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# Effect of particle creation on Kaluza-Klein cosmologies

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One of the attempts that try to explain the smallness of the internal space in Kaluza-Klein theories is the attractive idea of cosmological dimensional reduction proposed by Chodos and Detweiler and by Freund. In these theories, the internal space shrinks to an unobservable scale by the dynamical evolution of the anisotropic universe. However, if we consider a quantized matter field, the higher-dimensional anisotropic space-time may be isotropized by the effect of particle creation as suggested by Zel'dovich in the case of a conventional four-dimensional space-time. Here, we show that this isotropization process is rather rapid and the mechanism of cosmological dimensional reduction does not work well in the case of space-time with $M_4 \times S^1$ topology. We also show that the above result holds even if we take into account the adiabatic regularization term. We give the analytic solution in the case of space-time with $M_4 \times T^D$ topology under some approximation and show that the idea of cosmological dimensional reduction in this case is also broken.

## I. INTRODUCTION

The higher-dimensional theory of Kaluza and Klein is one of the most interesting ways of unifying gravitation and gauge interactions. Following the idea of this theory, we consider that the higher-dimensional space-time is reduced to a four-dimensional space-time plus a compact internal space. Gravity is expressed by the metric of the four-dimensional space-time, and gauge symmetry is induced from the symmetry of the internal space, the scale of which is unobservably small (near the Planck length $l_P$).[1] Recently, in order to investigate whether this scheme is valid, there have been many works on the method of reduction of higher dimensions such as Freund and Rubin's,[2] and on the existence and the stability of the solution of the higher-dimensional Einstein equations.[3]

These static vacuum solutions may explain what the present vacuum is, but they do not explain a large discrepancy between the scales of the internal space ($\sim l_P$) and the physical three-space (cosmological scale), and they do not explain how to get a vacuum with such a large discrepancy. As for ideas on how to get rid of this discrepancy, we now have two possibilities. One is a method of considering a quantum instability of the internal space for a contraction as shown by Appelquist and Chodos,[4] and the other is a method of using a dynamical evolution of the higher-dimensional universe as proposed by Chodos and Detweiler.[5] The latter one, which we call a cosmological dimensional reduction, is very attractive for us because the evolution of the Universe may explain the smallness of the internal space. We can consider the following scenario. At first, a topological compactification of higher-dimensional space-time occurs near the Planck time by an unknown mechanism (e.g., $M^{11} \to M^4 \times S^7$). At this time, the scale of the internal space is nearly the same as that of the physical three-space. Afterwards, by an anisotropic expansion of the Universe, which originates in the initial compactification, the internal space shrinks to an unobservable scale. This shrinking stops or slows down from the other effect (e.g., quantum effect) at a late stage of the Universe, because a rapid change is contradictory to the observation of changing fundamental constants. Chodos and Detweiler give a very simple model (Kasner-type solution) in order to explain this scheme. For example, in the case of five-dimensions the scale factor of the physical three-space expands as $a(t) \propto t^{1/2}$ and that of the five-dimensional space contracts as $a_5(t) \propto t^{-1/2}$.

In the conventional four-dimensional Einstein theory, Zel'dovich[6] proposed the isotropization of the Universe by a particle creation mechanism. If we consider a quantum effect of a matter field, the initial anisotropic expansion of the Universe, even if it exists, may be isotropized by the creation of particles. Zel'dovich and Starobinsky[7] and Hu and Parker[8] quantitatively analyzed the isotropization process of the anisotropic universe which starts from a Kasner-type solution initially. They show that the isotropization of the Universe is rather quick, i.e, the characteristic time of isotropization is less than $2 \times 10^3 t_P$ if the initial time $t_0 < 3t_P$, where $t_P$ is the Planck time.

The model given by Chodos and Detweiler is just a Kasner-type universe in higher dimensions. Also, in the models by Freund[9] and by Gleiser, Rajpoot, and Taylor,[10] which are the solutions in more realistic Kaluza-Klein theories, we treat the time-dependent anisotropic space-times. In realistic Kaluza-Klein unified theory, we may have to consider not only the Einstein Lagrangian but also the Lagrangian of a matter field such as an antisymmetric tensor field $A_{\mu\nu\rho}$ in 11-dimensional supergravity theory. Then, the isotropization mechanism due to the particle creation may destroy the idea of the cosmological dimensional reduction.

In my previous paper,[11] I showed that this isotropization process in five-dimensional space-time is rather rapid, as in the case of a four-dimensional universe by Hu and Parker. However, in that paper I did not take into account the closed behavior of the internal space and the regularization term. The purpose of this paper is mainly to investigate whether these effects change the above result. Also, I consider a higher-dimensional case without the curvature ($M_4 \times T^D$), where $T^D$ is a $D$-dimensional torus. And, I want to investigate whether the effect of the particle creation is also important in cases of space-times with the other topology. Therefore, in this paper, I shall start by dealing with general anisotropic higher-dimensional Einstein equations in the case of a space-time with some symmetries. Following the method of Hu and Parker, I give the energy density of created particles in Sec. III. In Sec. IV, as a simple example, I show by the numerical integration, how quickly the anisotropic universe is isotropized in space-time with $M_4 \times S^1$ topology. Also I consider the effect of the adiabatic regularization term. The energy density with the adiabatic regularization term is calculated in the Appendix. In Sec. V, I give the analytic solution in the case with topology $M_4 \times T^D$ under the approximation that the whole energy is created at initial time. The case with the other topology will be dealt with in a subsequent paper. In Sec. VI, I point out the problems when we consider the particle creation effect on more realistic Kaluza-Klein cosmologies.

## II. THE HIGHER-DIMENSIONAL EINSTEIN EQUATIONS

We consider a space-time manifold with one timelike and $(d+D)$ spacelike dimensions. The $(1+d+D)$-dimensional Einstein equations with a matter field and a cosmological constant $\Lambda$ are

$$\mathscr{R}_{\mu\nu} - \tfrac{1}{2} g_{\mu\nu} \mathscr{R} = 8\pi G \langle T_{\mu\nu} \rangle_{\text{reg}} - \Lambda g_{\mu\nu} , \tag{2.1}$$

where $G$ is a $(1+d+D)$-dimensional gravitational constant.[12] $\langle T_{\mu\nu} \rangle_{\text{reg}}$ is an expectation value of a regularized energy-momentum tensor of a quantized matter field. We assume the space-time manifold has the metric form:

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu = -dt^2 + a^2(t) \gamma_{mn}(x^p) dx^m dx^n + b^2(t) \tilde{\gamma}_{MN}(y^P) dy^M dy^N , \tag{2.2}$$

where $\gamma_{mn}$ and $\tilde{\gamma}_{MN}$ are the metrics of maximally symmetric $d$- and $D$-dimensional spaces, respectively. $a$ and $b$ are the time-dependent cosmological scale factors. It follows from this assumption that the nonvanishing components of $\langle T_{\mu\nu} \rangle$ are

$$\langle T^0_0 \rangle = -\rho, \quad \langle T^n_m \rangle \equiv P_a \delta^n_m ,$$

and (2.3)

$$\langle T^N_M \rangle \equiv P_b \delta^N_M ,$$

where $\rho$ is an energy density and $P_a$ and $P_b$ are principal pressures. The Einstein equations (2.1) become

$$\mathscr{G}^0_0 = -\frac{1}{2} \left[ \frac{d(d-1)}{a^2} k_d + \frac{D(D-1)}{b^2} k_D + \frac{1}{R^2} [(d\alpha' + D\beta')^2 - d(\alpha')^2 - D(\beta')^2] \right] = -8\pi G \rho - \Lambda , \tag{2.4}$$

$$\mathscr{R}^m_m = \frac{1}{R^2} \left[ \alpha'' + \frac{d+D-1}{d+D} \alpha'(d\alpha' + D\beta') \right] + \frac{d-1}{a^2} k_d = 8\pi G \left[ P_a - \frac{T}{d+D-1} \right] + \frac{2\Lambda}{d+D-1} , \tag{2.5}$$

$$\mathscr{R}^M_M = \frac{1}{R^2} \left[ \beta'' + \frac{d+D-1}{d+D} \beta'(d\alpha' + D\beta') \right] + \frac{D-1}{b^2} k_D = 8\pi G \left[ P_b - \frac{T}{d+D-1} \right] + \frac{2\Lambda}{d+D-1} , \tag{2.6}$$

where $\mathscr{G}^0_0 = \mathscr{R}^0_0 - \frac{1}{2}\mathscr{R}$, $\alpha \equiv \ln a$, $\beta \equiv \ln b$, $T = = \langle T^\mu_\mu \rangle$, and $k_d$ and $k_D$ are the curvature constants of $d$- and $D$-dimensional spaces, respectively [no sum on $m$ and $M$ in (2.5) and (2.6)]. A prime denotes a derivative with respect to the conformal time $\eta$, defined by

$$\eta = \int^t R^{-1}(t') dt' , \tag{2.7}$$

and $R \equiv (a^d b^D)^{1/(d+D)}$ is the geometrical mean of the scale factors.

From the Bianchi identity, we get a conservation equation for the energy-momentum tensor, $\langle T^{\mu\nu} \rangle_{;\nu} = 0$. From (2.2) and (2.3), this equation becomes

$$V^{-1}(\rho V)' + dP_a \alpha' + DP_b \beta' = 0 , \tag{2.8}$$

where $V \equiv a^d b^D = R^{d+D}$ is the proper volume.

