T_{GUT}, the (constant) vacuum energy associated with the GUT transition dominates the radiation energy. The universe enters a phase of exponential expansion. In so expanding, it first supercools but later reheats back to $T_{\text{reheat}}\sim O(M_{\text{GUT}})$. The exponential ex-

TABLE I. For $T_{\text{deex}} \le M_P$, $M_{\text{KK}} = R_c^{-1} \le f_1(c,a_0) M_p [S(r_0=1)/F_n]^{2/n(n-3)[1-1/(3-p)]} = f_1(c,a_0) M_{\text{KK}}^{\max}$, $T_{\min} \le f_2(c,a_0) \times M_{\max} [S(r_0=1)/F_n]^{-2/n(n-1)}$, $\tau_{\text{univ}}^{\text{MD}} \ge f_3(c,a_0) M_{\max}{}^{-1} [S(r_0=1)/F_n]^{(n+1)/n(n-1)}$, where $M_{\max}^{n-1} = (M_{\text{KK}}^{\max})^{n-3} M_p{}^2$ and f_i are parameter-dependent factors which are of order 1 over a range of parameter values. All the tabulated numbers correspond to $S(r_0=1)/F_n = 10^{83}$.

No. of spatial dimensions \ Interesting mass scales	$M_{\text{KK}}^{\max}$ (GeV)	$M_{\max} \left[\dfrac{S(r_0=1)}{F_n} \right]^{-2/n(n-1)}$ (GeV)	$M_{\max}{}^{-1} \left[\dfrac{S(r_0-1)}{F_n} \right]^{(n+1)/n(n-1)}$ (GeV^{-1})
10	$10^{14.8}$	$10^{13.9}$	$10^{-5.6}$
15	$10^{16.5}$	$10^{16.0}$	$10^{-10.5}$
20	$10^{17.0}$	$10^{16.7}$	$10^{-12.6}$
25	$10^{17.3}$	$10^{17.1}$	$10^{-13.3}$

pansion inflates the region R to a much bigger region R'. Since the horizon remains constant during the exponential expansion, R' is no longer within a single horizon. However, since R' emerged from the causally connected R, it is homogeneous and isotropic. Furthermore, the process of reheating generates an entropy in $R' \ge S_{\text{obs}}$.

Notice that the net effect of the inflation is to get at a high temperature a universe much larger than the FRW one. The multidimensional scenario provides an alternative mechanism for doing precisely this. However, it does so without generating entropy. The collapse of the extra dimensions allows r to increase without a drop in temperature, so that from the point of view of the $1+3$ dimensions which are eventually singled out, it feels as though entropy is being generated.[16] Furthermore, the increase in r close to τ_0 (which is where most of it takes place for the higher-dimensional picture) is faster than exponential: for this increase r becomes infinite in finite time rather than the infinite time required by exponential growth.

Now that we have understood the significance of the parameter A, we can conclude this section with specific numbers. We shall set $S(r_0=1)=10^{83}F_n$ and shall look at the interesting case in which deexcitation occurs before the universe reheats to the Planck temperature, i.e., before quantum gravity becomes of crucial importance.

If we supplement the expressions (5.10), (5.11), (5.20), and (5.22) for $T_{\min}$, M_{KK}, T_{deex}, and $\tau_{\text{univ}}^{\text{MD}}$, respectively with these inputs we get the values given in Table I for the relevant factors of $S(r_0=1)$ and M occurring in the latter. The exact values of the remaining factors depend on the specific choice for the arbitrary parameters a_0 and c. However, for a range of reasonable values those factors are of order 1.

VI. CONCLUDING REMARKS

A series of observations are in order at this point. To begin with, it is hoped that this scenario makes the concept of extra dimensions plausible, at the very least. The idea of expanding dimensions is altogether commonplace, while blueshifts, indicating contracting dimensions, are eventually expected if our universe is closed. In view of this, the idea of simultaneous expansion for some dimensions and contraction for others is hardly outlandish. Furthermore, if the universe undergoes such an evolution, the contracting dimensions can certainly be expected to deexcite, and to thus effectively disappear.

Secondly, for the case considered, viz., $k_d=0$, solutions in which r and R behave the same way do not exist. Indeed, if we set $r=R$ in Eqs. (2.7), we find that Eqs. (2.7b) and (2.7c) are mutually inconsistent.

Furthermore, although we have a variety of solutions to the field equations, they all behave in an identical fashion close to τ_0. And it is on this behavior[17] that the more important conclusions of the scenario, viz., the contraction and deexcitation of the extra dimensions, the rapid increase in r, etc., are based.

In any case, it is probably prudent not to take too seriously the behavior of the solutions near $\tau=0$. Indeed as our understanding of the extra dimensions grows, particularly in the context of supergravity theories,[18] we can hope to discover mechanisms which spontaneously induce on the universe a toroidal geometry with three spatial dimensions distinguished from the rest. Once such a geometry comes into existence, the above scenario takes over, and soon all but the three dimensions "disappear." In this sense, the scenario moves us one step closer to answering the question: "Why three spatial dimensions?"

It should be pointed out that the solutions investigated in this paper are generalizations of the Kantowski solutions Eq. (4.15) which had been worked out earlier for $1+3$ dimensions. Higher-dimensional generalizations[19] of the vacuum Kasner solutions have also been found to have some interesting features. It is unlikely that this exhausts the set of interesting generalizations. A more systematic search for the various possibilities should thus be carried out by extending the Petrov classification of

(1 + 3)-dimensional metrics to higher dimensions.

In summary, then, a higher-dimensional cosmology which collapses into a (1 + 3)-dimensional FRW universe has been developed. The underlying equations of motion have been solved quite generally for the case of a flat three-dimensional universe and it is found that the behavior producing the scenario is quite generic. The horizon problem has been solved modulo the fact that the free parameter A has to be given a large value, which is nonetheless very much smaller than in the (1 + 3)-dimensional case, and which decreases the larger the number of extra dimensions introduced. Finally, the explanation for there being only three dimensions has been advanced by one step.

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