We can show that the conservation equation (2.8) is equivalent to the constraint equation (2.4), if the constraint equation is satisfied at an initial time. Therefore, we use Eq. (2.8) instead of Eq. (2.4) except for the initial time. At the initial time we have to take into account Eq. (2.4). The trace of the energy-momentum tensor is

$$T \equiv \langle T^\mu_\mu \rangle = -\rho + dP_a + DP_b . \tag{2.9}$$

From (2.8) and (2.9), the pressures $P_a$ and $P_b$ are expressed by $\rho$ and $T$ as

$$P_a = \frac{\rho}{d+D} + \frac{1}{d(\beta' - \alpha')} [R^{-(d+D+1)} (\rho R^{d+D+1})' + \beta' T] \tag{2.10a}$$

and

$$P_b = \frac{\rho}{d+D} + \frac{1}{D(\alpha' - \beta')} [R^{-(d+D+1)} (\rho R^{d+D+1})' + \alpha' T] . \tag{2.10b}$$

The first term on the right-hand side of Eq. (2.10) [i.e., $\rho/(d+D)$] is the pressure by a relativistic isotropic fluid, like a photon gas, in $(d+D)$-dimensional space.

If $\rho$, $\rho'$, and $T$ are given as functionals of the scale factors $a$ and $b$, the Einstein equations (2.5) and (2.6) can be solved as a Cauchy problem. We shall present explicitly the functional forms of $\rho$ and $\rho'$ of the created particles in the anisotropic expansion in Sec. III.

The Einstein equations (2.4)—(2.6) are rewritten in a more convenient form by new variables, $y = \ln R$ and $z = \ln(b/a)$, as follows:

$$(d+D)(d+D-1)(y')^2-\frac{dD}{d+D}(z')^2+\left[\left[\frac{R}{a}\right]^2 d(d-1)k_d+\left[\frac{R}{b}\right]^2 D(D-1)k_D\right]=16\pi GR^2\rho+2R^2\Lambda\ , \tag{2.11}$$

$$y''+(d+D-1)(y')^2+\frac{1}{d+D}\left[\left[\frac{R}{a}\right]^2 d(d-1)k_d+\left[\frac{R}{b}\right]^2 D(D-1)k_D\right]$$
$$=8\pi GR^2\frac{1}{d+D}\left[dP_a+DP_b-\frac{d+D}{d+D-1}T\right]+\frac{2R^2\Lambda}{d+D-1}\ , \tag{2.12}$$

and

$$z''+(d+D-1)y'z'-\left[\left[\frac{R}{a}\right]^2(d-1)k_d-\left[\frac{R}{b}\right]^2(D-1)k_D\right]=8\pi GR^2(P_b-P_a)\ , \tag{2.13}$$

where

$$\left[\frac{R}{a}\right]^2=\exp\left[\frac{2D}{d+D}z\right]$$

and

$$\left[\frac{R}{b}\right]^2=\exp\left[-\frac{2d}{d+D}z\right]\ .$$

The terms of the matter field in these equations are, from (2.10),

$$dP_a+DP_b-\frac{d+D}{d+D-1}T=\rho-\frac{T}{d+D-1} \tag{2.14}$$

and

$$P_b-P_a=-\frac{d+D}{dD}\frac{1}{z'}[R^{-(d+D+1)}(\rho R^{d+D+1})'+y'T]\ . \tag{2.15}$$

In numerical integration, we use these equations (2.11)–(2.15).

## III. ENERGY DENSITY OF CREATED PARTICLES

We can consider quantum effects by two types of matter. One is an effect by a matter field with a coherent vacuum expectation value from spontaneous symmetry breaking as proposed by Freund and Rubin.[2] The other is an effect by created particles from an anisotropic expansion of a universe as considered in my previous paper.[11] We shall deal with the former case, including consideration of the effect of particle creation, in a subsequent paper. In this paper, from now on, we consider only the latter case, i.e., the energy-momentum tensor by particle creation.

For simplicity, we consider a quantized conformal massless scalar field $\phi$ as the matter field. The Lagrangian of the scalar field is

$$\mathscr{L}_M=-\tfrac{1}{2}\sqrt{-g}\,(g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi+\xi\mathscr{R}\phi^2)\ , \tag{3.1}$$

where

$$\xi=\frac{d+D-1}{4(d+D)}\ .$$

The field equation of $\phi$ is

$$\nabla_\mu\nabla^\mu\phi-\xi\mathscr{R}\phi=0\ , \tag{3.2}$$

where $\nabla_\mu$ is the covariant derivative. The energy-momentum tensor is given by

$$T_{\mu\nu}=(\nabla_\mu\phi)(\nabla_\nu\phi)-\tfrac{1}{2}g_{\mu\nu}(\nabla_\rho\phi)(\nabla^\rho\phi)+\xi\phi^2\mathscr{G}_{\mu\nu}$$
$$-\xi[\nabla_\mu\nabla_\nu(\phi^2)-g_{\mu\nu}\nabla_\rho\nabla^\rho(\phi^2)]\ , \tag{3.3}$$

where $\mathscr{G}_{\mu\nu}=\mathscr{R}_{\mu\nu}-\frac{1}{2}g_{\mu\nu}\mathscr{R}$. The trace of the energy-momentum tensor, $T\equiv T^\mu_\mu$, vanishes on a classical level, and the trace anomaly on a quantum level also does not appear in the case of odd-dimensional space-times. We consider odd-dimensional space-times, and then set $T=0$.

We shall quantize the scalar field and give the vacuum expectation value of the energy-momentum tensor, $\langle T_{\mu\nu}\rangle_{\rm reg}$. We introduce the harmonics $h_{\vec{l}}(x)$ and $H_{\vec{L}}(y)$ of $d$- and $D$-dimensional maximally symmetric spaces, respectively. $h_{\vec{l}}(x)$ and $H_{\vec{L}}(y)$ satisfy the equations

$$D_mD^mh_{\vec{l}}(x)=-k_l{}^2h_{\vec{l}}(x) \tag{3.4a}$$

and

$$\tilde{D}_M\tilde{D}^MH_{\vec{L}}(y)=-K_L{}^2H_{\vec{L}}(y) \tag{3.4b}$$

and the orthonormal conditions

$$\int d^dx\sqrt{\gamma}h^*_{\vec{l}}(x)h_{\vec{l}'}(x)=\delta_{\vec{l}\,\vec{l}'} \tag{3.5a}$$

and

$$\int d^Dy\sqrt{\tilde{\gamma}}H^*_{\vec{L}}(y)H_{\vec{L}'}(y)=\delta_{\vec{L}\,\vec{L}'}\ , \tag{3.5b}$$

respectively, where $\gamma=\det(\gamma_{mn})$, $\tilde{\gamma}=\det(\tilde{\gamma}_{MN})$, and $D_m$ and $\tilde{D}_M$ are the covariant derivatives with respect to $\gamma_{mn}$ and $\tilde{\gamma}_{MN}$, respectively. $\vec{l}$ and $\vec{L}$ are quantum numbers, and $k_l{}^2$ and $K_L{}^2$ are eigenvalues of harmonics $h_{\vec{l}}(x)$ and $H_{\vec{L}}(y)$, respectively.

We shall set

$$\phi=R(\eta)^{-(d+D-1)/2}\chi(\eta,x,y) \tag{3.6}$$

and expand $\chi$ by the above harmonics as

$$\chi=\sum_{\vec{l},\vec{L}}[A_{\vec{l},\vec{L}}\chi_{\vec{l},\vec{L}}(\eta)h_{\vec{l}}(x)H_{\vec{L}}(y)+A^{\dagger}_{\vec{l},\vec{L}}\chi^{*}_{\vec{l},\vec{L}}(\eta)h^{*}_{\vec{l}}(x)H^{*}_{\vec{L}}(y)]\ . \quad (3.7)$$

From the field equation (3.2), we get the equation of $\chi$, and then that of $\chi_{\vec{l},\vec{L}}(\eta)$ as follows:

$$\chi''+Q\chi-R^2(D_mD^m+\tilde{D}_M\tilde{D}^M)\chi+\xi R^2\left[\frac{\mathscr{R}^{(d)}}{a^2}+\frac{\tilde{\mathscr{R}}^{(D)}}{b^2}\right]\chi=0\ , \quad (3.8)$$

and

$$\chi''_{\vec{l},\vec{L}}+(\Omega_{l,L}{}^2+Q)\chi_{\vec{l},\vec{L}}=0\ , \quad (3.9)$$

where

$$Q\equiv\frac{dD(d+D-1)}{4(d+D)^2}(\alpha'-\beta')^2=\frac{dD(d+D-1)}{4(d+D)^2}(z')^2 \quad (3.10)$$

and

$$\Omega_{l,L}{}^2\equiv R^2\left[\frac{\omega_l{}^2}{a^2}+\frac{\omega_L{}^2}{b^2}\right] \quad (3.11)$$

with

$$\omega_l{}^2=k_l{}^2+d(d-1)\xi k_d \quad (3.12a)$$

and

$$\omega_L{}^2=K_L{}^2+D(D-1)\xi k_D\ . \quad (3.12b)$$

$\mathscr{R}^{(d)}$ and $\tilde{\mathscr{R}}^{(D)}$ are scalar curvatures of $d$- and $D$-dimensional maximally symmetric spaces, which are given by

$$\mathscr{R}^{(d)}=d(d-1)k_d \text{ and } \tilde{\mathscr{R}}^{(D)}=D(D-1)k_D\ .$$

Provided that the normalization condition of $\chi_{\vec{l},\vec{L}}$ is

$$\chi^{*\prime}_{\vec{l},\vec{L}}\chi_{\vec{l},\vec{L}}-\chi^{*}_{\vec{l},\vec{L}}\chi'_{\vec{l},\vec{L}}=i\ , \quad (3.13)$$

the canonical quantization condition gives the usual commutation relations of annihilation and creation operators for $A_{\vec{l},\vec{L}}$ and $A^{\dagger}_{\vec{l},\vec{L}}$. Here, we can define formally a "vacuum" state $|0_A\rangle$ by the annihilation operators $A_{\vec{l},\vec{L}}$, i.e.,

$$A_{\vec{l},\vec{L}}|0_A\rangle=0 \text{ for any } \vec{l} \text{ and } \vec{L}\ .$$

From (3.3), (3.6), (3.8), and (2.4) (the explicit form of $\mathscr{G}^0_0$), the energy density is given by

$$-T^0_0=\tfrac{1}{2}R^{-(d+D+1)}\left[(\chi')^2-Q\chi^2+R^2\left[\frac{1}{a^2}[(1-4\xi)(D_m\chi)(D^m\chi)-4\xi\chi D_mD^m\chi+\xi\mathscr{R}^{(d)}\chi^2]+\frac{1}{b^2}[(1-4\xi)(\tilde{D}_M\chi)(\tilde{D}^M\chi)-4\xi\chi\tilde{D}_M\tilde{D}^M\chi+\xi\tilde{\mathscr{R}}^{(D)}\chi^2]\right]\right]\ . \quad (3.14)$$

Since the vacuum expectation value of (3.14), $\rho=-\langle 0_A|T^0_0|0_A\rangle$, is independent of the spatial coordinates, $x^m$ and $y^M$, $\rho$ can be expressed as

$$\rho=-\frac{1}{V_dV_D}\int d^dx\sqrt{\gamma}\int d^Dy\sqrt{\tilde{\gamma}}\langle 0_A|T^0_0|0_A\rangle\ , \quad (3.15)$$

where $V_d=\int d^dx\sqrt{\gamma}$ and $V_D=\int d^Dy\sqrt{\tilde{\gamma}}$. Inserting (3.7) into (3.14), and then into (3.15), we get the vacuum expectation value of energy density formally as

$$\rho=\frac{1}{2V_dV_D}R^{-(d+D+1)}\sum_{\vec{l}}\sum_{\vec{L}}[|\chi'_{\vec{l},\vec{L}}|^2+(\Omega^2_{l,L}-Q)|\chi_{\vec{l},\vec{L}}|^2]\ . \quad (3.16)$$

We notice that if the $d$-dimensional space is topologically open ($k_d\le 0$), we have to change $\sum_{\vec{l}}$ into some integration with the measure, e.g., in the case that $k_d=0$,

$$\sum_{\vec{l}}\to\frac{V_d}{(2\pi)^d}\int d^dk\ .$$

Since the formal expression (3.16) includes the UV divergence, we have to regularize the energy-momentum tensor. However, in the case that a space-time is time dependent *and* is not conformally flat, we have no appropriate regularization method except for the adiabatic regularization.[13] Even by the adiabatic regularization method, it is difficult to give an energy-momentum tensor explicitly as a functional of metrics unless we know the explicit form of metrics. Therefore, we shall give a crude estimation of energy density by the higher-dimensional version of the method given by Hu and Parker.[8]

We decompose the energy density $\rho$ into a quantum part $\rho_{q(t)}$ and a classical part $\rho_{c(t)}$. We set $\underline{L}=(\vec{l},\vec{L})$. $q(t)$ is the quantum domain in $\underline{L}$-space over which $\omega_{\underline{L}}(\equiv R^{-1}\Omega_{\underline{L}})\lesssim r_H{}^{-1}(t)$ and $c(t)$ is the classical one with $\omega_{\underline{L}}>r_H^{-1}(t)$, where $r_H(t)$ is the horizon scale. In the low-frequency region $q(t)$ of momentum space, quantum effects such as particle creation are dominant. For the high-frequency region $c(t)$, where the WKB approximation is valid, the created particles can be treated as classical incoherent matter.

$\rho_{q(t)}$ can be estimated by Eq. (3.16). In the low-frequency limit, we can get the solution of Eq. (3.9) as follows. First, we define the functions $\alpha_{\vec{l},\vec{L}}(\eta)$ and $\beta_{\vec{l},\vec{L}}(\eta)$ by

$$\chi_{\vec{l},\vec{L}}=(2\Omega_{l,L})^{-1/2}\left[\alpha_{\vec{l},\vec{L}}\exp\left[-i\int_{\eta_0}^{\eta}\Omega_{l,L}d\eta'\right]+\beta_{\vec{l},\vec{L}}\exp\left[i\int_{\eta_0}^{\eta}\Omega_{l,L}d\eta'\right]\right], \tag{3.17a}$$

and

$$\chi'_{\vec{l},\vec{L}}=-i(\Omega_{l,L}/2)^{1/2}\left[\alpha_{\vec{l},\vec{L}}\exp\left[-i\int_{\eta_0}^{\eta}\Omega_{l,L}d\eta'\right]-\beta_{\vec{l},\vec{L}}\exp\left[i\int_{\eta_0}^{\eta}\Omega_{l,L}d\eta'\right]\right]. \tag{3.17b}$$

From Eqs. (3.9) and (3.17), we get the equations of $\alpha_{\vec{l},\vec{L}}$ and $\beta_{\vec{l},\vec{L}}$ as

$$\alpha'_{\vec{l},\vec{L}}=\frac{1}{\Omega_{l,L}}\left[-\frac{i}{2}Q\alpha_{\vec{l},\vec{L}}+\tfrac{1}{2}(\Omega'_{l,L}-iQ)\beta_{\vec{l},\vec{L}}\exp\left[2i\int_{\eta_0}^{\eta}\Omega_{l,L}d\eta'\right]\right], \tag{3.18a}$$

and

$$\beta'_{\vec{l},\vec{L}}=\frac{1}{\Omega_{l,L}}\left[\frac{i}{2}Q\beta_{\vec{l},\vec{L}}+\tfrac{1}{2}(\Omega'_{l,L}+iQ)\alpha_{\vec{l},\vec{L}}\exp\left[-2i\int_{\eta_0}^{\eta}\Omega_{l,L}d\eta'\right]\right]. \tag{3.18b}$$

Setting

$$\exp\left[2i\int_{\eta_0}^{\eta}\Omega_{l,L}d\eta'\right]\approx 1$$

in the low-frequency limit, we find the general solution of Eq. (3.18), i.e.,

$$\alpha_{\vec{l},\vec{L}}=C_1\left[\Omega_{l,L}{}^{1/2}-i\Omega_{l,L}^{-1/2}\int_{\eta_0}^{\eta}Qd\eta'\right]+C_2\Omega_{l,L}{}^{-1/2}, \tag{3.19a}$$

and

$$\beta_{\vec{l},\vec{L}}=C_1\left[\Omega_{l,L}{}^{1/2}+i\Omega_{l,L}{}^{-1/2}\int_{\eta_0}^{\eta}Qd\eta'\right]-C_2\Omega_{l,L}{}^{-1/2}. \tag{3.19b}$$

From $|\alpha_{\vec{l},\vec{L}}|^2-|\beta_{\vec{l},\vec{L}}|^2=1$, which is deduced from (3.13), the integral constants $C_1$ and $C_2$ satisfy

$$C_1C_2^*+C_1^*C_2=\tfrac{1}{2}. \tag{3.20}$$

We consider the initial state is vacuum. Then, we have to take the initial condition that $\chi_{\vec{l},\vec{L}}(\eta)$ becomes a positive frequency function at an initial time $\eta_0$, i.e., $\alpha_{\vec{l},\vec{L}}(\eta_0)=1$ and $\beta_{\vec{l},\vec{L}}(\eta_0)=0$. This condition determines the integral constants as

$$C_1=\tfrac{1}{2}\Omega_{l,L}(\eta_0)^{-1/2}\text{ and }C_2=\tfrac{1}{2}\Omega_{l,L}(\eta_0)^{1/2}. \tag{3.21}$$

Since we can approximate the wave function $\chi_{\vec{l},\vec{L}}(\eta)$ in the quantum region $q(t)$ by the above solution given by Eqs. (3.17), (3.19), and (3.21), we get the energy density, $\rho_{q(t)}$, inserting this into (3.16), as

$$\rho_{q(t)}=\frac{1}{2V_dV_D}R^{-(d+D+1)}\times\sum_{\underline{L}\in q(t)}\left\{\frac{1}{2\Omega_0}\left[\Omega^2-Q+\left[\int_{\eta_0}^{\eta}Qd\eta'\right]^2\right]+\frac{\Omega_0}{2}\right\}, \tag{3.22}$$

where the subscript $\vec{l}$ and $\vec{L}$ have been dropped and $\Omega_0\equiv\Omega(\eta_0)$.

We consider that classical particles do not exist initially, i.e.,

$$\rho_c(t_0)=0. \tag{3.23}$$

As the quantum domain $q(t)$ shrinks with time, some part of the quantum particles, which satisfy the condition $\omega_{\underline{L}}>r_H{}^{-1}$, start to behave as classical particles. Let $\underline{L}_m$ be the maximum quantum number which satisfies $\omega_{\underline{L}}<r_H{}^{-1}$. The part of energy density by the particles which become classical in the time interval from $\eta$ to $\eta+\Delta\eta$ is given by

$$\delta\rho_q(\eta)=\rho_q(\underline{L}_m(\eta),\eta)-\rho_q(\underline{L}_m(\eta+\Delta\eta),\eta). \tag{3.24}$$

As the classical particles in the domain $c(t)$, we assume a collision-dominated relativistic fluid with an "isotropic" pressure $P_c=\rho_c/(d+D)$. If newly created particles strongly interact with each other in the high-density fluid, this assumption may be valid. From this assumption and the conservation of energy momentum (2.8), $\rho_c(t)$ is proportional to $R^{-(d+D+1)}(t)$. Then, the total energy density at $\eta+\Delta\eta$ is given by

$$\begin{aligned}\rho(\eta+\Delta\eta)&=\left[\frac{R(\eta)}{R(\eta+\Delta\eta)}\right]^{d+D+1}[\rho_c(\eta)+\delta\rho_q(\eta)]\\&\quad+\rho_q(\underline{L}_m(\eta+\Delta\eta),\eta+\Delta\eta)\\&=\left[\frac{R(\eta)}{R(\eta+\Delta\eta)}\right]^{d+D+1}\rho(\eta)\\&\quad+\Delta\eta R^{-(d+D+1)}\frac{\partial}{\partial\eta}[\rho_q(\underline{L}_m,\eta)R^{d+D+1}].\end{aligned} \tag{3.25}$$

$\partial/\partial\eta$ denotes a partial derivative with respect to $\eta$ under fixing $\underline{L}_m(\eta)$. Then,

$$(\rho R^{d+D+1})'=\frac{\partial}{\partial\eta}(\rho_qR^{d+D+1}). \tag{3.26}$$

From (2.12)–(2.15) and (3.26), we can see that the source term by the matter field in the equation for the mean scale factor ($R$) is related to the total energy density $\rho$ and that for the ratio of scale factors ($b/a$) is related to the creation rate of the quantum particles, $\partial(\rho_qR^{d+D+1})/\partial\eta$. Giving the initial data, we can compute $\rho(\eta)$ and

$\partial(\rho_q R^{d+D+1})/\partial\eta$ at any time as the functionals of metrics from (3.22) and (3.25), and then can calculate the time-dependent behavior of metrics $a$ and $b$, including the back reaction on the metrics of the created particles, by Eqs. (2.12) and (2.13). From the behavior of $a$ and $b$, we can see how quickly the anisotropic expansion is isotropized by the particle creation mechanism. Here, the "isotropization" means that the expansion rate of the internal space ($\beta'$) becomes equal to that of the physical three-space ($\alpha'$), because this type of isotropization is enough to break the idea of the cosmological dimensional reduction. In Sec. IV, we show a simple example by numerical integration of Eqs. (2.12) and (2.13).

Hu and Parker adopted the adiabatic regularization method even in the estimation of the quantum part (the low-frequency region) of the energy density. However, the adiabatic regularization is valid in the high-frequency limit and we do not know whether this method is also valid in the low-frequency region. Therefore, we give in the Appendix only the energy density with some part of the adiabatic regularization terms, by which the initial energy density $\rho_q(t_0)$ vanishes and the initial state becomes a true vacuum. We can compare the cases with and without the adiabatic regularization terms, and can simultaneously investigate whether the isotropization occurs even in the case in which the Universe starts from a true vacuum state.

## IV. QUANTITATIVE ANALYSIS IN A SIMPLE MODEL

In this section, we consider first the space-time $M_4\times M_D$—where $M_4$ is a conventional four-dimensional flat universe and is not closed. $M_D$ is a $D$-dimensional closed internal space.

From (3.22),

$$\rho_{q(t)}=\frac{1}{16\pi^3 V_D}R^{-(D+4)}\times\sum_{\vec{L}}{}'\int{}'d^3k\left\{\frac{1}{2\Omega_0}\left[\Omega^2-Q+\left[\int_{\eta_0}^{\eta}Q\,d\eta'\right]^2\right]+\frac{\Omega_0}{2}\right\} \tag{4.1}$$

and then

$$\frac{\partial}{\partial\eta}(\rho_q R^{D+4})=\frac{1}{16\pi^3 V_D}\times\sum_{\vec{L}}{}'\int{}'d^3k\left[\frac{1}{2\Omega_0}\left[2\Omega\Omega'-Q'+Q\int_{\eta_0}^{\eta}Q\,d\eta'\right]\right], \tag{4.2}$$

where $\sum_{\vec{L}}'\int{}'d^3k$ denotes that the summation and the integration are carried out in the region $q(t)$.

From (3.10),

$$Q'=\frac{3D(D+2)}{2(D+3)^2}z'z''\,. \tag{4.3}$$

In order to solve the second-order differential equation (2.13) with (2.15) and (3.26), the source term $(\partial/\partial\eta)(\rho_q R^{D+4})$ must not include the term with the second-order derivative. Then, inserting Eq. (2.13) into (4.3), and then (4.3) into (4.2), we resolve the equation with respect to $(\partial/\partial\eta)(\rho_q R^{D+4})$ and re-express $(\partial/\partial\eta)(\rho_q R^{D+4})$ by the terms without the second-order derivative. We find

$$\frac{\partial}{\partial\eta}(\rho_q R^{D+4})=\frac{1}{16\pi^3 V_D}\left[1-\frac{G}{8\pi^2 V_D}\frac{(D+2)}{(D+3)}R^{-(D+2)}\sum_{\vec{L}}{}'\int{}'d^3k\frac{1}{\Omega_0}\right]^{-1}\times\sum_{\vec{L}}{}'\int{}'d^3k\left[\frac{1}{2\Omega_0}\left[2\Omega\Omega'-\tilde{Q}'+Q\int_{\eta_0}^{\eta}Q\,d\eta'\right]\right], \tag{4.4}$$

where

$$\tilde{Q}'=-\frac{3D(D+2)}{2(D+3)^2}z'\left[(D+2)y'z'+2k_D\left[\frac{R}{b}\right]^2\right].$$

Integrating over the wave number $k$, we get

$$\rho_q(t_0)=\frac{1}{16\pi^3 V_D}R^{-(D+4)}\left[\left[\frac{R_0}{a_0}\right]^2\sum_{\vec{L}}{}'I_2+\left[\frac{R_0}{b_0}\right]^2\sum_{\vec{L}}{}'(\omega_L{}^2 I_0)-\frac{Q}{2}\sum_{\vec{L}}{}'I_0\right] \tag{4.5}$$

and

$$\frac{\partial(\rho_q R^{D+4})}{\partial\eta}=\frac{1}{16\pi^3 V_D}\left[1-\frac{G}{8\pi^2 V_D}\frac{(D+2)}{(D+3)}R^{-(D+2)}\sum_{\vec{L}}{}' I_0\right]^{-1}\times\left\{\frac{z'}{D+3}\left[D\left[\frac{R}{a}\right]^2\sum_{\vec{L}}{}'I_2-3\left[\frac{R}{b}\right]^2\sum_{\vec{L}}{}'\omega_L{}^2 I_0\right]+\frac{1}{2}\left[-\tilde{Q}'+Q\int_{\eta_0}^{\eta}Q\,d\eta'\right]\sum_{\vec{L}}{}'I_0\right\}, \tag{4.6}$$

where $a_0=a(\eta_0)$, $b_0=b(\eta_0)$, and $R_0=R(\eta_0)$.

$$I_0\equiv \int' d^3k\frac{1}{\Omega_0}=2\pi\frac{a_0}{R_0}\left[k_m\left[k_m{}^2+\frac{a_0{}^2}{b_0{}^2}\omega_L{}^2\right]^{1/2}-\frac{a_0{}^2}{b_0{}^2}\omega_L{}^2\ln\left|\frac{k_m+[k_m{}^2+(a_0\omega_L/b_0)^2]^{1/2}}{a_0\omega_L/b_0}\right|\right] \tag{4.7}$$

and

$$I_2\equiv \int' d^3k\frac{k^2}{\Omega_0}$$

$$=\frac{\pi}{2}\frac{a_0}{R_0}\left[k_m\left[k_m{}^2+\frac{a_0{}^2}{b_0{}^2}\omega_L{}^2\right]^{1/2}\left[2k_m{}^2-3\frac{a_0{}^2}{b_0{}^2}\omega_L{}^2\right]+3\left[\frac{a_0{}^2}{b_0{}^2}\omega_L{}^2\right]^2\ln\left|\frac{k_m+[k_m{}^2+(a_0\omega_L/b_0)^2]^{1/2}}{a_0\omega_L/b_0}\right|\right], \tag{4.8}$$

where

$$k_m=a(r_H{}^{-2}-\omega_L{}^2/b^2)^{1/2}.$$

About the energy density, as mentioned before we need only its initial value, which is given by Eq. (4.5).

In the case of $D=1$ (i.e., $M_D=S^1$ and $k_D$ vanishes), $\omega_L{}^2=(2\pi/l)^2n^2$, where $l$ is the circumference of the fifth dimension, and

$$\sum_{\vec{L}}{}'=\sum_{\vec{n}}{}'=2\sum_{n=1}^{n_{\max}}+(n=0).$$

$n_{\max}$ is determined by

$$n_{\max}=\left[\frac{bl}{2\pi r_4}\right]. \tag{4.9}$$

In the case that $M_D=S^D$ ($D$-dimensional sphere), $\omega_L{}^2=[L(L+D-1)+2\xi D]/r_D{}^2$ ($r_D$ is the curvature radius of $D$-dimensional space) and

$$\sum_{\vec{L}}{}'=\sum_{L=0}^{L_{\max}}D_L,$$

where

$$D_L=\frac{(2L+D-1)(L+D-2)!}{(D-1)!L!}$$

is the degeneracy factor. $L_{\max}$ is given by the maximum integer which satisfies

$$\omega_L{}^2<b^2/r_H{}^2. \tag{4.10}$$

Here, we analyze numerically the case of $D=1$, without the cosmological constant, i.e., $M_4\times S^1$. This is too simple to draw some conclusions for more realistic Kaluza-Klein cosmologies, but this example may give some insight into the possibility or the difficulty of the cosmological dimensional reduction. We shall give the analysis of the more general cases in a subsequent paper.

The equations to be solved are, from Eqs. (2.11)—(2.15), (3.25), (3.26), and (4.6), as follows.

*(1) Einstein equations:*

Definition,

$$y=\ln R=\tfrac{1}{4}(3\ln a+\ln b)\text{ and }z=\ln(b/a).$$

Constraint equation,

$$(y')^2-(z'/4)^2=(4\pi/3)G\rho R^2. \tag{4.11}$$

Dynamical equations,

$$y''+3(y')^2=2\pi G\rho R^2 \tag{4.12}$$

and

$$z''+3y'z'=-(32\pi/3)G(z'R^3)^{-1}\partial(\rho_qR^5)/\partial\eta. \tag{4.13}$$

*(2) Energy density:*

$$\rho(\eta+\Delta\eta)=\left[\frac{R(\eta)}{R(\eta+\Delta\eta)}\right]^5\rho(\eta)+\Delta\eta R^{-5}\frac{\partial}{\partial\eta}(\rho_qR^5), \tag{4.14}$$

$$\frac{\partial}{\partial\eta}(\rho_qR^5)=\frac{1}{16\pi^3l}\left[1-\frac{3G}{32\pi^2l}R^{-3}\sum_{\vec{n}}{}'I_0\right]^{-1}$$

$$\times\left\{\frac{z'}{4}\left[\left[\frac{R}{a}\right]^2\sum_{\vec{n}}{}'I_2-3\left[\frac{R}{b}\right]^2\left[\frac{2\pi}{l}\right]^2\sum_{\vec{n}}{}'n^2I_0\right]+\frac{1}{2}\left[-\tilde{Q}'+Q\int_{\eta_0}^{\eta}Q\,d\eta'\right]\sum_{\vec{n}}{}'I_0\right\}, \tag{4.15}$$

where $\tilde{Q}'=-\frac{27}{32}y'(z')^2$ and $Q=\frac{9}{64}(z')^2$. $I_0$ and $I_2$ are given by Eqs. (4.7) and (4.8).

We shall see whether the mechanism of the cosmological dimensional reduction in the Chodos-Detweiler version works well. Then, we set the horizon scale $r_H=t$ and take the following initial conditions.

*(3) Initial conditions:*

*Case (a) Without regularization term.*

(i) The initial energy density, which does not vanish, is given by (4.5).

(ii) We put $a_0=b_0=1$. Because three-space has the scale invariance and the scale of the fifth dimension is expressed by $b_0 l$, where $l$ is an undetermined parameter.

(iii) The initial expansion rates, $y'_0$ and $z'_0$, are determined by the constraint equation (4.11) and the condition that $y'_0$ is the Kasner value, i.e., $y'_0=\frac{1}{3}\eta_0$. This condition is slightly different from that in my previous paper, where we imposed the condition of $\beta'_0$ being the Kasner value. This condition is better than the previous one in the case that the energy density is created instantaneously because the source term in the equation for $z$ is the time derivative of the energy density and $z'$ may deviate from the Kasner value even at initial time.

*Case (b) With the regularization term.*

(i) The initial energy density vanishes, i.e., $\rho(t_0)=0$.

(ii) Same as for case (a).

(iii) $y'_0$ and $z'_0$ are Kasner values, i.e.,

$$y'_0=\frac{1}{3}\eta_0 \text{ and } z'_0=-\frac{4}{3}\eta_0 \ . \tag{4.16}$$

As the undetermined parameters, we have $G$, $t_0$, and $l$. $G\sim 2\pi G_N R_5$, where $R_5(\gtrsim l_P)$ is the present scale of the fifth dimension and $G_N$ is the Newtonian gravitational constant. Here, we consider two cases, that of $R_5=l_P$ and $R_5=10l_P$. $t_0$ is the initial cosmic time when the compactification occurs and the anisotropy of the space-time appears, and then the particle creation starts. We assume that $t_0$ is near the Planck time. Then, we consider three cases, that of $t_0=t_P$, $2t_P$, and $3t_P$. $l$ is the initial scale of the fifth-dimensional space. In my previous paper, we assumed that the fifth-dimensional space is also not closed. This corresponds to the limit of $l\to\infty$. So, in this paper we compare the results for various values of $l$ in order to see the effect of the closedness.

First, we show the results of case (a). In Figs. 1(a) and 1(b), we show the behavior of a measure of anisotropy, defined by

$$\frac{\Delta H}{H}\equiv\frac{(d'-\beta')/2}{(3\alpha'+\beta')/4}=-\frac{z'}{2y'} \ . \tag{4.17}$$

If the expansion of space-time is isotropic, $\Delta H/H$ vanishes. If the expansion is exactly Kasner-type, $\Delta H/H=2$. Therefore, we define the characteristic isotropization time $t_F$, denoted by the symbol + in the figure, by the time when $\Delta H/H$ becomes unity. And we define another characteristic time $t_E$, denoted by the symbol ○ in the figure, by the epoch when the fifth dimension turns from a contraction to an expansion. We can consider that the cosmological dimensional reduction is broken if $t_F$ or $t_E$ is not so large. In Fig. 1(a), we take the parameters $r_0\equiv l/2\pi=l_P$ and $G=2\pi$ (i.e., $R_5=l_P$). When we take the parameter $t_0=t_P$, $t_F\sim 6.68t_P$, and $t_E\sim 18.1t_P$. When $t_0=2t_P$ and $t_0=3t_P$, $t_F\sim 123t_P$ and $t_E\sim 337t_P$, and $t_F\sim 575t_P$ and $t_E\sim 1579t_P$, respectively. The isotropization mechanism by the particle creation is rather effective. In Fig. 1(b), the dependence of the initial radius of fifth dimensional space $r_0(\equiv l/2\pi)$ is shown in the case of $t_0=2t_P$. The curves in the cases of $r_0=5l_P$, $10l_P$, and $10^2l_P$ coincide with each other. This means that the effect of the closedness of the fifth dimensions disappears for

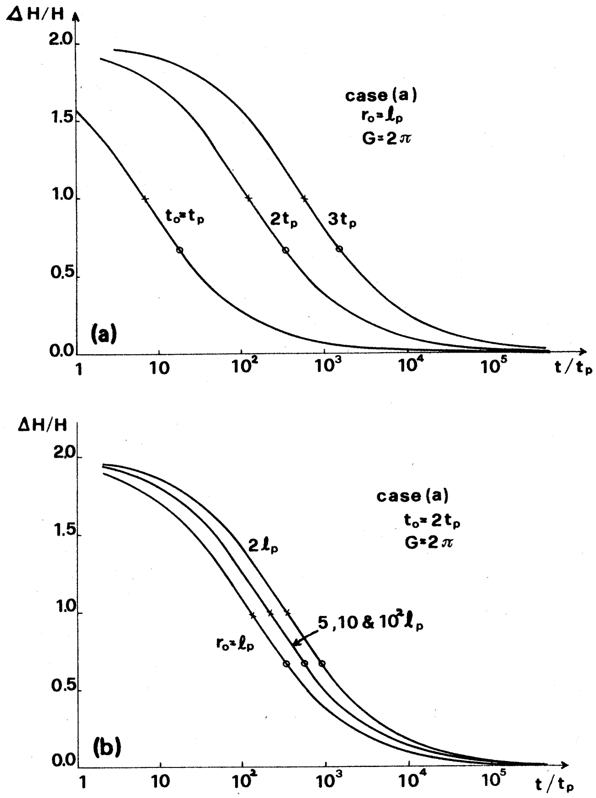


FIG. 1. A measure of anisotropy, $\Delta H/H$, is plotted with respect to $t/t_P$ for the initial time $t_0=t_P$, $2t_P$, and $3t_P$ [Fig. 1(a)], and for the initial scale of the fifth dimension $r_0=l_P$, $2l_P$, $5l_P$, $10l_P$, and $10^2l_P$ [Fig. 1(b)] in case (a). + and ○ denote the characteristic isotropization times $t_F$ and $t_E$, respectively.

$r_0>5l_P$, as mentioned in my previous paper. When $r_0=l_P$ and $2l_P$, we can see the effect of the closedness, but the result is not so different from that for the other $r_0$. In Fig. 2, we show the behavior of the cosmological scale factors $a$ and $b$. If the expansion is isotropic, i.e., $\alpha'=\beta'$, the scale factors behave as $R\propto a\propto b\propto t^{2/5}$. We show the line of $t^{2/5}$ by the dotted line as a reference. We can see that the fifth-dimensional scale $b$ shrinks at first,

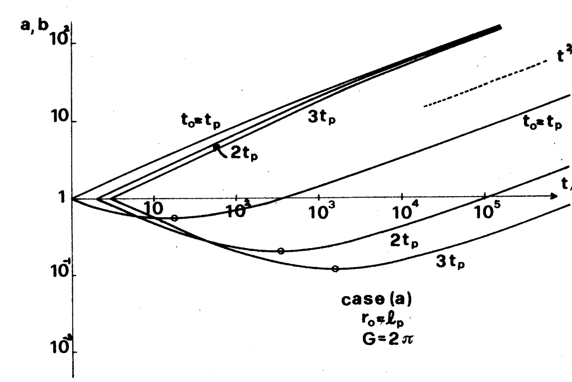


FIG. 2. The cosmological scale factors, $a$ and $b$, are plotted with respect to $t/t_P$ for the initial time $t_0=t_P$, $2t_P$, and $3t_P$ in case (a). The dashed line denotes the expansion law when the expansion is isotropic ($a,b\propto t^{2/5}$).

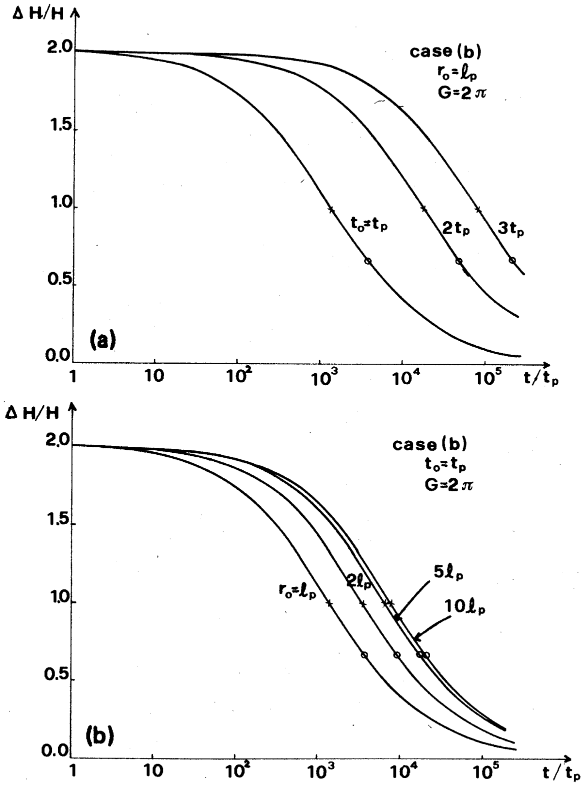


FIG. 3. $\Delta H/H$ is plotted with respect to $t/t_P$ for $t_0=t_P$, $2t_P$, and $3t_P$ [Fig. 3(a)], and for $r_0=l_P$, $2l_P$, $5l_P$, and $10l_P$ [Fig. 3(b)] in case (b) with $G=2\pi$.

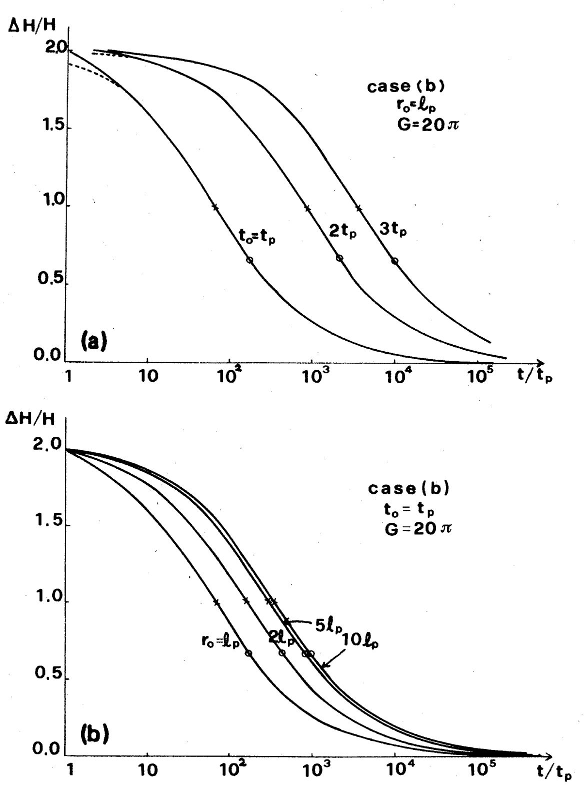


FIG. 4. The same figures as in Fig. 3 in case (b) with $G=20\pi$. The dashed lines denote the analytic solution under some approximation (see the text).

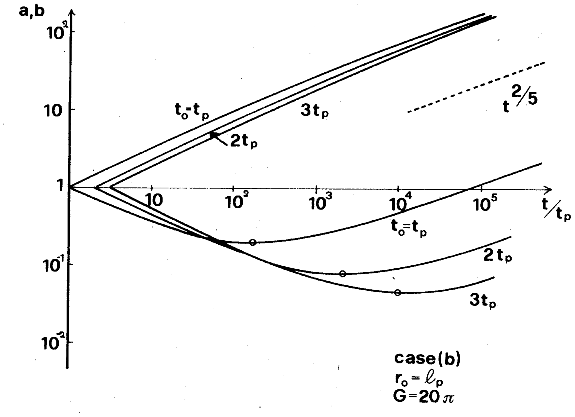


FIG. 5. $a$ and $b$ are plotted with respect to $t/t_P$ for $t_0=t_P$, $2t_P$, and $3t_P$ in case (b) with $G=20\pi$.

gradually turns from contraction to expansion, and asymptotically expands as $t^{2/5}$. The expansion of three-space is slightly decelerated and asymptotically is as $t^{2/5}$ too. The difference between the asymptotic scales of three-space and fifth-dimensional space is about order 2, which cannot explain the smallness of the fifth-dimensional space. Then, the idea of the cosmological dimensional reduction is broken by the mechanism of the particle creation.

Next, we show the result of case (b). The behavior of $\Delta H/H$ in the case of $G=2\pi$ is shown in Figs. 3(a) and 3(b). In Fig. 3(a) we set $r_0=l_P$. The isotropization times $t_F$ and $t_E$ are later than that of case (a), by the order 2. The reason is that the final energy density of created particles becomes smaller than that in case (a) by the order 1.5, by the subtraction of the regularization term. From Fig. 3(b), we can see that the effect of the scale of the fifth dimension becomes smaller as $r_0$ becomes larger, as in case (a). If we assume that the electromagnetic field is expressed by the metric of the fifth dimension, the present size of the fifth dimension is about $10l_P$. Then, we show the figures in the case of $G=20\pi$, i.e., $R_5=10l_P$ [Figs. 4(a) and 4(b)]. In this case, the created energy density is the same as the previous one, but the gravitational constant is larger and then the effect of the created matter is enhanced. Therefore, the isotropization times $t_F$ and $t_E$

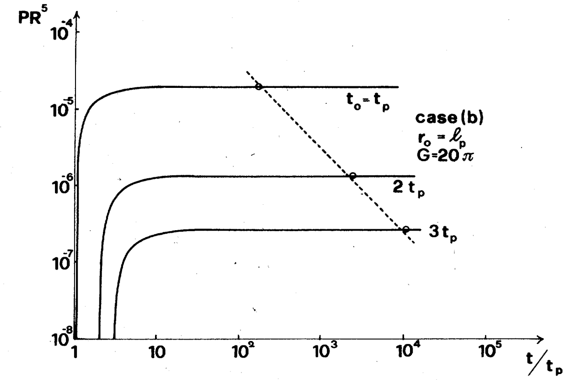


FIG. 6. The energy density $\rho R^5$ is plotted as a function of $t/t_P$ for $t_0=t_P$, $2t_P$, and $3t_P$ in case (b) with $G=20\pi$. The dashed line denotes the law $t_E \propto (\rho R^5)^{-11/12}$.

become smaller by the order 1, as seen in Fig. 4. Figure 5 corresponds to Fig. 2 in case (a). We can see the similar behavior of the isotropization for $a$ and $b$ with that in case (a). Finally, we show the energy density in Fig. 6. The isotropization point $t_E$ is denoted by the symbol ○. It seems that there is some relation between $t_E$ and the final product of $\rho R^5$, which is $t_E \propto (\rho R^5)_{\rm final}{}^{-11/12}$ (denoted by the dashed line in Fig. 6). The law is more precisely held in case (a). Also, there seems to be the relation that $(\rho R^5)_{\rm final} \propto t_0{}^{-4}$ (for $r_0 = l_P$). In Sec. V, we show that similar laws are held in a more general case (i.e., $M_4 \times T^D$) by the analytic solution.

## V. ANALYTIC SOLUTION

As known from Fig. 6, the energy density of the created particle becomes the classical one soon after on-set of the particle creation, i.e., $\rho R^5$ approaches to some constant value rapidly. The reason is that the quantum region decreases rapidly with the expansion of the Universe. Therefore we may approximate that $\rho R^5 = \text{const}$. In this section, we present the analytic solution in the case of zero-curvature constants under the approximation of $\rho R^{d+D+1} = \text{const}$ in order to compare with the numerical result and to investigate the case of topology $M_4 \times T^D$.

We set

$$F = \frac{d+D}{d+D-1} R^{d+D-1} , \tag{5.1}$$

$$\Gamma = \frac{[dD(d+D-1)]^{1/2}}{d+D} F z' , \tag{5.2}$$

and

$$K = 8\pi G \rho R^{d+D+1} . \tag{5.3}$$

The Einstein equations (2.11)–(2.15) become ($k_d = k_D = 0$ and $\Lambda = 0$)

$$(F')^2 - \Gamma^2 = 2KF , \tag{5.4}$$

$$F'' = K , \tag{5.5}$$

and

$$(\Gamma^2)' = -2K'F . \tag{5.6}$$

Since Eq. (5.6) is reduced from Eqs. (5.4) and (5.5), the independent equations are Eqs. (5.4) and (5.5). We assume $K = K_0 = \text{const}$. From Eq. (5.5),

$$F = \tfrac{1}{2} K_0 (\eta - \eta_0)^2 + F'_0 (\eta - \eta_0) + F_0 , \tag{5.7}$$

where $F'_0$ and $F_0$ are the initial values of $F'$ and $F$ at $\eta = \eta_0$. From Eq. (5.4)

$$\Gamma^2 = (F'_0)^2 - 2K_0 F_0 \equiv \Gamma_0{}^2 \tag{5.8}$$

is constant. The measure of anisotropy $\Delta H / H$ is given by

$$\begin{aligned} \frac{\Delta H}{H} &= \frac{d+D}{2} \left[ \frac{d+D-1}{dD} \right]^{1/2} \frac{\Gamma}{F'} \\ &= \frac{d+D}{2} \left[ \frac{d+D-1}{dD} \right]^{1/2} \frac{\Gamma_0}{K_0(\eta - \eta_0) + F'_0} . \end{aligned} \tag{5.9}$$

Since

$$z' = - \frac{d+D}{[dD(d+D-1)]^{1/2}} \frac{\Gamma}{F}$$

we find

$$z = z_0 - \frac{d+D}{[dD(d+D-1)]^{1/2}} \times \ln \left| \frac{(F'_0 + \Gamma_0)[K_0(\eta - \eta_0) + F'_0 - \Gamma_0]}{(F'_0 - \Gamma_0)[K_0(\eta - \eta_0) + F'_0 + \Gamma_0]} \right| . \tag{5.10}$$

The cosmological scale factors $a$ and $b$ are given by

$$a = R \exp \left[ -\frac{D}{d+D} z \right] \text{ and } b = R \exp \left[ \frac{d}{d+D} z \right] . \tag{5.11}$$

Inserting (5.7) and (5.10), we find

$$a = a_0 \left[ 1 + \frac{F'_0}{F_0}(\eta - \eta_0) + \frac{K_0}{2F_0}(\eta - \eta_0)^2 \right]^{1/(d+D-1)} \left[ \frac{(F'_0 + \Gamma_0)[K_0(\eta - \eta_0) + F'_0 - \Gamma_0]}{(F'_0 - \Gamma_0)[K_0(\eta - \eta_0) + F'_0 + \Gamma_0]} \right]^{D/[dD(d+D-1)]^{1/2}} , \tag{5.12a}$$

and

$$b = b_0 \left[ 1 + \frac{F'_0}{F_0}(\eta - \eta_0) + \frac{K_0}{2F_0}(\eta - \eta_0)^2 \right]^{1/(d+D-1)} \left[ \frac{(F'_0 + \Gamma_0)[K_0(\eta - \eta_0) + F'_0 - \Gamma_0]}{(F'_0 - \Gamma_0)[K_0(\eta - \eta_0) + F'_0 + \Gamma_0]} \right]^{-d/[dD(d+D-1)]^{1/2}} , \tag{5.12b}$$

where

$$a_0 = \left[ \frac{d+D-1}{d+D} F_0 \right]^{1/(d+D-1)} \exp \left[ -\frac{D}{d+D} z_0 \right] \tag{5.13a}$$

and

$$b_0 = \left[ \frac{d+D-1}{d+D} F_0 \right]^{1/(d+D-1)} \exp \left[ \frac{d}{d+D} z_0 \right] . \tag{5.13b}$$

Given $F_0$, $z_0$, and $F'_0$ as initial data, $a_0$, $b_0$, and $\Gamma_0$ are determined by Eqs. (5.8) and (5.13), and then the analytic solution (5.9) and (5.12) is given by using the conformal time $\eta$.

Since the cosmic time $t$ is related to $\eta$ by

$$t = t_0 + \int_{\eta_0}^{\eta} R \, d\eta' ,$$

we find

$$t=t_0+\left[\frac{d+D-1}{d+D}\right]^{1/(d+D-1)}\int_{\eta_0}^{\eta}F(\eta')^{1/(d+D-1)}d\eta'$$

$$=t_0+\frac{\Gamma_0}{K_0}\left[\frac{(d+D-1)\Gamma_0^2}{2(d+D)K_0}\right]^{1/(d+D-1)}$$

$$\times\int_{F_0'/\Gamma_0}^{(K_0/\Gamma_0)(\eta-\eta_0)+F_0'/\Gamma_0}(\zeta^2-1)^{1/(d+D-1)}d\zeta\ . \tag{5.14}$$

This integral is given by hypergeometric functions. By Eqs. (5.9), (5.12), and (5.14), we can compare the analytic solution with the numerical results. In case (a), if we take $K_0$ the same value as the numerical one, the behavior of $a$, $b$, and $\Delta H/H$ is exactly the same as the numerical one. The reason is that the energy density is almost created at initial time $\eta_0$ in case (a) and the approximation of $K$=const is just valid. Even in case (b), in which the energy density initially vanishes and is created afterwards, the behavior of $a$ and $b$ is exactly the same as the numerical one. Only the behavior of $\Delta H/H$ is slightly deviated from the numerical one at the early stage, as denoted by the dotted lines in Fig. 4(a), but it also coincides soon with the numerical one. This shows that the approximation of $K$=const is valid even in case (b).

Using this analytical solution, we shall investigate the space-time with $M_4\times T^D$ topology.

First, we have to estimate the created energy density from Eq. (4.5). We put $a_0=b_0=1$. We can consider two cases: one case is that $r_0<r_H(\sim t_0)$ and the other case is that $r_0>r_H$. In the former case, $n_{\max}=0$ and then

$$\rho R^{D+4}\sim\frac{1}{16\pi^3(2\pi r_0)^D}\pi\left[\frac{1}{t_0}\right]^4$$

$$=\frac{\pi^2}{(2\pi)^{D+4}}\frac{1}{r_0^D}\frac{1}{t_0^4}\ . \tag{5.15a}$$

In the latter case, $n_{\max}\sim r_0/t_0$ and then

$$\rho R^{D+4}\sim\frac{1}{16\pi^3(2\pi r_0)^D}\left[\frac{r_0}{t_0}\right]^D\pi\left[\frac{1}{t_0}\right]^4$$

$$=\frac{\pi^2}{(2\pi)^{D+4}}\frac{1}{t_0^{D+4}}\ . \tag{5.15b}$$

The factor $\pi^2/(2\pi)^{D+4}$ is $1.01\times10^{-3}$ for $D=1$. In the numerical calculation, $\rho R^5\sim9.2\times10^{-4}$ for $r_0=1$ and $t_0=1$ in case (a). So, the above estimate is not so bad.

Using Eq. (5.15), we shall estimate the characteristic isotropization time $t_E$. $t_E$ is defined by the time when $\beta'$ vanishes. Since $\beta'\equiv y'+[d/(d+D)]z'=0$, $\Delta H/H\equiv-z'/2y'=(d+D)/2d$ at $t_E$. From Eq. (5.9), we find

$$K_0(\eta_E-\eta_0)=\left[\frac{d(d+D-1)}{D}\right]^{1/2}\Gamma_0-F_0'\ , \tag{5.16}$$

and then, from Eq. (5.14),

$$t_E=t_0+\frac{\Gamma_0}{K_0}\left[\frac{(d+D-1)\Gamma_0^2}{2(d+D)K_0}\right]^{1/(d+D-1)}$$

$$\times\int_{F_0'/\Gamma_0}^{[d(d+D-1)/D]^{1/2}}(\zeta^2-1)^{1/(d+D-1)}d\zeta\ . \tag{5.17}$$

The initial values of $\Gamma_0$ and $F_0'$ are $t_0^{-1}$ if they are Kasner values. Since we shall start from the state near the Kasner universe, $F_0'/\Gamma_0\sim1$. Then, the integral of the above equation is almost independent of the initial condition and gives the numerical value of order 1. Then, we find

$$t_E\sim t_0+\left[\frac{d+D-1}{2(d+D)}\right]^{1/(d+D-1)}$$

$$\times K_0^{-(d+D)/(d+D-1)}\Gamma_0^{(d+D+1)/(d+D-1)}\ . \tag{5.18}$$

Setting $d=3$ and $\Gamma_0\sim t_0^{-1}$, and taking $K_0$ as the value given by Eq. (5.15), we get

$$t_E\sim t_0+\left[\frac{D+2}{2(D+3)}\right]^{1/(D+2)}$$

$$\times\left[\frac{1}{2\pi}\frac{G}{(2\pi r_0)^D}\right]^{-(D+3)/(D+2)}$$

$$\times\begin{cases}t_0^{(3D+8)/(D+2)} & \text{for } r_0<t_0\\ (t_0/r_0)^{(D+3)/(D+2)}t_0^{(3D+8)/(D+2)} & \text{for } r_0>t_0\ .\end{cases} \tag{5.19}$$

When $D=1$, $G=2\pi$, $r_0=l_P$, and $t_0=t_P$ (see Fig. 1), Eq. (5.19) gives $t_E\sim9.4t_P$. And the dependence of $t_0$ is $t_E\propto t_0^{11/3}$ and $K_0\sim t_0^{-4}$, and then we have $t_E\sim K_0^{-11/12}$, which is just the law from the numerical result. If we consider the regularization term, the energy density becomes smaller than that of Eq. (5.15) by the subtraction terms. However, the above law is held even in this case. Then, the dependence of $t_0$ etc., on the energy density may be kept and the qualitative behavior is quite similar to case (a). The difference may be only the delay of the isotropization time by the decrease of the energy density. We can see how much it is delayed from Eq. (5.18). If we put $D=7$, $G=(2\pi)^7$, $r_0=l_P$, and $t_0=t_P$, Eq. (5.19) gives $t_E\sim8.05t_P$ and $t_E\propto t_0^{29/9}$. From this, we expect that there is not such a large difference between the case of $D=1$ and the other case, but that they are almost the same. Therefore, also in the case of $M_4\times T^D$, the idea of the cosmological dimensional reduction may be broken.

## VI. CONCLUDING REMARKS

In Sec. V we concluded that if we consider the particle creation effect, the cosmological dimensional reduction is difficult in simple models ($M_4\times T^D$). However, in order to confirm this result in the more realistic Kaluza-Klein model we have to consider the following things:

(1) the regularization problem,

(2) the topological effect (or curvature effect),

(3) the effect of the anisotropic coherent matter (Freund-Rubin matter).

For (1), our method is too crude. Recently, Appelquist and Chodos have shown that the five-dimensional static Kaluza-Klein vacuum ($M^4 \times S^1$) is unstable for a contraction of the fifth dimension when we consider a quantum effect of gravitational field. This instability is due to the Casimir effect. We expect the Casimir effect also in the case of the scalar field. This Casimir effect may prevent the fifth dimension from turning to expansion by the particle creation effect. In order to treat both effects systematically, we have to regularize correctly the energy-momentum tensor or the effective action. However, we have no complete regularization method in the case that the space-time is time dependent and not conformally flat. Recently, Randjbar-Daemi, Salam, and Strathdee,[14] and Gilbert, McClain, and Rubin,[15] and M. Yoshimura[16] gave the effective action in time-dependent background geometry under some approximation. These methods are good approaches. However, from the point of view of the cosmological dimensional reduction, the actions of Refs. 14 and 15 are valid at a late stage, but may not be valid at an early stage, i.e., just after the compactification. (The scale of the internal space is nearly the same as that of physical three-space.) The action by Yoshimura may be useful even at an early stage, if the Universe is not in the vacuum state but in the thermal equilibrium state at initial time.

As another approach, we can consider the following. If the deviation from the conformal flatness is small, i.e., space-time has small anisotropy, we can treat the problem by the perturbation and can also regularize the effective action by the dimensional regularization.[17] This approach may confirm the isotropization by the particle creation dealt with in this paper. We shall show this result elsewhere.

Secondly, the topological effect (or curvature effect) may be more important, because the realistic Kaluza-Klein theory predicts the internal space has a curvature. The effect of curvature may change the dynamical behavior of the metric. Then, does the particle creation have an effect on this behavior? Also, the existence of anisotropic matter may change how the particles are created. These problems are in progress.

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## APPENDIX: ENERGY DENSITY WITH ADIABATIC REGULARIZATION TERM

Adopting the adiabatic regularization method proposed by Fulling, Parker, and Hu,[18] we write down the regularized energy density as follows:

$$\rho \equiv -\langle 0_A | T^0_0 | 0_A \rangle$$

$$= \frac{1}{2V_d V_D} R^{-(d+D+1)} \sum_{\vec{l}} \sum_{\vec{L}} \left\{ |\chi'_{\vec{l},\vec{L}}|^2 + (\Omega^2 - Q)|\chi_{\vec{l},\vec{L}}|^2 - \Omega - \frac{1}{2\Omega}\left[\frac{1}{4}\left[\frac{\Omega'}{\Omega}\right]^2 - Q\right] + \frac{1}{2\Omega}\left[\frac{1}{8}\left[\frac{\Omega'}{\Omega}\right]^2 \epsilon_{2(2)} - \frac{1}{4}\frac{\Omega'}{\Omega}\epsilon'_{2(3)} - \tfrac{1}{4}\Omega^2(\epsilon_{2(2)})^2 - \tfrac{1}{2}Q\epsilon_{2(2)}\right]\right\}, \quad \text{(A1)}$$

where the definitions of $\epsilon_{2(2)}$ and $\epsilon_{2(3)}$ are the same as in Eqs. (2.40) and (2.41) in Ref. 18 and $\Omega = \Omega_{l,L}$.

The energy density in the quantum region $q(t)$ is

$$\rho_{q(t)} = \frac{1}{2V_d V_D} R^{-(d+D+1)} \sum_{L \in q(t)} \left\{ \frac{1}{2\Omega_0}\left[\Omega^2 - Q + \left[\int_{\eta_0}^{\eta} Q\, d\eta'\right]^2\right] + \frac{\Omega_0}{2} - \Omega - \frac{1}{2\Omega}\left[\frac{1}{4}\left[\frac{\Omega'}{\Omega}\right]^2 - Q\right] + \frac{1}{2\Omega}[\cdots]\right\}, \quad \text{(A2)}$$

where $\Omega_0 = \Omega(\eta_0)$.

As mentioned in the text, we take into account one part of the regularization terms, i.e., $-\Omega + (1/2\Omega)Q$, by which the initial energy density vanishes. Following the text (Sec. IV), we assume the space-time is $M_4 \times M_D$, where $M_4$ is four-dimensional flat universe and is not closed. From (A2),

$$\rho_q(t) = \frac{1}{16\pi^3 V_D} R^{-(D+4)} \sum_{\vec{L}}{}' \int' d^3k \left\{ \frac{1}{2\Omega_0}\left[\Omega^2 - Q + \left[\int_{\eta_0}^{\eta} Q d\eta'\right]^2\right] + \frac{\Omega_0}{2} - \Omega + \frac{1}{2\Omega} Q \right\}, \quad \text{(A3)}$$

and then

$$\frac{\partial}{\partial \eta}(\rho_q R^{D+4}) = \frac{1}{16\pi^3 V_D} \sum_{\vec{L}}{}' \int' d^3k \left[ \frac{1}{2\Omega_0}\left[2\Omega\Omega' - Q' + Q\int_{\eta_0}^{\eta} Q d\eta'\right] - \frac{1}{2\Omega}\left[2\Omega\Omega' - Q' + \frac{\Omega'}{\Omega} Q\right]\right]. \quad \text{(A4)}$$

In the same way as in the text, we reexpress $\partial(\rho_q R^{D+4})/\partial\eta$ by the terms without the second-order derivative.

The result is the following:

$$\frac{\partial}{\partial\eta}(\rho_q R^{D+4})=\frac{1}{16\pi^3 V_D}\left[1-\frac{G}{8\pi^2 V_D}\frac{(D+2)}{(D+3)}R^{-(D+2)}\sum_{\vec{L}}{}'(I_0-J_0)\right]^{-1}$$
$$\times\left\{\frac{z'}{D+3}\left[D\left[\frac{R}{a}\right]^2\sum_{\vec{L}}{}'\left[I_2-J_2-\frac{Q}{2}K_2\right]-3\left[\frac{R}{b}\right]^2\sum_{\vec{L}}{}'\omega_L^2\left[I_0-J_0-\frac{Q}{2}K_0\right]\right]\right.$$
$$\left.-\tfrac{1}{2}\tilde{Q}'\sum_{\vec{L}}{}'(I_0-J_0)+\tfrac{1}{2}Q\int_{\eta_0}^{\eta}Q\,d\eta'\sum_{\vec{L}}{}'I_0\right\}, \tag{A5}$$

where $I_0$ and $I_2$ are the same as in the text,

$$J_0\equiv\int{}'d^3k\frac{1}{\Omega}=2\pi\left[\frac{a}{R}\right]\left[k_m\frac{a}{r_H}-\left[\frac{a^2}{b^2}\omega_L{}^2\right]^2\ln\left|\frac{k_m+a/r_H}{a\omega_L/b}\right|\right], \tag{A6}$$

$$J_2\equiv\int{}'d^3k\frac{k^2}{\Omega}=\frac{\pi}{2}\left[\frac{a}{R}\right]\left\{k_m\frac{a}{r_H}\left[2k_m{}^2-3\left[\frac{a^2}{b^2}\omega_L{}^2\right]^2\right]+3\left[\frac{a^2}{b^2}\omega_L{}^2\right]^2\ln\left|\frac{k_m+a/r_H}{a\omega_L/b}\right|\right\}, \tag{A7}$$

$$K_0\equiv\int{}'d^3k\frac{1}{\Omega^3}=4\pi\left[\frac{a}{R}\right]^3\left[-\frac{k_m r_H}{a}+\ln\left|\frac{k_m+a/r_H}{a\omega_L/b}\right|\right], \tag{A8}$$

and

$$K_2\equiv\int{}'d^3k\frac{k^2}{\Omega^3}=2\pi\left[\frac{a}{R}\right]^3\left\{k_m\left[\frac{a}{r_H}+2\left[\frac{a^2}{b^2}\omega_L{}^2\right]\frac{r_H}{a}\right]-3\left[\frac{a^2}{b^2}\omega_L{}^2\right]^2\ln\left|\frac{k_m+a/r_H}{a\omega_L/b}\right|\right\}. \tag{A9}$$

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